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Volume 96

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Second Edition

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To

Brigitta, Karen, and Kenneth Gasper

and

Babu, Raja, and to the memory of Parul S. Rahman

Contents

Foreword

My education was not much different from that of most mathematicians of my generation. It included courses on modern algebra, real and complex variables, both point set and algebraic topology, some number theory and projective geometry, and some specialized courses such as one on Riemann surfaces. In none of these courses was a hypergeometric function mentioned, and I am not even sure if the gamma function was mentioned after an advanced calculus course. The only time Bessel functions were mentioned was in an undergraduate course on differential equations, and the only thing done with them was to find a power series solution for the general Bessel equation. It is small wonder that with a similar education almost all mathematicians think of special functions as a dead subject which might have been interesting once. They have no idea why anyone would care about it now.

Fortunately there was one part of my education which was different. As a junior in college I read Widder's book *The Laplace Transform* and the manuscript of its very important sequel, Hirschman and Widder's *The Convolution Transform*. Then as a senior, I. I. Hirschman gave me a copy of a preprint of his on a multiplier theorem for Legendre series and suggested I extend it to ultraspherical series. This forced me to become acquainted with two other very important books, Gabor Szegő's great book *Orthogonal Polynomials*, and the second volume of *Higher Transcendental Functions*, the monument to Harry Bateman which was written by Arthur Erdélyi and his co-workers W. Magnus, F. Oberhettinger and F. G. Tricomi.

From this I began to realize that the many formulas that had been found, usually in the 18th or 19th century, but once in a while in the early 20th century, were useful, and started to learn about their structure. However, I had written my Ph.D. thesis and worked for three more years before I learned that not every fact about special functions I would need had already been found, and it was a couple of more years before I learned that it was essential to understand hypergeometric functions. Like others, I had been put off by all the parameters. If there were so many parameters that it was necessary to put subscripts on them, then there has to be a better way to solve a problem than this. That was my initial reaction to generalized hypergeometric functions, and a very common reaction to judge from the many conversations I have had on these functions in the last twenty years. After learning a little more about hypergeometric functions, I was very surprised to realize that they had occurred regularly in first year calculus. The reason for the subscripts on the parameters is that not all interesting polynomials are of degree one or two. For a generalized hypergeometric function has a series representation

$$\sum_{n=0}^{\infty} c_n \tag{1}$$

with c_{n+1}/c_n a rational function of n. These contain almost all the examples of infinite series introduced in calculus where the ratio test works easily. The ratio c_{n+1}/c_n can be factored, and it is usually written as

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)\cdots(n+a_p)x}{(n+b_1)\cdots(n+b_q)(n+1)}. \tag{2}$$

Introduce the shifted factorial

$$\begin{aligned} (a)_0 &= 1, \\ (a)_n &= a(a+1)\cdots(a+n-1), \quad n = 1, 2, \ldots . \end{aligned} \tag{3}$$

Then if $c_0 = 1$, equation (2) can be solved for c_n as

$$c_n = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} \tag{4}$$

and

$${}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} \tag{5}$$

is the usual notation.

The first important result for a ${}_pF_q$ with $p > 2,\ q > 1$ is probably Pfaff's sum

$${}_3F_2\left[\begin{matrix} -n,\ a,\ b \\ c,\ a+b+1-c-n \end{matrix}; 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n = 0, 1, \ldots . \tag{6}$$

This result from 1797, see Pfaff [1797], contains as a limit when $n \to \infty$, another important result usually attributed to Gauss [1813],

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\ (c-a-b) > 0. \tag{7}$$

The next instance is a very important result of Clausen [1828]:

$$\left\{{}_2F_1\left[\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix}; x\right]\right\}^2 = {}_3F_2\left[\begin{matrix} 2a,\ 2b,\ a+b \\ a+b+\frac{1}{2},\ 2a+2b \end{matrix}; x\right]. \tag{8}$$

Some of the interest in Clausen's formula is that it changes the square of a class of ${}_2F_1$'s to a ${}_3F_2$. In this direction it is also interesting because it was probably the first instance of anyone finding a differential equation satisfied by $[y(x)]^2,\ y(x)z(x)$ and $[z(x)]^2$ when $y(x)$ and $z(x)$ satisfy

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{9}$$

This problem was considered for (9) by Appell, see Watson [1952], but the essence of his general argument occurs in Clausen's paper. This is a common phenomenon, which is usually not mentioned when the general method is introduced to students, so they do not learn how often general methods come from specific problems or examples. See D. and G. Chudnovsky [1988] for an instance of the use of Clausen's formula, where a result for a ${}_2F_1$ is carried to a ${}_3F_2$ and from that to a very interesting set of expansions of π^{-1}. Those identities were first discovered by Ramanujan. Here is Ramanujan's most impressive example:

$$\frac{9801}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty}[1103 + 26390n]\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n n!}\frac{1}{(99)^{4n}}. \tag{10}$$

There is another important reason why Clausen's formula is important. It leads to a large class of ${}_3F_2$'s that are nonnegative for the power series variable between -1 and 1. The most famous use of this is in the final step of de Branges' solution of the Bieberbach conjecture, see de Branges [1985]. The integral of the ${}_2F_1$ or Jacobi polynomial he had is a ${}_3F_2$, and its positivity is an easy consequence of Clausen's formula, as Gasper had observed ten years earlier. There are other important results which follow from the positivity in Clausen's identity.

Once Kummer [1836] wrote his long and important paper on ${}_2F_1$'s and ${}_1F_1$'s, this material became well-known. It has been reworked by others. Riemann redid the ${}_2F_1$ using his idea that the singularities of a function go a long way toward determining the function. He showed that if the differential equation (9) has regular singularities at three points, and every other point in the extended complex plane is an ordinary point, then the equation is equivalent to the hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0, \tag{11}$$

which has regular singular points at $x = 0, 1, \infty$. Riemann's work was very influential, so much so that much of the mathematical community that considered hypergeometric functions studied them almost exclusively from the point of view of differential equations. This is clear in Klein's book [1933], and in the work on multiple hypergeometric functions that starts with Appell in 1880 and is summarized in Appell and Kampé de Fériet [1926].

The integral representations associated with the differential equation point of view are similar to Euler's integral representation. This is

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 (1-xt)^{-a}t^{b-1}(1-t)^{c-b-1}dt, \tag{12}$$

$|x| < 1$, $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, and includes related integrals with different contours. The differential equation point of view is very powerful where it works, but it does not work well for $p \geq 3$ or $q \geq 2$ as Kummer discovered. Thus there is a need to develop other methods to study hypergeometric functions.

In the late 19th and early 20th century a different type of integral representation was introduced. These two different types of integrals are best represented by Euler's beta integral

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \text{Re } (a,b) > 0 \tag{13}$$

and Barnes' beta integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)\ dt \tag{14}$$
$$= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \quad \text{Re } (a,b,c,d) > 0.$$

There is no direct connection with differential equations for integrals like (14), so it stands a better chance to work for larger values of p and q.

While Euler, Gauss, and Riemann and many other great mathematicians wrote important and influential papers on hypergeometric functions, the development of basic hypergeometric functions was much slower. Euler and Gauss did important work on basic hypergeometric functions, but most of Gauss' work was unpublished until after his death and Euler's work was more influential on the development of number theory and elliptic functions.

Basic hypergeometric series are series $\sum c_n$ with c_{n+1}/c_n a rational function of q^n for a fixed parameter q, which is usually taken to satisfy $|q| < 1$, but at other times is a power of a prime. In this Foreword $|q| < 1$ will be assumed.

Euler summed three basic hypergeometric series. The one which had the largest impact was

$$\sum_{-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2} = (q;q)_\infty, \tag{15}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{16}$$

If

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty \tag{17}$$

then Euler also showed that

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}, \quad |x| < 1, \tag{18}$$

and

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n}. \tag{19}$$

Eventually all of these were contained in the q-binomial theorem

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n, \quad |x| < 1. \tag{20}$$

While (18) is clearly the special case $a = 0$, and (19) follows easily on replacing x by xa^{-1} and letting $a \to \infty$, it is not so clear how to obtain (15) from (20). The easiest way was discovered by Cauchy and many others. Take $a = q^{-2N}$, shift n by N, rescale and let $N \to \infty$. The result is called the triple product, and can be written as

$$(x;q)_\infty (qx^{-1};q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n. \tag{21}$$

Then $q \to q^3$ and $x = q$ gives Euler's formula (15).

Gauss used a basic hypergeometric series identity in his first proof of the determination of the sign of the Gauss sum, and Jacobi used some to determine the number of ways an integer can be written as the sum of two, four, six and eight squares. However, this particular aspect of Gauss' work on Gauss sums was not very influential, as his hypergeometric series work had been, and Jacobi's work appeared in his work on elliptic functions, so its hypergeometric character was lost in the great interest in the elliptic function work. Thus neither of these led to a serious treatment of basic hypergeometric series. The result that seems to have been the crucial one was a continued fraction of Eisenstein. This along with the one hundredth anniversary of Euler's first work on continued fractions seem to have been the motivating forces behind Heine's introduction of a basic hypergeometric extension of ${}_2F_1(a,b;c;x)$. He considered

$${}_2\phi_1 \left[\begin{matrix} q^a, \ q^b \\ q^c \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(q^a;q)_n (q^b;q)_n}{(q^c;q)_n (q;q)_n} x^n, \quad |x| < 1. \tag{22}$$

Observe that

$$\lim_{q \to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n,$$

so

$$\lim_{q \to 1} {}_2\phi_1 \left[\begin{matrix} q^a, \ q^b \\ q^c \end{matrix} ; q, x \right] = {}_2F_1 \left[\begin{matrix} a, \ b \\ c \end{matrix} ; x \right].$$

Heine followed the pattern of Gauss' published paper on hypergeometric series, and so obtained contiguous relations and from them continued fraction expansions. He also obtained some series transformations, and the sum

$$ {}_2\phi_1\left[\begin{matrix} q^a, \ q^b \\ q^c \end{matrix}; q, q^{c-a-b}\right] = \frac{(q^{c-a};q)_\infty (q^{c-b};q)_\infty}{(q^c;q)_\infty (q^{c-a-b};q)_\infty}, \quad |q^{c-a-b}| < 1. \qquad (23) $$

This sum becomes (7) when $q \to 1$.

As often happens to path breaking work, this work of Heine was to a large extent ignored. When writing the second edition of *Kugelfunctionen* (Heine [1878]) Heine decided to include some of his work on basic hypergeometric series. This material was printed in smaller type, and it is clear that Heine included it because he thought it was important, and he wanted to call attention to it, rather than because he thought it was directly related to spherical harmonics, the subject of his book. Surprisingly, his inclusion of this material led to some later work, which showed there was a very close connection between Heine's work on basic hypergeometric series and spherical harmonics. The person Heine influenced was L. J. Rogers, who is still best known as the first discoverer of the Rogers–Ramanujan identities. Rogers tried to understand this aspect of Heine's work, and one transformation in particular. Thomae [1879] had observed this transformation of Heine could be written as an extension of Euler's integral representation (12), but Rogers was unaware of this explanation, and so discovered a second reason. He was able to modify the transformation so it became the permutation symmetry in a new series. While doing this he introduced a new set of polynomials which we now call the continuous q-Hermite polynomials. In a very important set of papers which were unjustly neglected for decades, Rogers discovered a more general set of polynomials and found some remarkable identities they satisfy, see Rogers [1893a,b, 1894, 1895]. For example, he found the linearization coefficients of these polynomials which we now call the continuous q-ultraspherical polynomials. These polynomials contain many of the spherical harmonics Heine studied. Contained within this product identity is the special case of the square of one of these polynomials as a double series. As Gasper and Rahman have observed, one of these series can be summed, and the resulting identity is an extension of Clausen's sum in the terminating case. Earlier, others had found a different extension of Clausen's identity to basic hypergeometric series, but the resulting identity was not satisfactory. The identity had the product of two functions, the same functions but one evaluated at x and the other at qx, and so was not a square. Thus the nonnegativity that is so useful in Clausen's formula was not true for the corresponding basic hypergeometric series. Rogers' result for his polynomials led directly to the better result which contains the appropriate nonnegativity. From this example and many others, one sees that orthogonal polynomials provide an alternative approach to the study of hypergeometric and basic hypergeometric functions. Both this approach and that of differential equations are most useful for small values of the degrees of the numerator and denominator polynomials in the ratio c_{n+1}/c_n, but orthogonal polynomials work for a larger class of series, and are much more useful for basic hypergeometric series. However, neither of these approaches is powerful enough to encompass all aspects of these functions. Direct series manipulations are surprisingly useful, when done by a master, or when a computer algebra system is used as an aid. Gasper and Rahman are both experts at symbolic calculations, and I regularly marvel at some of the formulas they have found. As quantum groups become better known, and as Baxter's work spreads to other parts of mathematics as it has started to do, there will be many people trying to learn how to deal with basic hypergeometric series. This book is where I would start.

For many years people have asked me what is the best book on special functions. My response was George Gasper's copy of Bailey's book, which was heavily annotated with useful results and remarks. Now others can share the information contained in these margins, and many other very useful results.

Richard Askey
University of Wisconsin

Preface

The study of basic hypergeometric series (also called q-hypergeometric series or q-series) essentially started in 1748 when Euler considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} (1-q^{k+1})^{-1}$ as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers (see §8.10). But it was not until about a hundred years later that the subject acquired an independent status when Heine converted a simple observation that $\lim_{q\to 1}[(1-q^a)/(1-q)] = a$ into a systematic theory of ${}_2\phi_1$ basic hypergeometric series parallel to the theory of Gauss' ${}_2F_1$ hypergeometric series. Heine's transformation formulas for ${}_2\phi_1$ series and his q-analogue of Gauss' ${}_2F_1(1)$ summation formula are derived in Chapter 1, along with a q-analogue of the binomial theorem, Jacobi's triple product identity, and some formulas for q-analogues of the exponential, gamma and beta functions.

Apart from some important work by J. Thomae and L. J. Rogers the subject remained somewhat dormant during the latter part of the nineteenth century until F. H. Jackson embarked on a lifelong program of developing the theory of basic hypergeometric series in a systematic manner, studying q-differentiation and q-integration and deriving q-analogues of the hypergeometric summation and transformation formulas that were discovered by A. C. Dixon, J. Dougall, L. Saalschütz, F. J. W. Whipple, and others. His work is so pervasive that it is impossible to cover all of his contributions in a single volume of this size, but we have tried to include many of his important formulas in the first three chapters. In particular, a derivation of his summation formula for an ${}_8\phi_7$ series is given in §2.6. During the 1930's and 1940's many important results on hypergeometric and basic hypergeometric series were derived by W. N. Bailey. Some mathematicians consider Bailey's greatest work to be the Bailey transform (an equivalent form of which is covered in Chapter 2), but equally significant are his nonterminating extensions of Jackson's ${}_8\phi_7$ summation formula and of Watson's transformation formula connecting very-well-poised ${}_8\phi_7$ series with balanced ${}_4\phi_3$ series. Much of the material on summation, transformation and expansion formulas for basic hypergeometric series in Chapter 2 is due to Bailey.

D. B. Sears, L. Carlitz, W. Hahn, and L. J. Slater were among the prominent contributors during the 1950's. Sears derived several transformation formulas for ${}_3\phi_2$ series, balanced ${}_4\phi_3$ series, and very-well-poised ${}_{n+1}\phi_n$ series. Simple proofs of some of his ${}_3\phi_2$ transformation formulas are given in Chapter 3. Three of his very-well-poised transformation formulas are derived in Chapter 4, where we follow G. N. Watson and Slater to develop the theory of basic hypergeometric series from a contour integral point of view, an idea first introduced by Barnes in 1907.

Chapter 5 is devoted to bilateral basic hypergeometric series, where the most fundamental formula is Ramanujan's ${}_1\psi_1$ summation formula. Substantial contributions were also made by Bailey, M. Jackson, Slater and others, whose works form the basis of this chapter.

During the 1960's R. P. Agarwal and Slater each published a book partially devoted to the theory of basic hypergeometric series, and G. E. Andrews initiated his work in number theory, where he showed how useful the summation and transformation formulas for basic hypergeometric series are in the theory of partitions. Andrews gave simpler proofs of many old results, wrote review articles pointing out many important applications and, during the mid 1970's, started a period of very fruitful collaboration with R. Askey. Thanks to these two mathematicians, basic hypergeometric series is an active field of research today. Since Askey's primary area of interest is orthogonal polynomials, q-series suddenly provided him and his co-workers with a very rich environment for deriving q-extensions of beta integrals and of the classical orthogonal polynomials of Jacobi, Gegenbauer, Legendre, Laguerre and Hermite. Askey and his students and collaborators who include W. A. Al-Salam, M. E. H. Ismail, T. H. Koornwinder, W. G. Morris, D. Stanton, and J. A. Wilson have produced a substantial amount of interesting work over the past fifteen years. This flurry of activity has been so infectious that many researchers found themselves hopelessly trapped by this alluring "q-disease", as it is affectionately called.

Our primary motivation for writing this book was to present in one modest volume the significant results of the past two hundred years so that they are readily available to students and researchers, to give a brief introduction to the applications to orthogonal polynomials that were discovered during the current renaissance period of basic hypergeometric series, and to point out important applications to other fields. Most of the material is elementary enough so that persons with a good background in analysis should be able to use this book as a textbook and a reference book. In order to assist the reader in developing a deeper understanding of the formulas and proof techniques and to include additional formulas, we have given a broad range of exercises at the end of each chapter. Additional information is provided in the Notes following the Exercises, particularly in relation to the results and relevant applications contained in the papers and books listed in the References. Although the References may have a bulky appearance, it is just an introduction to the vast literature available. Appendices I, II, and III are for quick reference, so that it is not necessary to page through the book in order to find the most frequently needed identities, summation formulas, and transformation formulas. It can be rather tedious to apply the summation and transformation formulas to the derivation of other formulas. But now that several symbolic computer algebraic systems are available, persons having access to such a system can let it do some of the symbolic manipulations, such as computing the form of Bailey's ${}_{10}\phi_9$ transformation formula when its parameters are replaced by products of other parameters.

Due to space limitations, we were unable to be as comprehensive in our coverage of basic hypergeometric series and their applications as we would have liked. In particular, we could not include a systematic treatment of basic hypergeometric series in two or more variables, covering F. H. Jackson's work on basic Appell series and the works of R. A. Gustafson and S. C. Milne on $U(n)$ multiple series generalizations of basic hypergeometric series referred to in the References. But we do highlight Askey and Wilson's fundamental work on their beautiful q-analogue of the classical beta integral in Chapter 6 and develop its connection with very-well-poised ${}_8\phi_7$ series. Chapter 7 is devoted to applications to orthogonal polynomials, mostly developed by Askey and his collaborators. We conclude the book with some further applications in Chapter 8, where we present part of our work on product and linearization formulas, Poisson kernels, and nonnegativity, and we also manage to point out some elementary facts about applications to the theory of partitions and the representations of integers as sums of squares of integers. The interested reader is referred to the books and papers of Andrews and N. J. Fine for additional applications to partition theory, and recent references are pointed out for applications to affine root systems (Macdonald identities), association schemes, combinatorics, difference equations, Lie algebras and groups, physics (such as representations of quantum groups and R. J. Baxter's work on the hard hexagon model of phase transitions in statistical mechanics), statistics, etc.

We use the common numbering system of letting (k.m.n) refer to the n-th numbered display in Section m of Chapter k, and letting (I.n), (II.n), and (III.n) refer to the n-th numbered display in Appendices I, II, and III, respectively. To refer to the papers and books in the References, we place the year of publication in square brackets immediately after the author's name. Thus Bailey [1935] refers to Bailey's 1935 book. Suffixes a, b, ... are used after the years to distinguish different papers by an author that appeared in the same year. Papers that have not yet been published are referred to with the year 2004, even though they might be published later due to the backlogs of journals. Since there are three Agarwals, two Chiharas and three Jacksons listed in the References, to minimize the use of initials we drop the initials of the author whose works are referred to most often. Hence Agarwal, Chihara, and Jackson refer to R. P. Agarwal, T. S. Chihara, and F. H. Jackson, respectively.

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Preface to the second edition

In 1990 it was beyond our wildest imagination that we would be working on a second edition of this book thirteen years later. In this day and age of rapid growth in almost every area of mathematics, in general, and in Orthogonal Polynomials and Special Functions, in particular, it would not be surprising if the book became obsolete by now and gathered dust on the bookshelves. All we hoped for is a second printing. Even that was only a dream since the main competitor of authors and publishers these days are not other books, but the ubiquitous copying machine. But here we are: bringing out a second edition with full support of our publisher.

The main source of inspiration, of course, has been the readers and users of this book. The response has been absolutely fantastic right from the first weeks the book appeared in print. The kind of warm reception we enjoyed far exceeded our expectations. Years later many of the leading researchers in the field kept asking us if an updated version would soon be forthcoming. Yes, indeed, an updated and expanded version was becoming necessary during the latter part of the 1990's in view of all the explosive growth that the subject was experiencing in many different areas of applications of basic hypergeometric series (also called q-series). However, the most important and significant impetus came from an unexpected source — Statistical Mechanics. In trying to find elliptic (doubly periodic meromorphic) solutions of the so-called Yang-Baxter equation arising out of an eight-vertex model in Statistical Mechanics the researchers found that the solutions are, in fact, a form of hypergeometric series $\sum a_n z^n$, where a_{n+1}/a_n is an elliptic function of n, with n regarded as a complex variable. In a span of only five years the study of elliptic hypergeometric series and integrals has become almost a separate area of research on its own, whose leading researchers include, in alphabetical order, R. J. Baxter, E. Date, J. F. van Diejen, G. Felder, P. J. Forrester, I. B. Frenkel, M. Jimbo, Y. Kajihara, K. Kajiwara, H. T. Koelink, A. Kuniba, T. Masuda, T. Miwa, Y. van Norden, M. Noumi, Y. Ohta, M. Okado, E. Rains, H. Rosengren, S. N. M. Ruijsenaars, M. Schlosser, V. P. Spiridonov, L. Stevens, V. G. Turaev, A. Varchenko, S. O. Warnaar, Y. Yamada, and A. S. Zhedanov.

Even though we had not had any research experience in this exciting new field, it became quite clear to us that a new edition could be justified only if we included a chapter on elliptic hypergeometric series (and modular and theta hypergeometric series), written in an expository manner so that it would be more accessible to non-experts and be consistent with the rest of the book. Chapter 11 is entirely devoted to that topic. Regrettably, we had to be ruthlessly selective about choosing one particular approach from many possible approaches, all of which are interesting and illuminating on their own. We were guided by the need for brevity and clarity at the same time, as well as consistency of notations.

In addition to Chapter 11 we added Chapter 9 on generating and bilinear generating functions in view of their central importance in the study of orthogonal polynomials, as well as Chapter 10, covering briefly the huge topic of multivariable q-series, restricted mostly to F. H. Jackson's q-analogues of the four Appell functions F_1, F_2, F_3, F_4, and some of their more recent extensions. In these chapters we have attempted to describe the basic methods and results, but left many important formulas as exercises. Some parts of Chapters 1–8 were updated by the addition of textual material and a number of exercises, which were added at the ends of the sections and exercises in order to retain the same numbering of the equations and exercises as in the first edition.

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1

BASIC HYPERGEOMETRIC SERIES

1.1 Introduction

Our main objective in this chapter is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas, basic integrals, and applications to orthogonal polynomials and to other fields that follow in the subsequent chapters. We begin by defining Gauss' ${}_2F_1$ hypergeometric series, the ${}_rF_s$ (generalized) hypergeometric series, and pointing out some of their most important special cases. Next we define Heine's ${}_2\phi_1$ basic hypergeometric series which contains an additional parameter q, called the base, and then give the definition and notations for ${}_r\phi_s$ basic hypergeometric series. Basic hypergeometric series are called q-analogues (basic analogues or q-extensions) of hypergeometric series because an ${}_rF_s$ series can be obtained as the $q \to 1$ limit case of an ${}_r\phi_s$ series.

Since the binomial theorem is at the foundation of most of the summation formulas for hypergeometric series, we then derive a q-analogue of it, called the q-binomial theorem, and use it to derive Heine's q-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are q-analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Kummer, Pfaff and Saalschütz, and to Karlsson and Minton. We also introduce q-analogues of the exponential, gamma and beta functions, as well as the concept of a q-integral that allows us to give a q-analogue of Euler's integral representation of a hypergeometric function. Many additional formulas and q-analogues are given in the exercises at the end of the chapter.

1.2 Hypergeometric and basic hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss [1813]) in which he considered the infinite series

$$1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots \quad (1.2.1)$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $\mathrm{Re}\ (c-a-b) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula (see (1.2.11) below) for the sum of this series when $z = 1$ and $\mathrm{Re}\ (c-a-b) > 0$.

Although Gauss used the notation $F(a, b, c, z)$ for his series, it is now customary to use $F(a, b; c; z)$ or either of the notations

$$ {}_2F_1(a,\ b;\ c;\ z), \qquad {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] $$

for this series (and for its sum when it converges), because these notations separate the numerator parameters a, b from the denominator parameter c and the variable z. In view of Gauss' paper, his series is frequently called *Gauss' series.* However, since the special case $a = 1, b = c$ yields the geometric series

$$ 1 + z + z^2 + z^3 + \cdots, $$

Gauss' series is also called the *(ordinary) hypergeometric series* or the *Gauss hypergeometric series.*

Some important functions which can be expressed by means of Gauss' series are

$$ \begin{aligned} (1+z)^a &= F(-a, b; b; -z), \\ \log(1+z) &= zF(1, 1; 2; -z), \\ \sin^{-1} z &= zF(1/2, 1/2; 3/2; z^2), \\ \tan^{-1} z &= zF(1/2, 1; 3/2; -z^2), \\ e^z &= \lim_{a\to\infty} F(a, b; b; z/a), \end{aligned} \tag{1.2.2} $$

where $|z| < 1$ in the first four formulas. Also expressible by means of Gauss' series are the classical orthogonal polynomials, such as the *Tchebichef polynomials of the first and second kinds*

$$ T_n(x) = F(-n, n; 1/2; (1-x)/2), \tag{1.2.3} $$

$$ U_n(x) = (n+1)F(-n, n+2; 3/2; (1-x)/2), \tag{1.2.4} $$

the *Legendre polynomials*

$$ P_n(x) = F(-n, n+1; 1; (1-x)/2), \tag{1.2.5} $$

the *Gegenbauer (ultraspherical) polynomials*

$$ C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} F(-n, n+2\lambda; \lambda+1/2; (1-x)/2), \tag{1.2.6} $$

and the more general *Jacobi polynomials*

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1;\ \alpha+1; (1-x)/2), \tag{1.2.7} $$

where $n = 0, 1, \ldots,$ and $(a)_n$ denotes the *shifted factorial* defined by

$$ (a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \ldots . \tag{1.2.8} $$

Before Gauss, Chu [1303] (see Needham [1959, p. 138], Takács [1973] and Askey [1975, p. 59]) and Vandermonde [1772] had proved the summation formula

$$ F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, \ldots, \tag{1.2.9} $$

which is now called *Vandermonde's formula* or the *Chu–Vandermonde formula*, and Euler [1748] had derived several results for hypergeometric series, including his transformation formula

$$F(a,b;c;z) = (1-z)^{c-a-b}\, F(c-a,c-b;c;z), \quad |z|<1. \tag{1.2.10}$$

Formula (1.2.9) is the terminating case $a=-n$ of the summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0, \tag{1.2.11}$$

which Gauss proved in his paper.

Thirty-three years after Gauss' paper, Heine [1846, 1847, 1878] introduced the series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\cdots, \tag{1.2.12}$$

where it is assumed that $q\neq 1$, $c\neq 0,-1,-2,\ldots$ and the principal value of each power of q is taken. This series converges absolutely for $|z|<1$ when $|q|<1$ and it tends (at least termwise) to Gauss' series as $q\to 1$, because

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a. \tag{1.2.13}$$

The series in (1.2.12) is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series.*

Analogous to Gauss' notation, Heine used the notation $\phi(a,b,c,q,z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b, or c is replaced by zero, it is now customary to define the *basic hypergeometric series* by

$$\begin{aligned}\phi(a,b;c;q,z) \equiv {}_2\phi_1(a,b;c;q,z) \equiv {}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right]\\ = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n,\end{aligned} \tag{1.2.14}$$

where

$$(a;q)_n = \begin{cases}1, & n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots,\end{cases} \tag{1.2.15}$$

is the *q-shifted factorial* and it is assumed that $c\neq q^{-m}$ for $m=0,1,\ldots$. Some other notations that have been used in the literature for the product $(a;q)_n$ are $(a)_{q,n}$, $[a]_n$, and even $(a)_n$ when (1.2.8) is not used and the base is not displayed.

Another generalization of Gauss' series is the *(generalized) hypergeometric series* with r numerator parameters $a_1,\ldots,a_r$ and s denominator parameters $b_1,\ldots,b_s$ defined by

$$\begin{aligned}&{}_rF_s(a_1,a_2,\ldots,a_r;\ b_1,\ldots,b_s;z) \equiv {}_rF_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}z^n.\end{aligned} \tag{1.2.16}$$

Some well-known special cases are the *exponential function*

$$e^z = {}_0F_0(—;—;z), \tag{1.2.17}$$

the *trigonometric functions*

$$\begin{aligned} \sin z &= z\,{}_0F_1(—;3/2;-z^2/4), \\ \cos z &= {}_0F_1(—;1/2;-z^2/4), \end{aligned} \tag{1.2.18}$$

the *Bessel function*

$$J_\alpha(z) = (z/2)^\alpha\ {}_0F_1(—;\alpha+1;-z^2/4)/\Gamma(\alpha+1), \tag{1.2.19}$$

where a dash is used to indicate the absence of either numerator (when $r = 0$) or denominator (when $s = 0$) parameters. Some other well-known special cases are the *Hermite polynomials*

$$H_n(x) = (2x)^n\ {}_2F_0(-n/2,(1-n)/2;—;-x^{-2}), \tag{1.2.20}$$

and the *Laguerre polynomials*

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x). \tag{1.2.21}$$

Generalizing Heine's series, we shall define an ${}_r\phi_s$ *basic hypergeometric series* by

$$\begin{aligned} &{}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q,z) \equiv {}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{1.2.22}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s+1$.

In (1.2.16) and (1.2.22) it is assumed that the parameters $b_1,\ldots,b_s$ are such that the denominator factors in the terms of the series are never zero. Since

$$(-m)_n = \left(q^{-m};q\right)_n = 0, \quad n = m+1, m+2, \ldots, \tag{1.2.23}$$

an ${}_rF_s$ series terminates if one of its numerator parameters is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0,1,2,\ldots$, and $q \neq 0$. Basic analogues of the classical orthogonal polynomials will be considered in Chapter 7 as well as in the exercises at the ends of the chapters.

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $|q| > 1$, then we can perform an inversion with respect to the base by setting $p = q^{-1}$ and using the identity

$$(a;q)_n = (a^{-1};p)_n(-a)^n p^{-\binom{n}{2}} \tag{1.2.24}$$

to convert the series (1.2.22) to a similar series in base p with $|p| < 1$ (see Ex. 1.4(i)). The inverted series will have a finite radius of convergence if the original series does.

Observe that if we denote the terms of the series (1.2.16) and (1.2.22) which contain z^n by u_n and v_n, respectively, then

$$\frac{u_{n+1}}{u_n} = \frac{(a_1+n)(a_2+n)\cdots(a_r+n)}{(1+n)(b_1+n)\cdots(b_s+n)}\, z \tag{1.2.25}$$

is a rational function of n, and

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\left(-q^n\right)^{1+s-r} z \tag{1.2.26}$$

is a rational function of q^n. Conversely, if $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are power series with $u_0 = v_0 = 1$ such that u_{n+1}/u_n is a rational function of n and v_{n+1}/v_n is a rational function of q^n, then these series are of the forms (1.2.16) and (1.2.22), respectively.

By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \le s$, and for $|z| < 1$ if $r = s+1$. By an extension of the ratio test (Bromwich [1959, p. 241]), it converges absolutely for $|z| = 1$ if $r = s+1$ and Re $[b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0$. If $r > s+1$ and $z \neq 0$ or $r = s+1$ and $|z| > 1$, then this series diverges, unless it terminates.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$. It diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$, unless it terminates. As is customary, the ${}_rF_s$ and ${}_r\phi_s$ notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *hypergeometric functions* and *basic hypergeometric functions*, respectively) outside the circle of convergence.

Observe that the series (1.2.22) has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is again of the form (1.2.22) with r replaced by $r-1$. Because this is not the case for the ${}_r\phi_s$ series defined without the factors $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$ in the books of Bailey [1935] and Slater [1966] and we wish to be able to handle such limit cases, we have chosen to use the series defined in (1.2.22). There is no loss in generality since the Bailey and Slater series can be obtained from the $r = s+1$ case of (1.2.22) by choosing s sufficiently large and setting some of the parameters equal to zero.

An ${}_{r+1}F_r$ series is called *k-balanced* if $b_1 + b_2 + \cdots + b_r = k + a_1 + a_2 + \cdots + a_{r+1}$ and $z = 1$; a 1-balanced series is called *balanced* (or *Saalschützian*). Analogously, an ${}_{r+1}\phi_r$ series is called *k-balanced* if $b_1b_2\cdots b_r = q^k a_1a_2\cdots a_{r+1}$ and $z = q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). We will first encounter balanced series in §1.7, where we derive a summation formula for such a series.

For negative subscripts, the *shifted factorial* and the *q-shifted factorials* are defined by

$$(a)_{-n} = \frac{1}{(a-1)(a-2)\cdots(a-n)} = \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}, \tag{1.2.27}$$

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{1.2.28}$$

where $n = 0, 1, \ldots$. We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1.2.29}$$

for $|q| < 1$. Since the infinite product in (1.2.29) diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. The following easily verified identities will be frequently used in this book:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.2.30}$$

$$(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}, \tag{1.2.31}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk}, \tag{1.2.32}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \tag{1.2.33}$$

$$(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}, \tag{1.2.34}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}, \tag{1.2.35}$$

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}, \tag{1.2.36}$$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{1.2.37}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}, \tag{1.2.38}$$

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n, \tag{1.2.39}$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n, \tag{1.2.40}$$

where n and k are integers. A more complete list of useful identities is given in Appendix I at the end of the book.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1, a_2, \ldots, a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.2.41}$$

$$(a_1, a_2, \ldots, a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{1.2.42}$$

The ratio $(1-q^a)/(1-q)$ considered in (1.2.13) is called a *q-number* (or *basic number*) and it is denoted by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1. \tag{1.2.43}$$

It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the *q-number factorial* $[n]_q!$ is defined for a nonnegative integer n by

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \tag{1.2.44}$$

and the corresponding *q-number shifted factorial* is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1} [a+k]_q. \tag{1.2.45}$$

Clearly,

$$\lim_{q\to 1} [n]_q! = n!, \quad \lim_{q\to 1} [a]_q = a, \tag{1.2.46}$$

and

$$[a]_{q;n} = (1-q)^{-n}(q^a;q)_n, \quad \lim_{q\to 1} [a]_{q;n} = (a)_n. \tag{1.2.47}$$

Corresponding to (1.2.41) we can use the compact notation

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n}[a_2]_{q;n} \cdots [a_m]_{q;n}. \tag{1.2.48}$$

Since

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q! [b_1, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \\ &= {}_r\phi_s\left(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, z(1-q)^{1+s-r}\right), \end{aligned} \tag{1.2.49}$$

anyone working with q-numbers and the *q-number hypergeometric series* on the left-hand side of (1.2.49) can use the formulas for ${}_r\phi_s$ series in this book that have no zero parameters by replacing the parameters by q^{th} powers and applying (1.2.49).

As in Frenkel and Turaev [1995] one can define a *trigonometric number* $[a;\sigma]$ by

$$[a;\sigma] = \frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)} \tag{1.2.50}$$

for noninteger values of σ and view $[a;\sigma]$ as a *trigonometric deformation* of a since $\lim_{\sigma\to 0}[a;\sigma] = a$. The corresponding ${}_rt_s$ *trigonometric hypergeometric series* can be defined by

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; \sigma, z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r; \sigma]_n}{[n;\sigma]! [b_1, \ldots, b_s; \sigma]_n} \left[(-1)^n e^{\pi i \sigma \binom{n}{2}}\right]^{1+s-r} z^n, \end{aligned} \tag{1.2.51}$$

where

$$[n;\sigma]! = \prod_{k=1}^{n}[k;\sigma], \quad [a;\sigma]_n = \prod_{k=0}^{n-1}[a+k;\sigma], \tag{1.2.52}$$

and

$$[a_1, a_2, \ldots, a_m;\sigma]_n = [a_1;\sigma]_n[a_2;\sigma]_n\cdots[a_m;\sigma]_n. \tag{1.2.53}$$

From

$$[a;\sigma] = \frac{e^{\pi i\sigma a} - e^{-\pi i\sigma a}}{e^{\pi i\sigma} - e^{-\pi i\sigma}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = \frac{1-q^a}{1-q}q^{(1-a)/2}, \tag{1.2.54}$$

where $q = e^{2\pi i\sigma}$, it follows that

$$[a;\sigma]_n = \frac{(q^a;q)_n}{(1-q)^n}q^{n(1-a)/2-n(n-1)/4}, \tag{1.2.55}$$

and hence

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s;\sigma, z) \\ &= {}_r\phi_s(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, cz) \end{aligned} \tag{1.2.56}$$

with

$$c = (1-q)^{1+s-r}q^{r/2-s/2+(b_1+\cdots+b_s)/2-(a_1+\cdots+a_r)/2}, \tag{1.2.57}$$

which shows that the ${}_rt_s$ series is equivalent to the ${}_r\phi_s$ series in (1.2.49).

Elliptic numbers $[a;\sigma,\tau]$, which are a one-parameter generalization (deformation) of trigonometric numbers, are considered in §1.6, and the corresponding elliptic (and theta) hypergeometric series and their summation and transformation formulas are considered in Chapter 11.

We close this section with two identities involving ordinary binomial coefficients, which are particularly useful in handling some powers of q that arise in the derivations of many formulas containing q-series:

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn\,, \tag{1.2.58}$$

$$\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k - kn\,. \tag{1.2.59}$$

1.3 The q-binomial theorem

One of the most important summation formulas for hypergeometric series is given by the *binomial theorem*:

$${}_2F_1(a, c; c; z) = {}_1F_0(a; —; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n = (1-z)^{-a}, \tag{1.3.1}$$

where $|z| < 1$. We shall show that this formula has the following q-analogue

$${}_1\phi_0(a; —; q, z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1,\ |q| < 1, \tag{1.3.2}$$

which was derived by Cauchy [1843], Heine [1847] and by other mathematicians. See Askey [1980a], which also cites the books by Rothe [1811] and Schweins [1820], and the remark on p. 491 of Andrews, Askey, and Roy [1999] concerning the terminating form of the q-binomial theorem in Rothe [1811].

Heine's proof of (1.3.2), which can also be found in the books Heine [1878], Bailey [1935, p. 66] and Slater [1966, p. 92], is better understood if one first follows Askey's [1980a] approach of evaluating the sum of the binomial series in (1.3.1), and then carries out the analogous steps for the series in (1.3.2).

Let us set

$$f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{1.3.3}$$

Since this series is uniformly convergent in $|z| \le \epsilon$ when $0 < \epsilon < 1$, we may differentiate it termwise to get

$$\begin{aligned} f_a'(z) &= \sum_{n=1}^{\infty} \frac{n(a)_n}{n!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{n!} z^n = a f_{a+1}(z). \end{aligned} \tag{1.3.4}$$

Also

$$\begin{aligned} f_a(z) - f_{a+1}(z) &= \sum_{n=1}^{\infty} \frac{(a)_n - (a+1)_n}{n!} z^n \\ = \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{n!} &[a - (a+n)]\, z^n = -\sum_{n=1}^{\infty} \frac{n(a+1)_{n-1}}{n!} z^n \\ = -\sum_{n=0}^{\infty} \frac{(a+1)_n}{n!} z^{n+1} &= -z f_{a+1}(z). \end{aligned} \tag{1.3.5}$$

Eliminating $f_{a+1}(z)$ from (1.3.4) and (1.3.5), we obtain the first order differential equation

$$f_a'(z) = \frac{a}{1-z} f_a(z), \tag{1.3.6}$$

subject to the initial condition $f_a(0) = 1$, which follows from the definition (1.3.3) of $f_a(z)$. Solving (1.3.6) under this condition immediately gives that $f_a(z) = (1-z)^{-a}$ for $|z| < 1$.

Analogously, let us now set

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1,\ |q| < 1. \tag{1.3.7}$$

Clearly, $h_{q^a}(z) \to f_a(z)$ as $q \to 1$. Since $h_{aq}(z)$ is a q-analogue of $f_{a+1}(z)$, we first compute the difference

$$h_a(z) - h_{aq}(z) = \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} [1 - a - (1 - aq^n)] z^n$$

$$= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n$$

$$= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n = -azh_{aq}(z), \tag{1.3.8}$$

giving an analogue of (1.3.5). Observing that

$$f'(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} \tag{1.3.9}$$

for a differentiable function f, we next compute the difference

$$h_a(z) - h_a(qz) = \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n)$$

$$= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1}$$

$$= (1-a)zh_{aq}(z). \tag{1.3.10}$$

Eliminating $h_{aq}(z)$ from (1.3.8) and (1.3.10) gives

$$h_a(z) = \frac{1-az}{1-z} h_a(qz). \tag{1.3.11}$$

Iterating this relation $n-1$ times and then letting $n \to \infty$ we obtain

$$h_a(z) = \frac{(az;q)_n}{(z;q)_n} h_a(q^n z)$$

$$= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.3.12}$$

since $q^n \to 0$ as $n \to \infty$ and $h_a(0) = 1$ by (1.3.7), which completes the proof of (1.3.2).

One consequence of (1.3.2) is the product formula

$${}_1\phi_0(a;—;q,z)\ {}_1\phi_0(b;—;q,az) = {}_1\phi_0(ab;—;q,z), \tag{1.3.13}$$

which is a q-analogue of $(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$.

In the special case $a = q^{-n}, n = 0, 1, 2, \ldots$, (1.3.2) gives

$${}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n = (-z)^n q^{-n(n+1)/2}(q/z;q)_n, \tag{1.3.14}$$

where, by analytic continuation, z can be any complex number. From now on, unless stated othewise, whenever $q^{-j}, q^{-k}, q^{-m}, q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers.

If we set $a = 0$ in (1.3.2), we get

$${}_1\phi_0(0;—;q,z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \ |z| < 1, \tag{1.3.15}$$

which is a q-analogue of the exponential function e^z. Another q-analogue of e^z can be obtained from (1.3.2) by replacing z by $-z/a$ and then letting $a \to \infty$ to get

$$ {}_0\phi_0(—;—;q,-z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty. \tag{1.3.16} $$

Observe that if we denote the *q-exponential* functions in (1.3.15) and (1.3.16) by $e_q(z)$ and $E_q(z)$, respectively, then $e_q(z)E_q(-z) = 1$, $e_{q^{-1}}(z) = E_q(-qz)$ by (1.2.24), and

$$ \lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z. \tag{1.3.17} $$

In deriving q-analogues of various formulas we shall sometimes use the observation that

$$ \frac{(q^a z;q)_\infty}{(z;q)_\infty} = {}_1\phi_0(q^a;—;q,z) \to {}_1F_0(a;—;z) = (1-z)^{-a} \text{ as } q \to 1^-. \tag{1.3.18} $$

Thus

$$ \lim_{q\to 1^-} \frac{(q^a z;q)_\infty}{(z;q)_\infty} = (1-z)^{-a}, \quad |z| < 1, \quad a \text{ real}. \tag{1.3.19} $$

By analytic continuation this holds for z in the complex plane cut along the positive real axis from 1 to ∞, with $(1-z)^{-a}$ positive when z is real and less than 1.

Let Δ and ∇ be the *forward* and *backward q-difference operators*, respectively, defined by

$$ \Delta f(z) = f(qz) - f(z), \quad \nabla f(z) = f(q^{-1}z) - f(z), \tag{1.3.20} $$

where we take $0 < q < 1$, without any loss of generality. Then the unique analytic solutions of

$$ \frac{\Delta f(z)}{\Delta z} = f(z), \quad f(0) = 1 \quad \text{and} \quad \frac{\nabla g(z)}{\nabla z} = g(z), \quad g(0) = 1, \tag{1.3.21} $$

are

$$ f(z) = e_q(z(1-q)) \quad \text{and} \quad g(z) = E_q(z(1-q)). \tag{1.3.22} $$

The *symmetric q-difference operator* δ_q is defined by

$$ \delta_q f(z) = f(zq^{1/2}) - f(zq^{-1/2}). \tag{1.3.23} $$

If we seek an analytic solution of the initial-value problem

$$ \frac{\delta_q f(z)}{\delta_q z} = f(z), \quad f(0) = 1, \tag{1.3.24} $$

in the form $\sum_{n=0}^{\infty} a_n z^n$, then we find that

$$ a_{n+1} = \frac{1-q}{1-q^{n+1}} q^{n/2} a_n, \quad a_0 = 1, \tag{1.3.25} $$

$n = 0, 1, 2, \ldots$. Hence, $a_n = (1-q)^n q^{(n^2-n)/4}/(q;q)_n$, and we have a third q-exponential function

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n q^{(n^2-n)/4}}{(q;q)_n} z^n = \sum_{n=0}^{\infty} \frac{1}{[n;\sigma]!} z^n \tag{1.3.26}$$

with $q = e^{2\pi i\sigma}$. This q-exponential function has the properties

$$\exp_{q^{-1}}(z) = \exp_q(z), \qquad \lim_{q\to 1} \exp_q(z) = e^z, \tag{1.3.27}$$

and it is an entire function of z of order zero with an infinite product representation in terms of its zeros. See Nelson and Gartley [1994], and Atakishiyev and Suslov [1992a]. The multi-sheet Riemann surface associated with the q-*logarithm* inverse function $z = \ln_q(w)$ of $w = \exp_q(z)$ is considered in Nelson and Gartley [1996].

Ismail and Zhang [1994] found an extension of $\exp_q(z)$ in the form

$$f(z) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(aq^{\frac{1-m}{2}+z}, aq^{\frac{1-m}{2}-z}; q\right)_m b^m, \tag{1.3.28}$$

which has the property

$$\frac{\delta f(z)}{\delta x(z)} = f(z), \quad \delta f(z) = f(z+1/2) - f(z-1/2), \tag{1.3.29}$$

where

$$x(z) = C(q^z + q^{-z}) \tag{1.3.30}$$

with $C = -abq^{1/4}/(1-q)$ is the so-called q-*quadratic lattice*, and a and b are arbitrary complex parameters such that $|ab| < 1$. In the particular case $q^z = e^{-i\theta}$, $0 \le \theta \le \pi$, $x = \cos\theta$, the q-exponential function in (1.3.28) becomes the function

$$\mathcal{E}_q(x;a,b) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(q^{\frac{1-m}{2}} ae^{i\theta}, q^{\frac{1-m}{2}} ae^{-i\theta}; q\right)_m b^m. \tag{1.3.31}$$

Ismail and Zhang showed that

$$\lim_{q\to 1} \mathcal{E}_q(x;a,b(1-q)) = \exp[(1+a^2-2ax)b], \tag{1.3.32}$$

and that $\mathcal{E}_q(x;a,b)$ is an entire function of x when $|ab| < 1$. From (1.3.32) they observed that $\mathcal{E}_q(x;-i,-it/2)$ is a q-analogue of e^{xt}. It is now standard to use the notation in Suslov [2003] for the slightly modified q-exponential function

$$\mathcal{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} (-i\alpha)^m \left(-iq^{\frac{1-m}{2}} e^{i\theta}, -iq^{\frac{1-m}{2}} e^{-i\theta}; q\right)_m, \tag{1.3.33}$$

which, because of the normalizing factor that he introduced, has the nice property that $\mathcal{E}_q(0;\alpha) = 1$ (see Suslov [2003, p. 17]).

1.4 Heine's transformation formulas for ${}_2\phi_1$ series

Heine [1847, 1878] showed that

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\ {}_2\phi_1(c/b,z;az;q,b), \tag{1.4.1}$$

where $|z|<1$ and $|b|<1$. To prove this transformation formula, first observe from the q-binomial theorem (1.3.2) that

$$\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m.$$

Hence, for $|z|<1$ and $|b|<1$,

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty}z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(zq^m)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \\
&= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\ {}_2\phi_1(c/b,z;az;q,b)
\end{aligned}$$

by (1.3.2), which gives (1.4.1).

Heine also showed that Euler's transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z) \tag{1.4.2}$$

has a q-analogue of the form

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\ {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.3}$$

A short way to prove this formula is just to iterate (1.4.1) as follows

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\ {}_2\phi_1(c/b,z;az;q,b) \tag{1.4.4}$$

$$ = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\ {}_2\phi_1(abz/c,b;bz;q,c/b) \tag{1.4.5}$$

$$ = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\ {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.6}$$

1.5 Heine's q-analogue of Gauss' summation formula

In order to derive Heine's [1847] q-analogue of Gauss' summation formula (1.2.11) it suffices to set $z = c/ab$ in (1.4.1), assume that $|b| < 1$, $|c/ab| < 1$, and observe that the series on the right side of

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \, {}_1\phi_0(c/ab; —; q, b) $$

can be summed by (1.3.2) to give

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{1.5.1} $$

By analytic continuation, we may drop the assumption that $|b| < 1$ and require only that $|c/ab| < 1$ for (1.5.1) to be valid.

For the terminating case when $a = q^{-n}$, (1.5.1) reduces to

$$ {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \frac{(c/b; q)_n}{(c; q)_n}. \tag{1.5.2} $$

By inversion or by changing the order of summation it follows from (1.5.2) that

$$ {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{1.5.3} $$

Both (1.5.2) and (1.5.3) are q-analogues of Vandermonde's formula (1.2.9). These formulas can be used to derive other important formulas such as, for example, Jackson's [1910a] transformation formula

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(az; q)_\infty}{(z; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, c/b; q)_k}{(q, c, az; q)_k} (-bz)^k q^{\binom{k}{2}} \\ &= \frac{(az; q)_\infty}{(z; q)_\infty} \, {}_2\phi_2(a, c/b; c, az; q, bz). \end{aligned} \tag{1.5.4} $$

This formula is a q-analogue of the Pfaff–Kummer transformation formula

$$ {}_2F_1(a, b; c; z) = (1 - z)^{-a} \, {}_2F_1(a, c - b; c; z/(z - 1)). \tag{1.5.5} $$

To prove (1.5.4), we use (1.5.2) to write

$$ \frac{(b; q)_k}{(c; q)_k} = \sum_{n=0}^{k} \frac{(q^{-k}, c/b; q)_n}{(q, c; q)_n} \left(bq^k\right)^n $$

and hence

$$\begin{aligned}
&{}_2\phi_1(a,b;c;q,z)\\
&=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k\sum_{n=0}^{k}\frac{(q^{-k},c/b;q)_n}{(q,c;q)_n}\left(bq^k\right)^n\\
&=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(a;q)_k(c/b;q)_n}{(q;q)_{k-n}(q,c;q)_n}z^k(-b)^n q^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{k+n}(c/b;q)_n}{(q;q)_k(q,c;q)_n}(-bz)^n z^k q^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c;q)_n}(-bz)^n q^{\binom{n}{2}}\sum_{k=0}^{\infty}\frac{(aq^n;q)_k}{(q;q)_k}z^k\\
&=\frac{(az;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c,az;q)_n}(-bz)^n q^{\binom{n}{2}},
\end{aligned}$$

by (1.3.2). Also see Andrews [1973]. If $a=q^{-n}$, then the series on the right side of (1.5.4) can be reversed (by replacing k by $n-k$) to yield Sears' [1951c] transformation formula

$$\begin{aligned}
&{}_2\phi_1(q^{-n},b;c;q,z)\\
&=\frac{(c/b;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n {}_3\phi_2(q^{-n},q/z,c^{-1}q^{1-n};bc^{-1}q^{1-n},0;q,q). \quad (1.5.6)
\end{aligned}$$

1.6 Jacobi's triple product identity, theta functions, and elliptic numbers

Jacobi's [1829] well-known *triple product identity* (see Andrews [1971])

$$(zq^{\frac12},q^{\frac12}/z,q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}z^n,\quad z\neq 0, \tag{1.6.1}$$

can be easily derived by using Heine's summation formula (1.5.1).

First, set $c=bzq^{\frac12}$ in (1.5.1) and then let $b\to 0$ and $a\to\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2/2}}{(q;q)_n}z^n=(zq^{\frac12};q)_\infty. \tag{1.6.2}$$

Similarly, setting $c=zq$ in (1.5.1) and letting $a\to\infty$ and $b\to\infty$ we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q,zq;q)_n}=\frac{1}{(zq;q)_\infty}. \tag{1.6.3}$$

Now use (1.6.2) to find that

$$\begin{aligned}
&(zq^{\frac12},q^{\frac12}/z;q)_\infty\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}q^{(m^2+n^2)/2}}{(q;q)_m(q;q)_n}z^{m-n}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^{-n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}. \tag{1.6.4}$$

Formula (1.6.1) then follows from (1.6.3) by observing that

$$\frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk} = \frac{1}{(q;q)_n (q^{n+1};q)_\infty} = \frac{1}{(q;q)_\infty}.$$

An important application of (1.6.1) is that it can be used to express the *theta functions* (Whittaker and Watson [1965, Chapter 21])

$$\vartheta_1(x,q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x, \tag{1.6.5}$$

$$\vartheta_2(x,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x, \tag{1.6.6}$$

$$\vartheta_3(x,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nx, \tag{1.6.7}$$

$$\vartheta_4(x,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx \tag{1.6.8}$$

in terms of infinite products. Just replace q by q^2 in (1.6.1) and then set z equal to $qe^{2ix}, -qe^{2ix}, -e^{2ix}, e^{2ix}$, respectively, to obtain

$$\vartheta_1(x,q) = 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n}\cos 2x + q^{4n}), \tag{1.6.9}$$

$$\vartheta_2(x,q) = 2q^{1/4} \cos x \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n}\cos 2x + q^{4n}), \tag{1.6.10}$$

$$\vartheta_3(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n-1}\cos 2x + q^{4n-2}), \tag{1.6.11}$$

and

$$\vartheta_4(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n-1}\cos 2x + q^{4n-2}). \tag{1.6.12}$$

It is common to write $\vartheta_k(x)$ for $\vartheta_k(x,q)$, $k = 1, \ldots, 4$.

Since, from (1.6.9) and (1.6.10),

$$\lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_1(x,q) = \sin x, \quad \lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_2(x,q) = \cos x, \tag{1.6.13}$$

one can think of the theta functions $\vartheta_1(x,q)$ and $\vartheta_2(x,q)$ as one-parameter deformations (generalizations) of the trigonometric functions $\sin x$ and $\cos x$, respectively. This led Frenkel and Turaev [1995] to define an *elliptic number* $[a;\sigma,\tau]$ by

$$[a;\sigma,\tau] = \frac{\vartheta_1(\pi\sigma a, e^{\pi i\tau})}{\vartheta_1(\pi\sigma, e^{\pi i\tau})}, \tag{1.6.14}$$

where a is a complex number and the *modular parameters* σ and τ are fixed complex numbers such that Im $(\tau) > 0$ and $\sigma \neq m + n\tau$ for integer values of m and n, so that the denominator $\vartheta_1(\pi\sigma, e^{\pi i\tau})$ in (1.6.14) is never zero. Then, from (1.6.9) it is clear that $[a;\sigma,\tau]$ is well-defined, $[-a;\sigma,\tau] = -[a;\sigma,\tau]$, $[1;\sigma,\tau] = 1$, and

$$\lim_{\operatorname{Im}\tau\to\infty}[a;\sigma,\tau] = \frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)} = [a;\sigma]. \tag{1.6.15}$$

Hence, the elliptic number $[a;\sigma,\tau]$ is a one-parameter deformation of the trigonometric number $[a;\sigma]$ and a two-parameter deformation of the number a. Notice that $[a;\sigma,\tau]$ is called an "elliptic number" even though it is not an elliptic (doubly periodic and meromorphic) function of a. However, $[a;\sigma,\tau]$ is a quotient of ϑ_1 functions and, as is well-known (see Whittaker and Watson [1965, §21.5]), any (doubly periodic meromorphic) elliptic function can be written as a constant multiple of a quotient of products of ϑ_1 functions. The corresponding elliptic hypergeometric series are considered in Chapter 11.

1.7 A q-analogue of Saalschütz's summation formula

Pfaff [1797] discovered the summation formula

$${}_3F_2(a,b,-n;c,1+a+b-c-n;1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n = 0,1,\ldots, \tag{1.7.1}$$

which sums a terminating balanced ${}_3F_2(1)$ series with argument 1. It was rediscovered by Saalschütz [1890] and is usually called *Saalschütz formula* or the *Pfaff–Saalschütz formula*; see Askey [1975]. To derive a q-analogue of (1.7.1), observe that since, by (1.3.2),

$$\frac{(abz/c;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty}\frac{(ab/c;q)_k}{(q;q)_k}z^k$$

the right side of (1.4.3) equals

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ab/c;q)_k(c/a,c/b;q)_m}{(q;q)_k(q,c;q)_m}\left(\frac{ab}{c}\right)^m z^{k+m},$$

and hence, equating the coefficients of z^n on both sides of (1.4.3) we get

$$\sum_{j=0}^{n}\frac{(q^{-n},c/a,c/b;q)_j}{(q,c,cq^{1-n}/ab;q)_j}q^j = \frac{(a,b;q)_n}{(c,ab/c;q)_n}.$$

Replacing a,b by $c/a, c/b$, respectively, this gives the following sum of a terminating balanced ${}_3\phi_2$ series

$${}_3\phi_2(a,b,q^{-n};c,abc^{-1}q^{1-n};q,q) = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}, \quad n = 0,1,\ldots, \tag{1.7.2}$$

which was first derived by Jackson [1910a]. It is easy to see that (1.7.1) follows from (1.7.2) by replacing a, b, c in (1.7.2) by q^a, q^b, q^c, respectively, and letting $q \to 1$. Note that letting $a \to \infty$ in (1.7.2) gives (1.5.2), while letting $a \to 0$ gives (1.5.3).

1.8 The Bailey–Daum summation formula

Bailey [1941] and Daum [1942] independently discovered the summation formula

$$ {}_2\phi_1(a, b; aq/b; q, -q/b) = \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}, \tag{1.8.1}$$

which is a q-analogue of Kummer's formula

$$ {}_2F_1(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.8.2}$$

Formula (1.8.1) can be easily obtained from (1.4.1) by using the identity (1.2.40) and a limiting form of (1.2.39), namely, $(a;q)_\infty = (a, aq; q^2)_\infty$, to see that

$$\begin{aligned} &{}_2\phi_1(a, b; aq/b; q, -q/b) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty}\, {}_2\phi_1(q/b, -q/b; -q; q, a) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \sum_{n=0}^\infty \frac{(q^2/b^2; q^2)_n}{(q^2; q^2)_n} a^n \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \frac{(aq^2/b^2; q^2)_\infty}{(a; q^2)_\infty} \qquad \text{by (1.3.2)} \\ &= \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}. \end{aligned}$$

1.9 q-analogues of the Karlsson–Minton summation formulas

Minton [1970] showed that if a is a negative integer and $m_1, m_2, \ldots, m_r$ are nonnegative integers such that $-a \ge m_1 + \cdots + m_r$, then

$$\begin{aligned} &{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, b_1+m_1, \ldots, b_r+m_r \\ b+1, b_1, \ldots, b_r \end{matrix}; 1\right] \\ &= \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(b_1-b)_{m_1} \cdots (b_r-b)_{m_r}}{(b_1)_{m_1} \cdots (b_r)_{m_r}} \end{aligned} \tag{1.9.1}$$

where, as usual, it is assumed that none of the factors in the denominators of the terms of the series is zero. Karlsson [1971] showed that (1.9.1) also holds when a is not a negative integer provided that the series converges, i.e., if $\mathrm{Re}(-a) > m_1 + \cdots + m_r - 1$, and he deduced from (1.9.1) that

$$ {}_{r+1}F_r\left[\begin{matrix} a, b_1+m_1, \ldots, b_r+m_r \\ b_1, \ldots, b_r \end{matrix}; 1\right] = 0, \quad \mathrm{Re}\,(-a) > m_1 + \cdots + m_r, \tag{1.9.2}$$

$$
{}_{r+1}F_r\left[\begin{matrix}-(m_1+\cdots+m_r), b_1+m_1,\ldots,b_r+m_r\\ b_1,\ldots,b_r\end{matrix};1\right]
= (-1)^{m_1+\cdots+m_r}\frac{(m_1+\cdots+m_r)!}{(b_1)_{m_1}\cdots(b_r)_{m_r}}. \tag{1.9.3}
$$

These formulas are particularly useful for evaluating sums that appear as solutions to some problems in theoretical physics such as the Racah coefficients. They were also used by Gasper [1981b] to prove the orthogonality on $(0, 2\pi)$ of certain functions that arose in Greiner's [1980] work on spherical harmonics on the Heisenberg group. Here we shall present Gasper's [1981a] derivation of q-analogues of the above formulas. Some of the formulas derived below will be used in Chapter 7 to prove the orthogonality relation for the continuous q-ultraspherical polynomials.

Observe that if m and n are nonnegative integers with $m \geq n$, then

$$
{}_2\phi_1(q^{-n}, q^{-m}; b_r; q, q) = \frac{(b_r q^m; q)_n}{(b_r; q)_n} q^{-mn}
$$

by (1.5.3), and hence

$$
\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix}a_1,\ldots,a_r,b_rq^m\\ b_1,\ldots,b_{r-1},b_r\end{matrix};q,z\right]\\
&= \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_{r-1};q)_n}z^n\sum_{k=0}^{n}\frac{(q^{-n},q^{-m};q)_k}{(q,b_r;q)_k}q^{mn+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{m}\frac{(a_1,\ldots,a_r;q)_n(q^{-m};q)_k}{(b_1,\ldots,b_{r-1};q)_n(q;q)_{n-k}(q,b_r;q)_k}z^n(-1)^k q^{mn+k-nk+\binom{k}{2}}\\
&= \sum_{k=0}^{m}\frac{(q^{-m},a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_r;q)_k}(-zq^m)^k q^{-\binom{k}{2}}\\
&\quad\times {}_r\phi_{r-1}\left[\begin{matrix}a_1q^k,\ldots,a_rq^k\\ b_1q^k,\ldots,b_{r-1}q^k\end{matrix};q,zq^{m-k}\right],\qquad |z|<1.
\end{aligned} \tag{1.9.4}
$$

This expansion formula is a q-analogue of a formula in Fox [1927, (1.11)] and independently derived by Minton [1970, (4)].

When $r = 2$, formulas (1.9.4), (1.5.1) and (1.5.3) yield

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix}a,b,b_1q^m\\ bq,b_1\end{matrix};q,a^{-1}q^{1-m}\right] &= \frac{(q,bq/a;q)_\infty}{(bq,q/a;q)_\infty}\,{}_2\phi_1(q^{-m},b;b_1;q,q)\\
&= \frac{(q,bq/a;q)_\infty(b_1/b;q)_m}{(bq,q/a;q)_\infty(b_1;q)_m}b^m,
\end{aligned} \tag{1.9.5}
$$

provided that $|a^{-1}q^{1-m}| < 1$. By induction it follows from (1.9.4) and (1.9.5) that if $m_1, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$, then

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1}\left[\begin{matrix}a,b,b_1q^{m_1},\ldots,b_rq^{m_r}\\ bq,b_1,\ldots,b_r\end{matrix};q,a^{-1}q^{1-(m_1+\cdots+m_r)}\right]\\
&= \frac{(q,bq/a;q)_\infty}{(bq,q/a;q)_\infty}\frac{(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}}b^{m_1+\cdots+m_r}
\end{aligned} \tag{1.9.6}
$$

which is a q-analogue of (1.9.1). Formula (1.9.1) can be derived from (1.9.6) by replacing $a, b, b_1, \ldots, b_r$ by $q^a, q^b, q^{b_1}, \ldots, q^{b_r}$, respectively, and letting $q \to 1$.

Setting $b_r = b, m_r = 1$ and then replacing r by $r+1$ in (1.9.6) gives

$$ {}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{-(m_1+\cdots+m_r)} \right] = 0, \; |a^{-1} q^{-(m_1+\cdots+m_r)}| < 1, \tag{1.9.7} $$

while letting $b \to \infty$ in the case $a = q^{-(m_1+\cdots+m_r)}$ of (1.9.6) gives

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-(m_1+\cdots+m_r)}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] = \frac{(-1)^{m_1+\cdots+m_r} (q;q)_{m_1+\cdots+m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{-(m_1+\cdots+m_r)(m_1+\cdots+m_r+1)/2}, \tag{1.9.8} $$

which are q-analogues of (1.9.2) and (1.9.3). Another q-analogue of (1.9.3) can be derived by letting $b \to 0$ in (1.9.6) to obtain

$$ {}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{1-(m_1+\cdots+m_r)} \right] = \frac{(-1)^{m_1+\cdots+m_r} (q;q)_\infty b_1^{m_1} \cdots b_r^{m_r}}{(q/a;q)_\infty (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{\binom{m_1}{2}+\cdots+\binom{m_r}{2}}, \tag{1.9.9} $$

when $|a^{-1} q^{1-(m_1+\cdots+m_r)}| < 1$.

In addition, if $a = q^{-n}$ and n is a nonnegative integer then we can reverse the order of summation of the series in (1.9.6), (1.9.7) and (1.9.9) to obtain

$$ {}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-n}, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix} ; q, q \right] = \frac{b^n (q;q)_n (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq;q)_n (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \quad n \geq m_1 + \cdots + m_r, \tag{1.9.10} $$

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q \right] = 0, \quad n > m_1 + \cdots + m_r, \tag{1.9.11} $$

and the following generalization of (1.9.8)

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] = \frac{(-1)^n (q;q)_n q^{-n(n+1)/2}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \tag{1.9.12} $$

where $n \geq m_1 + \cdots + m_r$, which also follows by letting $b \to \infty$ in (1.9.10). Note that the $b \to 0$ limit case of (1.9.10) is (1.9.11) when $n > m_1 + \cdots + m_r$, and it is the $a = q^{-(m_1+\cdots+m_r)}$ special case of (1.9.9) when $n = m_1 + \cdots + m_r$.

1.10 The q-gamma and q-beta functions

The q-*gamma* function

$$ \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \; 0 < q < 1, \tag{1.10.1} $$

was introduced by Thomae [1869] and later by Jackson [1904e]. Heine [1847] gave an equivalent definition, but without the factor $(1-q)^{1-x}$. When $x = n+1$ with n a nonnegative integer, this definition reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{1.10.2}$$

which clearly approaches $n!$ as $q \to 1^-$. Hence $\Gamma_q(n+1)$ tends to $\Gamma(n+1) = n!$ as $q \to 1^-$. The definition of $\Gamma_q(x)$ can be extended to $|q| < 1$ by using the principal values of q^x and $(1-q)^{1-x}$ in (1.10.1).

To show that

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x) \tag{1.10.3}$$

we shall give a simple, formal proof due to Gosper; see Andrews [1986]. From (1.10.1),

$$\begin{aligned}\Gamma_q(x+1) &= \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{n+1})^x}{(1-q^{n+x})(1-q^n)^x}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{q\to 1^-} \Gamma_q(x+1) &= \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{n+1}{n}\right)^x \\ &= x\left[x^{-1}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^x\right] \\ &= x\Gamma(x) = \Gamma(x+1)\end{aligned}$$

by Euler's product formula (see Whittaker and Watson [1965, §12.11]) and the well-known functional equation for the gamma function

$$\Gamma(x+1) = x\,\Gamma(x), \quad \Gamma(1) = 1. \tag{1.10.4}$$

For a rigorous justification of the above steps see Koornwinder [1990]. From (1.10.1) it is easily seen that, analogous to (1.10.4), $\Gamma_q(x)$ satisfies the functional equation

$$f(x+1) = \frac{1-q^x}{1-q} f(x), \quad f(1) = 1. \tag{1.10.5}$$

Askey [1978] derived analogues of many of the well-known properties of the gamma function, including its log-convexity (see the exercises at the end of this chapter), which show that (1.10.1) is a natural q-analogue of $\Gamma(x)$.

It is obvious from (1.10.1) that $\Gamma_q(x)$ has poles at $x = 0, -1, -2, \ldots$. The residue at $x = -n$ is

$$\begin{aligned}\lim_{x\to -n}(x+n)\Gamma_q(x) &= \frac{(1-q)^{n+1}}{(1-q^{-n})(1-q^{1-n})\cdots(1-q^{-1})} \lim_{x\to -n}\frac{x+n}{1-q^{x+n}} \\ &= \frac{(1-q)^{n+1}}{(q^{-n};q)_n \log q^{-1}}. \end{aligned} \tag{1.10.6}$$

The q-gamma function has no zeros, so its reciprocal is an entire function with zeros at $x = 0, -1, -2, \ldots$. Since

$$\frac{1}{\Gamma_q(x)} = (1-q)^{x-1} \prod_{n=0}^{\infty} \frac{1-q^{n+x}}{1-q^{n+1}}, \tag{1.10.7}$$

the function $1/\Gamma_q(x)$ has zeros at $x = -n \pm 2\pi i k/\log q$, where k and n are nonnegative integers.

A q-analogue of Legendre's duplication formula

$$\Gamma(2x)\Gamma\left(\frac{1}{2}\right) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \tag{1.10.8}$$

can be easily derived by observing that

$$\begin{aligned}\frac{\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)} &= \frac{(q, q^2; q^2)_\infty}{(q^{2x}, q^{2x+1}; q^2)_\infty}\,(1-q^2)^{1-2x} \\ &= \frac{(q;q)_\infty}{(q^{2x};q)_\infty}(1-q^2)^{1-2x} = (1+q)^{1-2x}\Gamma_q(2x)\end{aligned}$$

and hence

$$\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right). \tag{1.10.9}$$

Similarly, it can be shown that the Gauss multiplication formula

$$\Gamma(nx)(2\pi)^{(n-1)/2} = n^{nx-\frac{1}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right) \tag{1.10.10}$$

has a q-analogue of the form

$$\begin{aligned}&\Gamma_q(nx)\Gamma_r\left(\frac{1}{n}\right)\Gamma_r\left(\frac{2}{n}\right)\cdots\Gamma_r\left(\frac{n-1}{n}\right) \\ &\quad = (1+q+\cdots+q^{n-1})^{nx-1}\Gamma_r(x)\Gamma_r\left(x+\frac{1}{n}\right)\cdots\Gamma_r\left(x+\frac{n-1}{n}\right)\end{aligned} \tag{1.10.11}$$

with $r = q^n$; see Jackson [1904e, 1905d]. The q-gamma function for $q > 1$ is considered in Exercise 1.23. For other interesting properties of the q-gamma function see Askey [1978] and Moak [1980a,b] and Ismail, Lorch and Muldoon [1986].

Since the *beta function* is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.10.12}$$

it is natural to define the *q-beta function* by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{1.10.13}$$

which tends to $B(x, y)$ as $q \to 1^-$. By (1.10.1) and (1.3.2),

$$
\begin{aligned}
B_q(x, y) &= (1-q)\frac{(q, q^{x+y}; q)_\infty}{(q^x, q^y; q)_\infty} \\
&= (1-q)\frac{(q;q)_\infty}{(q^y;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^y;q)_n}{(q;q)_n}\,q^{nx} \\
&= (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{n+y};q)_\infty}q^{nx}, \quad \text{Re}\, x,\ \text{Re}\, y > 0. \quad (1.10.14)
\end{aligned}
$$

This series expansion will be used in the next section to derive a q-integral representation for $B_q(x, y)$.

1.11 The q-integral

Thomae [1869, 1870] and Jackson [1910c, 1951] introduced the *q-integral*

$$\int_0^1 f(t)\ d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n \quad (1.11.1)$$

and Jackson gave the more general definition

$$\int_a^b f(t)\ d_qt = \int_0^b f(t)\ d_qt - \int_0^a f(t)\ d_qt, \quad (1.11.2)$$

where

$$\int_0^a f(t)\ d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n. \quad (1.11.3)$$

Jackson also defined an integral on $(0, \infty)$ by

$$\int_0^\infty f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \quad (1.11.4)$$

The *bilateral q-integral* is defined by

$$\int_{-\infty}^\infty f(t)\ d_qt = (1-q)\sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]\,q^n. \quad (1.11.5)$$

If f is continuous on $[0, a]$, then it is easily seen that

$$\lim_{q\to 1}\int_0^a f(t)\ d_qt = \int_0^a f(t)\ dt \quad (1.11.6)$$

and that a similar limit holds for (1.11.4) and (1.11.5) when f is suitably restricted. By (1.11.1), it follows from (1.10.14) that

$$B_q(x, y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\ d_qt,\ \text{Re}\, x > 0, \quad y \neq 0, -1, -2, \ldots, \quad (1.11.7)$$

which clearly approaches the beta function integral

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ dt, \quad \text{Re}\, x,\ \text{Re}\, y > 0, \quad (1.11.8)$$

as $q \to 1^-$. Thomae [1869] rewrote Heine's formula (1.4.1) in the q-integral form

$$ {}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tzq^a, tq; q)_\infty}{(tz, tq^{c-b}; q)_\infty} \, d_q t, \qquad (1.11.9) $$

which is a q-analogue of Euler's integral representation

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \, dt, \qquad (1.11.10) $$

where $|\arg(1-z)| < \pi$ and $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$.

The q-integral notation is, as we shall see later, quite useful in simplifying and manipulating various formulas involving sums of series.

Exercises

1.1 Verify the identities (1.2.30)–(1.2.40), and show that

(i) $$ (aq^{-n}; q)_n = (q/a; q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}, $$

(ii) $$ (aq^{-k-n}; q)_n = \frac{(q/a; q)_{n+k}}{(q/a; q)_k} \left(-\frac{a}{q}\right)^n q^{-nk-\binom{n}{2}}, $$

(iii) $$ \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_n} = \frac{1-aq^{2n}}{1-a}, $$

(iv) $$ (a; q)_{2n} = (a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}; q)_n, $$

(v) $$ (a; q)_n (q/a; q)_{-n} = (-a)^n q^{\binom{n}{2}}, $$

(vi) $$ (q, -q, -q^2; q^2)_\infty = 1. $$

1.2 The *q-binomial coefficient* is defined by

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} $$

for $k = 0, 1, \ldots, n$, and by

$$ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1}, q^{\alpha-\beta+1}; q)_\infty}{(q, q^{\alpha+1}; q)_\infty} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)} $$

for complex α and β when $|q| < 1$. Verify that

(i) $$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q, $$

(ii) $$ \begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-q^\alpha)^k q^{-\binom{k}{2}}, $$

(iii) $$ \begin{bmatrix} k+\alpha \\ k \end{bmatrix}_q = \frac{(q^{\alpha+1}; q)_k}{(q; q)_k}, $$

(iv) $$\begin{bmatrix} -\alpha \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q (-q^{-\alpha})^k q^{-\binom{k}{2}},$$

(v) $$\begin{bmatrix} \alpha+1 \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^k + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q q^{\alpha+1-k},$$

(vi) $$(z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k q^{\binom{k}{2}},$$

when k and n are nonnegative integers.

1.3 (i) Show that the *binomial theorem*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $n = 0, 1, \ldots,$ has a q-analogue of the form

$$\begin{aligned}(ab;q)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k (b;q)_{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (a;q)_k (b;q)_{n-k}.\end{aligned}$$

(ii) Extend the above formula to the *q-multinomial theorem*

$$\begin{aligned}&(a_1 a_2 \cdots a_{m+1};q)_n \\ &= \sum_{\substack{0\le k_1,\ldots,0\le k_m \\ k_1+\cdots+k_m\le n}} \begin{bmatrix} n \\ k_1,\ldots,k_m \end{bmatrix}_q a_2^{k_1} a_3^{k_1+k_2} \cdots a_{m+1}^{k_1+k_2+\cdots+k_m} \\ &\quad \times (a_1;q)_{k_1} (a_2;q)_{k_2} \cdots (a_m;q)_{k_m} (a_{m+1};q)_{n-(k_1+\cdots+k_m)},\end{aligned}$$

where $m = 1, 2, \ldots,\ n = 0, 1, \ldots,$ and

$$\begin{bmatrix} n \\ k_1,\ldots,k_m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{k_1} \cdots (q;q)_{k_m} (q;q)_{n-(k_1+\cdots+k_m)}}$$

is the *q-multinomial coefficient*.

1.4 (i) Prove the inversion formula

$$\begin{aligned}&{}_r\phi_s \left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1^{-1},\ldots,a_r^{-1};q^{-1})_n}{(q^{-1},b_1^{-1},\ldots,b_s^{-1};q^{-1})_n} \left(\frac{a_1\cdots a_r z}{b_1\cdots b_s q} \right)^n .\end{aligned}$$

(ii) By reversing the order of summation, show that

$$\begin{aligned}&{}_{r+1}\phi_s \left[\begin{matrix} a_1,\ldots,a_r,q^{-n} \\ b_1,\ldots,b_s \end{matrix} ; q, z \right] \\ &= \frac{(a_1,\ldots,a_r;q)_n}{(b_1\ldots,b_s;q)_n} \left(\frac{z}{q} \right)^n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r-1} \\ &\quad \times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1,\ldots,q^{1-n}/b_s,q^{-n};q)_k}{(q,q^{1-n}/a_1,\ldots,q^{1-n}/a_r;q)_k} \left(\frac{b_1\cdots b_s}{a_1\cdots a_r} \frac{q^{n+1}}{z} \right)^k\end{aligned}$$

when $n = 0, 1, \ldots$.

(iii) Show that

$$
{}_{r+1}\phi_r \left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, q^z \right]
= \frac{(a_1, \ldots, a_{r+1}; q)_\infty}{(1-q)(q, b_1, \ldots, b_r; q)_\infty} \int_0^1 t^{z-1} \frac{(qt, b_1 t, \ldots, b_r t; q)_\infty}{(a_1 t, \ldots, a_{r+1} t; q)_\infty} d_q t,
$$

when $0 < q < 1$, Re $z > 0$, and the series on the left side does not terminate.

1.5 Show that

$$
\frac{(c, bq^n; q)_m}{(b; q)_m} = \frac{(b/c; q)_n}{(b; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, c; q)_k q^k}{(q, cq^{1-n}/b; q)_k} (cq^k; q)_m.
$$

1.6 Prove the summation formulas

(i) ${}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2, q^2) = \dfrac{(b^2; q^2)_n}{(b^2; q)_n} q^{-\binom{n}{2}}$,

(ii) ${}_1\phi_1(a; c; q, c/a) = \dfrac{(c/a; q)_\infty}{(c; q)_\infty}$,

(iii) ${}_2\phi_0(a, q^{-n}; -; q, q^n/a) = a^{-n}$,

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q; q)_n^2} = \frac{2}{(q; q)_\infty}$,

(v) ${}_1\phi_0(a; -; p, z) = \dfrac{(zp^{-1}; p^{-1})_\infty}{(azp^{-1}; p^{-1})_\infty}$, $\quad |p| > 1,\ |azp^{-1}| < 1$,

(vi) ${}_2\phi_1(a, b; c; p, p) = \dfrac{(a/c, b/c; p^{-1})_\infty}{(1/c, ab/c; p^{-1})_\infty}$, $\quad |p| > 1$.

1.7 Show that, for $|z| < 1$,

$$
{}_2\phi_1(a^2, aq; a; q, z) = (1 + az) \frac{(a^2 qz; q)_\infty}{(z; q)_\infty}.
$$

1.8 Show that, when $|a| < 1$ and $|bq/a^2| < 1$,

$$
{}_2\phi_1(a^2, a^2/b; b; q^2, bq/a^2)
= \frac{(a^2, q; q^2)_\infty}{2(b, bq/a^2; q^2)_\infty} \left[\frac{(b/a; q)_\infty}{(a; q)_\infty} + \frac{(-b/a; q)_\infty}{(-a; q)_\infty} \right].
$$

(Andrews and Askey [1977])

1.9 Let $\phi(a, b, c)$ denote the series ${}_2\phi_1(a, b; c; q, z)$. Verify Heine's [1847] *q-contiguous relations*:

(i) $\phi(a, b, cq^{-1}) - \phi(a, b, c) = cz \dfrac{(1-a)(1-b)}{(q-c)(1-c)} \phi(aq, bq, cq)$,

(ii) $\phi(aq, b, c) - \phi(a, b, c) = az \dfrac{1-b}{1-c} \phi(aq, bq, cq)$,

(iii) $\phi(aq, b, cq) - \phi(a, b, c) = az\dfrac{(1-b)(1-c/a)}{(1-c)(1-cq)}\phi(aq, bq, cq^2)$,

(iv) $\phi(aq, bq^{-1}, c) - \phi(a, b, c) = az\dfrac{(1-b/aq)}{1-c}\phi(aq, b, cq)$.

1.10 Denoting ${}_2\phi_1(a, b; c; q, z)$, ${}_2\phi_1(aq^{\pm 1}, b; c, q, z)$, ${}_2\phi_1(a, bq^{\pm 1}; c; q, z)$ and ${}_2\phi_1(a, b; cq^{\pm 1}; q, z)$ by $\phi, \phi(aq^{\pm 1}), \phi(bq^{\pm 1})$ and $\phi(cq^{\pm 1})$, respectively, show that

(i) $b(1-a)\phi(aq) - a(1-b)\phi(bq) = (b-a)\phi$,

(ii) $a(1-b/c)\phi(bq^{-1}) - b(1-a/c)\phi(aq^{-1}) = (a-b)(1-abz/cq)\phi$,

(iii) $q(1-a/c)\phi(aq^{-1}) + (1-a)(1-abz/c)\phi(aq)$
$= [1+q-a-aq/c+a^2z(1-b/a)/c]\phi$,

(iv) $(1-c)(q-c)(abz-c)\phi(cq^{-1}) + (c-a)(c-b)z\phi(cq)$
$= (c-1)[c(q-c)+(ca+cb-ab-abq)z]\phi$.

(Heine [1847])

1.11 Let $g(\theta; \lambda, \mu, \nu) = (\lambda e^{i\theta}, \mu\nu; q)_\infty \; {}_2\phi_1(\mu e^{-i\theta}, \nu e^{-i\theta}; \mu\nu; q, \lambda e^{i\theta})$. Prove that $g(\theta; \lambda, \mu, \nu)$ is symmetric in λ, μ, ν and is even in θ.

1.12 Let $\mathcal{D}_q$ be the q-*derivative operator* defined for fixed q by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z},$$

and let $\mathcal{D}_q^n u = \mathcal{D}_q(D_q^{n-1}u)$ for $n = 1, 2, \ldots$. Show that

(i) $\lim_{q\to 1} \mathcal{D}_q f(z) = \dfrac{d}{dz} f(z)$ if f is differentiable at z,

(ii) $\mathcal{D}_q^n \; {}_2\phi_1(a, b; c; q, z) = \dfrac{(a, b; q)_n}{(c; q)_n (1-q)^n} \; {}_2\phi_1(aq^n, bq^n; cq^n; q, z)$,

(iii) $\mathcal{D}_q^n \left\{ \dfrac{(z; q)_\infty}{(abz/c; q)_\infty} \; {}_2\phi_1(a, b; c; q, z) \right\}$
$= \dfrac{(c/a, c/b; q)_n}{(c; q)_n (1-q)^n} \left(\dfrac{ab}{c}\right)^n \dfrac{(zq^n; q)_\infty}{(abz/c; q)_\infty} \; {}_2\phi_1(a, b; cq^n; q, zq^n)$.

(iv) Prove the q-Leibniz formula

$$\mathcal{D}_q^n [f(z)g(z)] = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{D}_q^{n-k} f(zq^k) \mathcal{D}_q^k g(z).$$

1.13 Show that $u(z) = {}_2\phi_1(a, b; c; q, z)$ satisfies (for $|z| < 1$ and in the formal power series sense) the second order q-differential equation

$$z(c - abqz)\mathcal{D}_q^2 u + \left[\frac{1-c}{1-q} + \frac{(1-a)(1-b) - (1-abq)}{1-q} z\right] \mathcal{D}_q u - \frac{(1-a)(1-b)}{(1-q)^2} u = 0,$$

where $\mathcal{D}_q$ is defined as in Ex. 1.12. By replacing a, b, c, respectively, by q^a, q^b, q^c and then letting $q \to 1^-$ show that the above equation tends to the second order differential equation

$$z(1-z)v'' + [c - (a+b+1)z]v' - abv = 0$$

for the hypergeometric function $v(z) = {}_2F_1(a, b; c; z)$, where $|z| < 1$. (Heine [1847])

1.14 Let $|x| < 1$ and let $e_q(x)$ and $E_q(x)$ be as defined in §1.3. Define

$$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}.$$

Also define

$$\mathrm{Sin}_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}, \quad \mathrm{Cos}_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}.$$

Show that

(i) $$e_q(ix) = \cos_q(x) + i \sin_q(x),$$

(ii) $$E_q(ix) = \mathrm{Cos}_q(x) + i\mathrm{Sin}_q(x),$$

(iii) $$\sin_q(x)\mathrm{Sin}_q(x) + \cos_q(x)\mathrm{Cos}_q(x) = 1,$$

(iv) $$\sin_q(x)\mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\cos_q(x) = 0.$$

For these identities and other identities involving q-analogues of $\sin x$ and $\cos x$, see Jackson [1904a] and Hahn [1949c].

1.15 Prove the transformation formulas

(i) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(bzq^{-n}/c;q)_\infty}{(bz/c;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, c/b, 0 \\ c, cq/bz \end{matrix}; q, q\right],$$

(ii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n\, {}_3\phi_1\left[\begin{matrix} q^{-n}, b, q/z \\ bq^{1-n}/c \end{matrix}; q, z/c\right],$$

(iii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c,\ 0 \end{matrix}; q, q\right].$$

(See Jackson [1905a, 1927])

1.16 Show that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{n(n+1)/2} = (-q;q)_\infty (aq;q^2)_\infty.$$

1.17 Show that

$$\sum_{k=0}^{n} \frac{(a,b;q)_k}{(q;q)_k} (-ab)^{n-k} q^{(n-k)(n+k-1)/2}$$

$$= (a;q)_{n+1} \sum_{k=0}^{n} \frac{(-b)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k} (1 - aq^{n-k})}. \qquad \text{(Carlitz [1974])}$$

1.18 Show that

(i) $(c;q)_\infty\ {}_1\phi_1(a;c;q,z) = (z;q)_\infty\ {}_1\phi_1(az/c;z;q,c)$,
and deduce that ${}_1\phi_1(-bq;0;q,-q) = (-bq^2;q^2)_\infty/(q;q^2)_\infty$,

(ii) $(z;q)_\infty\ {}_2\phi_1(a,0;c;q,z) = (az;q)_\infty\ {}_1\phi_2(a;c,az;q,cz)$,

(iii) $$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,a^2;q)_n} q^{\binom{n}{2}} (at/z)^n\ {}_2\phi_1(q^{-n},a;q^{1-n}/a;q,qz^2/a)$$
$$= (-zt;q)_\infty\ {}_2\phi_1(a,a/z^2;a^2;q,-zt), \quad |zt|<1.$$

1.19 Using (1.5.4) show that

(i) $${}_2\phi_2\left[\begin{matrix} a,q/a \\ -q,b \end{matrix};q,-b\right] = \frac{(ab,bq/a;q^2)_\infty}{(b;q)_\infty},$$

(ii) $${}_2\phi_2\left[\begin{matrix} a^2,b^2 \\ abq^{\frac{1}{2}},-abq^{\frac{1}{2}} \end{matrix};q,-q\right] = \frac{(a^2q,b^2q;q^2)_\infty}{(q,a^2b^2q;q^2)_\infty}.$$

(Andrews [1973])

1.20 Prove that if Re $x>0$ and $0<q<1$, then

(i) $$\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{nx}}{(q;q)_n},$$

(ii) $$\frac{1}{\Gamma_q(x)} = \frac{(1-q)^{x-1}}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{nx}}{(q;q)_n} q^{\binom{n}{2}}.$$

1.21 For $0<q<1$ and $x>0$, show that
$$\frac{d^2}{dx^2}\log\Gamma_q(x) = (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{n+x}}{(1-q^{n+x})^2},$$
which proves that $\log\Gamma_q(x)$ is convex for $x>0$ when $0<q<1$.

1.22 Conversely, prove that if $f(x)$ is a positive function defined on $(0,\infty)$which satisfies
$$f(x+1) = \frac{1-q^x}{1-q} f(x) \text{ for some } q, 0<q<1,$$
$$f(1) = 1,$$
and $\log f(x)$ is convex for $x>0$, then $f(x) = \Gamma_q(x)$. This is Askey's [1978] q-analogue of the Bohr-Mollerup [1922] theorem for $\Gamma(x)$. For two extensions to the $q>1$ case (with $\Gamma_q(x)$ defined as in the next exercise), see Moak [1980b].

1.23 For $q>1$ the *q-gamma function* is defined by
$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty} (q-1)^{1-x} q^{x(x-1)/2}.$$

Show that this function also satisfies the functional equation (1.10.5) and that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^+$. Show that for $q > 1$ the residue of $\Gamma_q(x)$ at $x = -n$ is

$$\frac{(q-1)^{n+1} q^{\binom{n+1}{2}}}{(q;q)_n \log q}.$$

1.24 Jackson [1905a,b,e] gave the following q-analogues of Bessel functions:

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \ {}_2\phi_1(0,0;q^{\nu+1};q,-x^2/4),$$

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \ {}_0\phi_1\left(—;q^{\nu+1};q,-\frac{x^2q^{\nu+1}}{4}\right),$$

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \ {}_1\phi_1(0;q^{\nu+1};q,qx^2/4),$$

where $0 < q < 1$. The above notations for the q-*Bessel functions* are due to Ismail [1981, 1982, 2003c].
Show that

$$J_\nu^{(2)}(x;q) = \left(-x^2/4;q\right)_\infty J_\nu^{(1)}(x;q), \quad |x| < 2, \quad \text{(Hahn [1949c])}$$

and

$$\lim_{q\to 1} J_\nu^{(k)}(x(1-q);q) = J_\nu(x), \qquad k = 1,2,3.$$

1.25 For the q-Bessel functions defined as in Exercise 1.24 prove that

(i) $$q^\nu J_{\nu+1}^{(k)}(x;q) = \frac{2(1-q^\nu)}{x} J_\nu^{(k)}(x;q) - J_{\nu-1}^{(k)}(x;q),\ k = 1,2;$$

(ii) $$J_\nu^{(1)}(xq^{\frac{1}{2}};q) = q^{\nu/2}\left(J_\nu^{(1)}(x;q) + \frac{x}{2}J_{\nu+1}^{(1)}(x;q)\right);$$

(iii) $$J_\nu^{(1)}(xq^{\frac{1}{2}};q) = q^{-\nu/2}\left(J_\nu^{(1)}(x;q) - \frac{x}{2}J_{\nu-1}^{(1)}(x;q)\right).$$

(iv) $$q^{\nu+1}J_{\nu+1}^{(3)}(xq^{1/2};q) = \frac{2(1-q^\nu)}{x}J_\nu^{(3)}(x;q) - J_{\nu-1}^{(3)}(x;q).$$

1.26 (i) Following Ismail [1982], let

$$f_\nu(x) = J_\nu^{(1)}(x;q)J_{-\nu}^{(1)}(xq^{\frac{1}{2}};q) - J_{-\nu}^{(1)}(x;q)J_\nu^{(1)}(xq^{\frac{1}{2}};q).$$

Show that

$$f_\nu(xq^{\frac{1}{2}}) = \left(1 + \frac{x^2}{4}\right) f_\nu(x)$$

and deduce that, for non-integral ν,

$$f_\nu(x) = q^{-\nu/2}(q^\nu, q^{1-\nu};q)_\infty/(q,q,-x^2/4;q)_\infty.$$

(ii) Show that

$$g_\nu(qx) + \left(x^2/4 - q^\nu - q^{-\nu}\right)g_\nu(x) + g_\nu(xq^{-1}) = 0$$

with $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2}; q^2)$ and deduce that

$$g_\nu(x)g_{-\nu}(xq^{-1}) - g_{-\nu}(x)g_\nu(xq^{-1}) = \frac{(q^{2\nu}, q^{1-2\nu}; q^2)_\infty}{(q^2, q^2; q^2)_\infty} q^{\nu(\nu-1)}.$$

(Ismail [2003c])

1.27 Show that

$$\text{(i)} \qquad \sum_{n=-\infty}^{\infty} t^n J_n^{(2)}(x; q) = \left(-x^2/4; q\right)_\infty e_q(xt/2) e_q(-x/2t),$$

$$\text{(ii)} \qquad \sum_{n=-\infty}^{\infty} t^n J_n^{(3)}(x; q) = e_q(xt/2) E_q(-qx/2t).$$

Both of these are q-analogues of the generating function

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{x(t-t^{-1})/2}.$$

1.28 The *continuous q-Hermite polynomials* are defined in Askey and Ismail [1983] by

$$H_n(x|q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$; see Szegő [1926], Carlitz [1955, 1957a, 1958, 1960] and Rogers [1894, 1917]. Derive the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q; q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \quad |t| < 1. \quad \text{(Rogers [1894])}$$

1.29 The *continuous q-ultraspherical polynomials* are defined in Askey and Ismail [1983] by

$$C_n(x; \beta|q) = \sum_{k=0}^{n} \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$. Show that

$$\begin{aligned} C_n(x; \beta|q) &= \frac{(\beta; q)_n}{(q; q)_n} e^{in\theta} \, {}_2\phi_1 \left[\begin{matrix} q^{-n}, \beta \\ \beta^{-1} q^{1-n} \end{matrix} ; q, q\beta^{-1} e^{-2i\theta} \right] \\ &= \frac{(\beta^2; q)_n}{(q; q)_n} e^{-in\theta} \beta^{-n} \, {}_3\phi_2 \left[\begin{matrix} q^{-n}, \beta, \beta e^{2i\theta} \\ \beta^2, 0 \end{matrix} ; q, q \right], \\ & \lim_{q \to 1} C_n(x; q^\lambda|q) = C_n^\lambda(x), \end{aligned}$$

and

$$\sum_{n=0}^{\infty} C_n(x; \beta|q) t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \ |t| < 1. \quad \text{(Rogers [1895])}$$

1.30 Show that if $m_1, \ldots, m_r$ are nonnegative integers, then

(i)
$$ {}_{r+1}\phi_{r+1}\left[\begin{matrix} b, & b_1q^{m_1}, \ldots, b_rq^{m_r} \\ & bq, b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right] $$
$$ = \frac{(q;q)_\infty (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq;q)_\infty (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r}, $$

(ii)
$$ {}_r\phi_r\left[\begin{matrix} b_1q^{m_1}, \ldots, b_rq^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{-(m_1+\cdots+m_r)}\right] = 0, $$

(iii)
$$ {}_r\phi_r\left[\begin{matrix} b_1q^{m_1}, \ldots, b_rq^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right] $$
$$ = \frac{(-1)^{m_1+\cdots+m_r} (q;q)_\infty b_1^{m_1} \cdots b_r^{m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{\binom{m_1}{2}+\cdots+\binom{m_r}{2}}. $$

(Gasper [1981a])

1.31 Let Δ_b denote the *q-difference operator* defined for a fixed q by

$$ \Delta_b f(z) = bf(qz) - f(z). $$

Then Δ_1 is the Δ operator defined in (1.3.20). Show that

$$ \Delta_b x^n = (bq^n - 1)x^n $$

and, if

$$ v_n(z) = \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} z^n, $$

then

$$ (\Delta\Delta_{b_1/q}\Delta_{b_2/q} \cdots \Delta_{b_s/q}) v_n(z) $$
$$ = z(\Delta_{a_1}\Delta_{a_2} \cdots \Delta_{a_r}) v_{n-1}(zq^{1+s-r}), \quad n = 1, 2, \ldots . $$

Use this to show that the basic hypergeometric series

$$ v(z) = {}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) $$

satisfies (in the sense of formal power series) the q-difference equation

$$ (\Delta\Delta_{b_1/q}\Delta_{b_2/q} \cdots \Delta_{b_s/q}) v(z) = z(\Delta_{a_1} \cdots \Delta_{a_r}) v(zq^{1+s-r}). $$

This is a q-analogue of the formal differential equation for generalized hypergeometric series given, e.g. in Henrici [1974, Theorem (1.5)] and Slater [1966, (2.1.2.1)]. Also see Jackson [1910d, (15)].

1.32 The *little q-Jacobi polynomials* are defined by

$$ p_n(x; a, b; q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx). $$

Show that these polynomials satisfy the orthogonality relation

$$ \sum_{j=0}^{\infty} \frac{(bq;q)_j}{(q;q)_j} (aq)^j p_n(q^j; a, b; q) p_m(q^j; a, b; q) $$
$$ = \begin{cases} 0, & \text{if } m \neq n, \\ \dfrac{(q, bq; q)_n (1-abq)(aq)^n}{(aq, abq; q)_n (1-abq^{2n+1})} \dfrac{(abq^2;q)_\infty}{(aq;q)_\infty}, & \text{if } m = n. \end{cases} $$

(Andrews and Askey [1977])

1.33 Show for the above little q-Jacobi polynomials that the formula

$$p_n(x;c,d;q) = \sum_{k=0}^{n} a_{k,n} p_k(x;a,b;q)$$

holds with

$$a_{k,n} = (-1)^k q^{\binom{k+1}{2}} \frac{(q^{-n}, aq, cdq^{n+1};q)_k}{(q, cq, abq^{k+1};q)_k} \, {}_3\phi_2 \left[\begin{matrix} q^{k-n}, cdq^{n+k+1}, aq^{k+1} \\ cq^{k+1}, abq^{2k+2} \end{matrix} ; q, q \right].$$

(Andrews and Askey [1977])

1.34 (i) If $m, m_1, m_2, \ldots, m_r$ are arbitrary nonnegative integers and $|a^{-1}q^{m+1-(m_1+\cdots+m_r)}| < 1$, show that

$$\begin{aligned} &{}_{r+2}\phi_{r+1} \left[\begin{matrix} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq^{1+m}, b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{m+1-(m_1+\cdots+m_r)} \right] \\ &= \frac{(q, bq/a;q)_\infty (bq;q)_m (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq, q/a;q)_\infty (q;q)_m (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r-m} \\ &\quad \times {}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-m}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix} ; q, q \right]; \end{aligned}$$

(ii) if $m_1, m_2, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$, $|cq| < 1$, show that

$$\begin{aligned} &{}_{r+2}\phi_{r+1} \left[\begin{matrix} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bcq, b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{1-(m_1+\cdots+m_r)} \right] \\ &= \frac{(bq/a, cq;q)_\infty}{(bcq, q/a;q)_\infty} \frac{(b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r} \\ &\quad \times {}_{r+2}\phi_{r+1} \left[\begin{matrix} c^{-1}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix} ; q, cq \right]. \end{aligned}$$

(Gasper [1981a])

1.35 Use Ex. 1.2(v) to prove that if x and y are indeterminates such that $xy = qyx$, q commutes with x and y, and the associative law holds, then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k}.$$

(See Cigler [1979], Feinsilver [1982], Koornwinder [1989], Potter[1950], Schützenberger [1953], and Yang [1991]).

1.36 Verify that if x and y are indeterminates satisfying the conditions in Ex. 1.35, then

(i) $\quad e_q(y)e_q(x) = e_q(x+y), \quad e_q(x)e_q(y) = e_q(x+y-yx);$

(ii) $\quad E_q(x)E_q(y) = E_q(x+y), \quad E_q(y)E_q(x) = E_q(x+y+yx).$

(Fairlie and Wu [1997]; Koornwinder [1997], where q-exponentials with q-Heisenberg relations and other relations are also considered.)

1.37 Show that

$$\mathcal{E}_q(z;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\left\{ {}_2\phi_1\left[\begin{matrix}-qe^{2i\theta}, -qe^{-2i\theta} \\ q\end{matrix}; q^2, \alpha^2\right] + \frac{2q^{1/4}}{1-q}\alpha\cos\theta\, {}_2\phi_1\left[\begin{matrix}-q^2e^{2i\theta}, -q^2e^{-2i\theta} \\ q^3\end{matrix}; q^2, \alpha^2\right]\right\}$$

with $z = \cos\theta$.

1.38 Extend Jacobi's triple product identity to the transformation formula

$$1 + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}(a^n + b^n) = (q, a, b; q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n} q^n}{(q, a, b, ab; q)_n}.$$

Deduce that

$$1 + 2\sum_{n=1}^{\infty} a^n q^{2n^2} = (q;q)_\infty (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a;q)_{2n} q^n}{(q, -aq; q)_n (aq;q^2)_n}.$$

(Warnaar [2003a])

Notes

§§1.1 and 1.2 For additional material on hypergeometric series and orthogonal polynomials see, e.g., the books by Erdélyi [1953], Rainville [1960], Szegő [1975], Whittaker and Watson [1965], Agarwal [1963], Carlson [1977], T.S. Chihara [1978], Henrici [1974], Luke [1969], Miller [1968], Nikiforov and Uvarov [1988], Vilenkin [1968], and Watson [1952]. Some techniques for using symbolic computer algebraic systems such as Mathematica, Maple, and Macsyma to derive formulas containing hypergeometric and basic hypergeometric series are discussed in Gasper [1990]. Also see Andrews [1984d, 1986, 1987b], Andrews, Crippa and Simon [1997], Andrews and Knopfmacher [2001], Andrews, Knopfmacher, Paule and Zimmermann [2001], Andrews, Paule and Riese [2001a,b], Askey [1989f, 1990], Askey, Koepf and Koornwinder [1999], Böing and Koepf [1999], Garoufalidis [2003], Garoufalidis, Le and Zeilberger [2003], Garvan [1999], Garvan and Gonnet [1992], Gosper [2001], Gosper and Suslov [2000], Koepf [1998], Koornwinder [1991b, 1993a, 1998], Krattenthaler [1995b], Paule and Riese [1997], Petkovsek, Wilf and Zeilberger [1996], Riese [2003], Sills [2003c], Wilf and Zeilberger [1990], and Zeilberger [1990b].

§§1.3–1.5 The q-binomial theorem was also derived in Jacobi [1846], along with the q-Vandermonde formula. Bijective proofs of the q-binomial theorem, Heine's ${}_2\phi_1$ transformation and q-analogue of Gauss' summation formula, the q-Saalschütz formula, and of other formulas are presented in Joichi and Stanton [1987]. Rahman and Suslov [1996a] used the method of first order linear difference equations to prove the q-binomial and q-Gauss formulas. Bender [1971] used partitions to derive an extension of the q-Vandermonde sum in the form of a generalized q-binomial Vandermonde convolution. The even and odd parts of the infinite series on the right side of (1.3.33) appeared in Atakishiyev and Suslov [1992a], but without any explicit reference to the q-exponential function. Also see Suslov [1998–2003] and the q-convolutions in Carnovale [2002], Carnovale and Koornwinder [2000], and Rogov [2000].

§1.6 Other proofs of Jacobi's triple product identity and/or applications of it are presented in Adiga *et al.* [1985], Alladi and Berkovich [2003], Andrews [1965], Cheema [1964], Ewell [1981], Gustafson [1989], Joichi and Stanton [1989], Kac [1978, 1985], Lepowsky and Milne [1978], Lewis [1984], Macdonald [1972], Menon [1965], Milne [1985a], Sudler [1966], Sylvester [1882], and Wright [1965]. Concerning theta functions, see Adiga *et al.* [1985], Askey [1989c], Bellman [1961], and Jensen's use of theta functions in Pólya [1927] to derive necessary and sufficient conditions for the Riemann hypothesis to hold.

§1.7 Some applications of the q-Saalschütz formula are contained in Carlitz [1969b] and Wright [1968].

§1.9 Formulas (1.9.3) and (1.9.8) were rediscovered by Gustafson [1987a, Theorems 3.15 and 3.18] while working on multivariable orthogonal polynomials.

§1.11 Also see Jackson [1917, 1951] and, for fractional q-integrals and q-derivatives, Al-Salam [1966] and Agarwal [1969b]. Toeplitz [1963, pp. 53–55] pointed out that around 1650 Fermat used a q-integral type Riemann sum to evaluate the integral of x^k on the interval $[0, b]$. Al-Salam and Ismail [1994] evaluated a q-beta integral on the unit circle and found corresponding systems of biorthogonal rational functions.

Ex. 1.2 The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which is also called the Gaussian binomial coefficient, counts the number of k dimensional subspaces of an n dimensional vector space over a field $GF(q)$, q a prime power (Goldman and Rota [1970]), and it is the generating function, in powers of q, for partitions into at most k parts not exceeding $n-k$ (Sylvester [1882]). It arises in such diverse fields as analysis, computer programming, geometry, number theory, physics, and statistics. See, e.g., Aigner [1979], Andrews [1971a, 1976], M. Baker and Coon [1970], Baxter and Pearce [1983], Berman and Fryer [1972], Dowling [1973], Dunkl [1981], Garvan and Stanton [1990], Handa and Mohanty [1980], Ihrig and Ismail [1981], Jimbo [1985, 1986], van Kampen [1961], Kendall and Stuart [1979, §31.25], Knuth [1971, 1973], Pólya [1970], Pólya and Alexanderson [1970], Szegő [1975, §2.7], and Zaslavsky [1987]. Sylvester [1878] used the invariant theory that he and Cayley developed to prove that the coefficients of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q = \Sigma a_j q^j$ are unimodal. A constructive proof was recently given by O'Hara [1990]. Also see Bressoud [1992] and Zeilberger [1989a,b, 1990b]. The unimodality of the sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}_q : k = 0, 1, \ldots, n\right)$ is explicitly displayed in Aigner [1979, Proposition 3.13], and Macdonald [1995, Example 4 on p. 137].

Ex. 1.3 Cigler [1979] derived an operator form of the q-binomial theorem. MacMahon [1916, Arts. 105–107] showed that if a multiset is permuted, then the generating function for inversions is the q-multinomial coefficient. Also see Carlitz [1963a], Kadell [1985a], and Knuth [1973, p. 33, Ex. 16]. Gasper derived the q-multinomial theorem in part (ii) several years ago by using the q-binomial theorem and mathematical induction. Andrews observed in a 1988 letter that it can also be derived by using the expansion formula for the q-Lauricella function Φ_D stated in Andrews [1972, (4.1)] and the q-Vandermonde sum. Some sums of q-multinomial coefficients are considered in Bressoud [1978, 1981c]. See also Agarwal [1953a].

Ex. 1.8 Jain [1980c] showed that the sum in this exercise is equivalent to the sum of a certain ${}_2\psi_2$ series, and summed some other ${}_2\psi_2$ series.

Ex. 1.10 Analogous recurrence relations for ${}_1\phi_1$ series are given in Slater [1954c].

Exercises 1.12 and 1.13 The notations Δ_q, ϑ_q, and D_q are also employed in the literature for this q-derivative operator. We employed the script $\mathcal{D}_q$ operator notation to distinguish this q-derivative operator from the q-derivative operator defined in (7.7.3) and the q-difference operator defined in Ex. 1.31. Additional results involving q-derivatives and q-difference equations are contained in Adams [1931], Agarwal [1953d], Andrews [1968, 1971a], Bowman [2002], Carmichael [1912], Di Vizio [2002, 2003], Faddeev and Kashaev [2002], Faddeev, Kashaev and Volkov [2001], Hahn [1949a,c, 1950, 1952, 1953], Ismail, Merkes and Styer [1990], Jackson [1905c, 1909a, 1910b,d,e], Miller [1970], Mimachi [1989], Sauloy [2003], Starcher [1931], and Trjitzinsky [1933]. For fractional q-derivatives and q-integrals see Agarwal [1969b] and Al-Salam and Verma [1975a,b]. Some "q-Taylor series" are considered in Jackson [1909b,c] and Wallisser [1985]. A q-Taylor theorem based on the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\phi_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, $x = \cos\theta$, was obtained by Ismail and Stanton [2003a,b] along with some interesting applications.

Ex. 1.14 For q-tangent and q-secant numbers and some of their properties, see Andrews and Foata [1980] and Foata [1981]. A discussion of q-trigonometry is given in Gosper [2001]. See also Bustoz and Suslov [1998] and Suslov [2003].

Exercises 1.20–1.23 Ismail and Muldoon [1994] studied some inequalities and monotonicity properties of the gamma and q-gamma functions.

Ex. 1.22 Also see Artin [1964, pp. 14–15]. A different characterization of Γ_q is presented in Kairies and Muldoon [1982].

Exercises 1.24–1.27 Other formulas involving q-Bessel functions are contained in Jackson [1904a–d, 1908], Ismail and Muldoon [1988], Rahman [1987, 1988c, 1989b,c], and Swarttouw and Meijer [1994]. It was pointed out by Ismail in an unpublished preprint in 1999 (rewritten for publication as Ismail [2003c]) that $J_\nu^{(3)}(x;q)$ was actually introduced by Jackson [1905a], contrary to the claim in Swarttouw [1992] that a special case of it was first discovered by Hahn [1953] and then in full generality by Exton [1978].

Ex. 1.28 See the generating functions for the continuous q-Hermite polynomials derived in Carlitz [1963b, 1972] and Bressoud [1980b], and the applications to modular forms in Bressoud [1986]. An extension of these q-Bessel functions to a q-quadratic grid is given in Ismail, Masson and Suslov [1999].

Ex. 1.32 Masuda *et al.* [1991] showed that the matrix elements that arise in the representations of certain quantum groups are expressible in terms of little q-Jacobi polynomials, and that this and a form of the Peter-Weyl theorem imply the orthogonality relation for these polynomials. Padé approximants for the moment generating function for the little q-Jacobi polynomials are employed in Andrews, Goulden and D.M. Jackson [1986] to explain and extend Shank's method for accelerating the convergence of sequences. Padé approximations for some q-hypergeometric functions are considered in Ismail, Perline and Wimp [1992].

2

SUMMATION, TRANSFORMATION, AND EXPANSION FORMULAS

2.1 Well-poised, nearly-poised, and very-well-poised hypergeometric and basic hypergeometric series

The hypergeometric series

$$ {}_{r+1}F_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; z\right] \tag{2.1.1}$$

is called *well-poised* if its parameters satisfy the relations

$$1 + a_1 = a_2 + b_1 = a_3 + b_2 = \ldots = a_{r+1} + b_r, \tag{2.1.2}$$

and it is called *nearly-poised* if all but one of the above pairs of parameters (regarding 1 as the first denominator parameter) have the same sum. The series (2.1.1) is called a *nearly-poised series of the first kind* if

$$1 + a_1 \neq a_2 + b_1 = a_3 + b_2 = \cdots = a_{r+1} + b_r, \tag{2.1.3}$$

and it is called a *nearly-poised series of the second kind* if

$$1 + a_1 = a_2 + b_1 = a_3 + b_2 = \cdots = a_r + b_{r-1} \neq a_{r+1} + b_r. \tag{2.1.4}$$

The order of summation of a terminating nearly-poised series can be reversed so that the resulting series is either of the first kind or of the second kind.

Kummer's summation formula (1.8.2) gives the sum of a well-poised ${}_2F_1$ series with argument -1. Another example of a summable well-poised series is provided by Dixon's [1903] formula

$$\begin{aligned} &{}_3F_2\left[a, b, c; 1 + a - b, 1 + a - c; 1\right] \\ &= \frac{\Gamma(1 + \frac{1}{2}a)\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(1 + a)\Gamma(1 + \frac{1}{2}a - b)\Gamma(1 + \frac{1}{2}a - c)\Gamma(1 + a - b - c)}, \end{aligned} \tag{2.1.5}$$

Re $(1 + \frac{1}{2}a - b - c) > 0$, which reduces to Kummer's formula (1.8.2) by letting $c \to -\infty$.

If the series (2.1.1) is well-poised and $a_2 = 1 + \frac{1}{2}a_1$, then it is called a *very-well-poised* series. Dougall's [1907] summation formulas

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a,\ 1 + \frac{1}{2}a,\ b,\ c,\ d,\ e,\ -n \\ \frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + n \end{matrix}; 1\right] \\ &= \frac{(1 + a)_n(1 + a - b - c)_n(1 + a - b - d)_n(1 + a - c - d)_n}{(1 + a - b)_n(1 + a - c)_n(1 + a - d)_n(1 + a - b - c - d)_n} \end{aligned} \tag{2.1.6}$$

when the series is 2-balanced (i.e, $1+2a+n=b+c+d+e$), and

$$
{}_5F_4\left[\begin{matrix} a, 1+\frac{1}{2}a,\ b,\ c,\ d \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d \end{matrix}; 1\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}, \tag{2.1.7}
$$

when Re $(1+a-b-c-d)>0$, illustrate the importance of very-well-poised hypergeometric series. Note that Dixon's formula (2.1.5) follows from (2.1.7) by setting $d=\frac{1}{2}a$.

Analogous to the hypergeometric case, we shall call the basic hypergeometric series

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots\ , b_r \end{matrix}; q, z\right] \tag{2.1.8}
$$

well-poised if the parameters satisfy the relations

$$
qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r; \tag{2.1.9}
$$

very-well-poised if, in addition, $a_2 = qa_1^{\frac{1}{2}}, a_3 = -qa_1^{\frac{1}{2}}$; a *nearly-poised series of the first kind* if

$$
qa_1 \neq a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r,
$$

and a *nearly-poised series of the second kind* if

$$
qa_1 = a_2b_1 = a_3b_2 = \cdots = a_rb_{r-1} \neq a_{r+1}b_r. \tag{2.1.10}
$$

In this chapter we shall be primarily concerned with the summation and transformation formulas for very-well-poised basic hypergeometric series. To help simplify some of the displays involving very-well-poised ${}_{r+1}\phi_r$ series which arise in the proofs in this and the subsequent chapters we shall frequently replace

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \ldots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{matrix}; q, z\right]
$$

by the more compact notation

$$
{}_{r+1}W_r\left(a_1; a_4, a_5, \ldots, a_{r+1}; q, z\right). \tag{2.1.11}
$$

In the displays of the main formulas, however, we shall continue to use the ${}_{r+1}\phi_r$ notation, since in most applications one needs to know the denominator parameters.

In all of the very-well-poised ${}_6\phi_5$, ${}_8\phi_7$ and ${}_{10}\phi_9$ series in Appendix II and Appendix III the parameters $a_1, a_4, \ldots, a_{r+1}$ and the argument z satisfy the *very-well-poised balancing condition*

$$
(a_4a_5\cdots a_{r+1})z = (\pm(a_1q)^{\frac{1}{2}})^{r-3} \tag{2.1.12}
$$

with either the plus or minus sign, where $(a_1q)^{\frac{1}{2}}$ is the principal value of the square root of a_1q. Thus, we will call a ${}_{r+1}W_r$ series *very-well-poised-balanced* (or, for brevity, *VWP-balanced*) if (2.1.12) holds. A special case of this condition was stated at the end of §2.4 in the first edition of this book. It follows from (2.1.12) that a VWP-balanced series is balanced (1-balanced) when $z=q$, a balanced ${}_{r+1}W_r$ series is VWP-balanced when r is even, but it is not necessarily VWP-balanced when r is odd since then we only know that the squares of both sides of (2.1.12) (with $z = q$) are equal (see the comments in the first paragraph of §2.6). When $r + 1 = 2k$ is an even integer, which is usually the case, (2.1.12) simplifies to the condition

$$(a_4 a_5 \cdots a_{2k})z = (a_1 q)^{k-2}. \tag{2.1.13}$$

Notice that if we set $a_{2k} = \pm(a_1 q)^{\frac{1}{2}}$, then the ${}_{2k}W_{2k-1}$ series reduces to a ${}_{2k-1}W_{2k-2}$ series that satisfies (2.1.12) with $r = 2k - 2$ and the corresponding plus or minus sign.

Since any well-poised ${}_{r+1}\phi_r$ series that satisfies the relations in (2.1.9) can be written as a very-well-poised series in the form

$${}_{r+5}W_{r+4}(a_1; a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_2,\ a_3, \ldots, a_{r+1}; q, z), \tag{2.1.14}$$

if follows that the very-well-poised balancing condition for the series in (2.1.14) is consistent with the *well-poised balancing condition*

$$(a_1 a_2 \cdots a_{r+1})z = -(\pm(a_1 q)^{\frac{1}{2}})^{r+1} \tag{2.1.15}$$

for a well-poised ${}_{r+1}\phi_r$ series to be *well-poised-balanced (WP-balanced)*. In particular, the ${}_2\phi_1$ series in the Bailey-Daum summation formula (1.8.1) is WP-balanced. Clearly, every VWP-balanced basic hypergeometric series is WP-balanced and, by the above observations, every WP-balanced basic hypergeometric series can be rewritten to be a VWP-balanced series of the form in (2.1.14).

2.2 A general expansion formula

Let a, b, c be arbitrary parameters and k be a nonnegative integer. Then, by the q-Saalschütz formula (1.7.2)

$$\begin{aligned} &{}_3\phi_2\left(q^{-k}, aq^k, aq/bc; aq/b, aq/c; q, q\right) \\ &= \frac{(c, q^{1-k}/b; q)_k}{(aq/b, cq^{-k}/a; q)_k} = \frac{(b, c; q)_k}{(aq/b, aq/c; q)_k} \left(\frac{aq}{bc}\right)^k, \end{aligned} \tag{2.2.1}$$

so that

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(b, c, q^{-n}; q)_k}{(q, aq/b, aq/c; q)_k} A_k \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(aq/bc, aq^k, q^{-k}; q)_j (q^{-n}; q)_k}{(q, aq/b, aq/c; q)_j (q; q)_k} q^j \left(\frac{bc}{aq}\right)^k A_k \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-j} \frac{\left(aq/bc, aq^{i+j}, q^{-i-j}; q\right)_j (q^{-n}; q)_{i+j}}{(q, aq/b, aq/c; q)_j (q; q)_{i+j}} q^j \left(\frac{bc}{aq}\right)^{i+j} A_{i+j} \\ &= \sum_{j=0}^{n} \frac{\left(aq/bc, aq^j, q^{-n}; q\right)_j}{(q, aq/b, aq/c; q)_j} (-1)^j q^{-\binom{j}{2}} \end{aligned}$$

$$\times \sum_{i=0}^{n-j} \frac{\left(q^{j-n}, aq^{2j}; q\right)_i}{(q, aq^j; q)_i} q^{-ij} \left(\frac{bc}{aq}\right)^{i+j} A_{i+j}, \tag{2.2.2}$$

where $\{A_k\}$ is an arbitrary sequence. This is equivalent to Bailey's [1949] lemma. Choosing

$$A_k = \frac{(a, a_1, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_{r+1}; q)_k} z^k, \tag{2.2.3}$$

we obtain the expansion formula

$$\begin{aligned} &{}_{r+4}\phi_{r+3} \left[\begin{matrix} a, b, c, a_1, a_2, \ldots, a_r, q^{-n} \\ aq/b, aq/c, b_1, b_2, \ldots, b_r, b_{r+1} \end{matrix} ; q, z \right] \\ &= \sum_{j=0}^{n} \frac{(aq/bc, a_1, a_2, \ldots, a_r, q^{-n}; q)_j}{(q, aq/b, aq/c, b_1, \ldots, b_r, b_{r+1}; q)_j} \left(-\frac{bcz}{aq}\right)^j q^{-\binom{j}{2}} (a;q)_{2j} \\ &\times {}_{r+2}\phi_{r+1} \left[\begin{matrix} aq^{2j}, a_1q^j, a_2q^j, \ldots, a_rq^j, q^{j-n} \\ b_1q^j, b_2q^j, \ldots, b_rq^j, b_{r+1}q^j \end{matrix} ; q, \frac{bcz}{aq^{j+1}} \right]. \end{aligned} \tag{2.2.4}$$

This is a q-analogue of Bailey's formula [1935, 4.3(1)]. The most important property of (2.2.4) is that it enables one to reduce the $_{r+4}\phi_{r+3}$ series to a sum of $_{r+2}\phi_{r+1}$ series. Consequently, if the above $_{r+2}\phi_{r+1}$ series is summable for some values of the parameters then (2.2.4) gives a transformation formula for the corresponding $_{r+4}\phi_{r+3}$ series in terms of a single series.

2.3 A summation formula for a terminating very-well-poised $_4\phi_3$ series

Setting $b = qa^{\frac{1}{2}}, c = -qa^{\frac{1}{2}}$ and $a_k = b_k, k = 1, 2, \ldots, r, b_{r+1} = aq^{n+1}$, we obtain from (2.2.4) that

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq^{n+1} \end{matrix} ; q, z \right] \\ &= \sum_{j=0}^{n} \frac{\left(-q^{-1}, q^{-n}; q\right)_j (a;q)_{2j}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq^{n+1}; q)_j} (qz)^j q^{-\binom{j}{2}} \\ &\times {}_2\phi_1 \left(aq^{2j}, q^{j-n}; aq^{j+n+1}; q, -zq^{1-j}\right). \end{aligned} \tag{2.3.1}$$

If we set $z = q^n$ so that the above $_4\phi_3$ series is VWP-balanced, then the $_2\phi_1$ series on the right of (2.3.1) can be summed by means of the Bailey-Daum summation formula (1.8.1), which gives

$$_2\phi_1 \left(aq^{2j}, q^{j-n}; aq^{j+n+1}; q, -q^{1+n-j}\right) = \frac{(-q;q)_\infty \left(aq^{2j+1}, aq^{2n+2}; q^2\right)_\infty}{(aq^{n+j+1}, -q^{1+n-j}; q)_\infty}. \tag{2.3.2}$$

Hence, using the identities (1.2.32), (1.2.39) and (1.2.40), and simplifying, we obtain the transformation formula

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq^{n+1} \end{matrix} ; q, q^n \right] \\ &= \frac{(aq, -q; q)_n}{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_n} {}_2\phi_1 \left(q^{-n}, -q^{-1}; -q^{-n}; q, q\right). \end{aligned} \tag{2.3.3}$$

Clearly, both sides of (2.3.3) are equal to 1 when $n = 0$. By (1.5.3) the ${}_2\phi_1$ series on the right of (2.3.3) has the sum $\left(q^{1-n};q\right)_n \left(-q^{-1}\right)^n / (-q^{-n};q)_n$ when $n = 0, 1, \ldots$. Since $\left(q^{1-n};q\right)_n = 0$ unless $n = 0$, it follows that

$$ {}_4\phi_3 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq^{n+1} \end{matrix} ; q, q^n \right] = \delta_{n,0} , \tag{2.3.4} $$

where

$$ \delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases} \tag{2.3.5} $$

is the Kronecker delta function. This summation formula will be used in the next section to obtain the sum of a ${}_6\phi_5$ series.

2.4 A summation formula for a terminating very-well-poised ${}_6\phi_5$ series

Let us now set $a_1 = qa^{\frac{1}{2}}$, $a_2 = -qa^{\frac{1}{2}}, b_1 = a^{\frac{1}{2}}$, $b_2 = -a^{\frac{1}{2}}, b_{r+1} = aq^{n+1}$ and $a_k = b_k$, for $k = 3, 4, \ldots, r$. Then (2.2.4) gives

$$ \begin{aligned} &{}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, z \right] \\ &= \sum_{j=0}^{n} \frac{(aq/bc, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n}; q)_j (a;q)_{2j}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1}; q)_j} \left(-\frac{bcz}{aq} \right)^j q^{-\binom{j}{2}} \\ &\times {}_4\phi_3 \left[\begin{matrix} aq^{2j}, q^{j+1}a^{\frac{1}{2}}, -q^{j+1}a^{\frac{1}{2}}, q^{j-n} \\ q^j a^{\frac{1}{2}}, -q^j a^{\frac{1}{2}}, aq^{j+n+1} \end{matrix} ; q, \frac{bcz}{aq^{j+1}} \right]. \end{aligned} \tag{2.4.1} $$

If $z = aq^{n+1}/bc$, then we can sum the above ${}_4\phi_3$ series by means of (2.3.4) to obtain the summation formula

$$ \begin{aligned} &{}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] \\ &= \frac{(aq/bc, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q^{-n}; q)_n \ (a;q)_{2n}}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1}; q)_n} (-1)^n q^{n(n+1)/2} \\ &= \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \end{aligned} \tag{2.4.2} $$

Note that the above ${}_6\phi_5$ series is VWP-balanced and that this summation formula reduces to (2.3.4) when $bc = aq$. In the next two sections, like climbing the steps of a ladder, we will use (2.4.2) to extend it to a transformation formula and a summation formula for very-well-poised ${}_8\phi_7$ series.

2.5 Watson's transformation formula for a terminating very-well-poised ${}_8\phi_7$ series

We shall now use (2.4.2) to prove Watson's [1929a] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series as a multiple of a terminating balanced ${}_4\phi_3$ series:

$$
{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2q^{2+n}}{bcde}\right]
= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, d, e, aq/bc \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right]. \tag{2.5.1}
$$

It suffices to observe that from (2.2.4) we have

$$
{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2q^{2+n}}{bcde}\right]
= \sum_{j=0}^{n} \frac{(aq/bc, qa^{\frac12}, -qa^{\frac12}, d, e, q^{-n}; q)_j\ (a;q)_{2j}}{(q, a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q)_j} \left(-\frac{aq^{n+1}}{de}\right)^j q^{-\binom{j}{2}}
\times {}_6\phi_5\left[\begin{matrix} aq^{2j}, q^{j+1}a^{\frac12}, -q^{j+1}a^{\frac12}, dq^j, eq^j, q^{j-n} \\ q^ja^{\frac12}, -q^ja^{\frac12}, aq^{j+1}/d, aq^{j+1}/e, aq^{j+n+1} \end{matrix}; q, \frac{aq^{1+n-j}}{de}\right], \tag{2.5.2}
$$

which gives formula (2.5.1) by using (2.4.2) to sum the above ${}_6\phi_5$ series.

Clearly, the ${}_8\phi_7$ series in (2.5.1) is VWP-balanced and the transformation formula (2.5.1) reduces to the summation formula (2.4.2) when $bc = aq$.

2.6 Jackson's sum of a terminating very-well-poised balanced ${}_8\phi_7$ series

The ${}_8\phi_7$ series in (2.5.1) is balanced when the six parameters a, b, c, d, e and n satisfy the condition

$$
a^2q^{n+1} = bcde, \tag{2.6.1}
$$

which is the very-well-poised balancing condition. For such a series Jackson [1921] showed that

$$
{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, q\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{2.6.2}
$$

when $n = 0, 1, 2, \ldots$. This formula follows directly from (2.5.1), since the ${}_4\phi_3$ series on the right of (2.5.1) becomes a balanced ${}_3\phi_2$ series when (2.6.1) holds, and therefore can be summed by the q-Saalschütz formula. Notice that the very-well-poised series in (2.6.2) is also balanced if $a^2q^{n+1} = -bcde \neq 0$, but then it is not VWP-balanced and it is not summable as a quotient of products of q-shifted factorials. Hence, not every balanced terminating very-well-poised ${}_8\phi_7$ series is summable.

Formula (2.6.2) is a q-analogue of (2.1.6), as can be seen by replacing a, b, c, d, e by q^a, q^b, q^c, q^d, q^e, respectively, and then letting $q \to 1$. It should be observed that the series ${}_8\phi_7$ in (2.6.2) is balanced, while the limiting series ${}_7F_6$ in (2.1.6) is 2-balanced. The reason for this apparent discrepancy is that the appropriate q-analogue of the term $(1 + \frac12 a)_k/(\frac12 a)_k = (a + 2k)/a$ in the ${}_7F_6$ series is not $(qa^{\frac12}; q)_k/(a^{\frac12}; q)_k = (1 - a^{\frac12}q^k)/(1 - a^{\frac12})$

but $(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_k/(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_k$, which introduces an additional q-factor in the ratio of the products of the numerator and denominator parameters. Analogous to (2.1.12), one could call the very-well-poised ${}_7F_6$ series in (2.1.6) VWP-balanced if $(b+c+d+e-n)+1 = 2(a+1)$, so that then every terminating very-well-poised ${}_7F_6(1)$ series is summable whenever it is VWP-balanced.

2.7 Some special and limiting cases of Jackson's and Watson's formulas: the Rogers–Ramanujan identities

Many of the known summation formulas for basic hypergeometric series are special or limiting cases of Jackson's formula (2.6.2). For example, if we take $d \to \infty$ in (2.6.2) we get (2.4.2). On the other hand, taking the limit $a \to 0$ after replacing d by aq/d gives the q-Saalschütz formula (1.7.2). Let us now take the limit $n \to \infty$ in (2.6.2) to obtain the following summation formula for a non-terminating VWP-balanced ${}_6\phi_5$ series

$$
{}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d \end{matrix} ; q, \frac{aq}{bcd} \right]
= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \tag{2.7.1}
$$

where, for convergence, it is required that $|aq/bcd| < 1$. This formula is clearly a q-analogue of Dougall's ${}_5F_4$ summation formula (2.1.7), and it reduces to (2.3.4) when $bc = aq$ and $d = q^{-n}$. If $cd = aq$, then (2.7.1) reduces to the summation formula ${}_4W_3(a; b; q, 1/b) = 0$, where $|b| > 1$. Setting $d = a^{\frac{1}{2}}$ in (2.7.1), we get a q-analogue of Dixon's formula (2.1.5)

$$
{}_4\phi_3 \left[\begin{matrix} a, -qa^{\frac{1}{2}}, b, c \\ -a^{\frac{1}{2}}, aq/b, aq/c \end{matrix} ; q, \frac{qa^{\frac{1}{2}}}{bc} \right]
= \frac{(aq, aq/bc, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c; q)_\infty}{(aq/b, aq/c, qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/bc; q)_\infty}, \tag{2.7.2}
$$

provided $|qa^{\frac{1}{2}}/bc| < 1$.

Watson [1929a] used his transformation formula (2.5.1) to give a simple proof of the famous Rogers–Ramanujan identities (Hardy [1937]):

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty}, \tag{2.7.3}
$$

$$
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q;q)_\infty}, \tag{2.7.4}
$$

where $|q| < 1$. First let $b, c, d, e \to \infty$ in (2.5.1) to obtain

$$
\sum_{k=0}^{n} \frac{(a;q)_k \ (1-aq^{2k}) \ (q^{-n};q)_k}{(q;q)_k \ (1-a) \ (aq^{n+1};q)_k} q^{2k^2} (a^2q^n)^k
= (aq;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \left(-aq^{n+1}\right)^k q^{k(k-1)/2}. \tag{2.7.5}
$$

Since the series on both sides are finite this limiting procedure is justified as long as the term-by-term limits are assumed to exist. However, our next step is to take the limit $n \to \infty$ on both sides of (2.7.5), which we can justify by applying the dominated convergence theorem. Thus we have

$$1 + \sum_{k=1}^{\infty} \frac{(aq;q)_{k-1}(1-aq^{2k})}{(q;q)_k}(-1)^k a^{2k} q^{k(5k-1)/2} = (aq;q)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q;q)_k}. \tag{2.7.6}$$

In view of Jacobi's triple product identity (1.6.1) the series on the left side of (2.7.6) can be summed in the cases $a = 1$ and $a = q$ by observing that

$$1 + \sum_{k=1}^{\infty} \frac{(q;q)_{k-1}(1-q^{2k})}{(q;q)_k}(-1)^k q^{k(5k-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2} = \left(q^2, q^3, q^5; q^5\right)_\infty \tag{2.7.7}$$

and

$$\sum_{k=0}^{\infty}(1-q^{2k+1})(-1)^k q^{k(5k+3)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+3)/2} = \left(q, q^4, q^5; q^5\right)_\infty. \tag{2.7.8}$$

The identities (2.7.3) and (2.7.4) now follow immediately by using (2.7.6). For an early history of these identities see Hardy [1940, pp. 90–99].

2.8 Bailey's transformation formulas for terminating ${}_5\phi_4$ and ${}_7\phi_6$ series

Using Jackson's formula (2.6.2), it can be easily shown that

$$\frac{(a,b,c;q)_k}{(q,aq/b,aq/c;q)_k} = \frac{(\lambda bc/a;q)_k}{(qa^2/\lambda bc;q)_k} \times \sum_{j=0}^{k} \frac{(\lambda;q)_j(1-\lambda q^{2j})(\lambda b/a, \lambda c/a, aq/bc;q)_j}{(q;q)_j(1-\lambda)(aq/b, aq/c, \lambda bc/a;q)_j} \times \frac{(a;q)_{k+j}(a/\lambda;q)_{k-j}}{(\lambda q;q)_{k+j}\ (q;q)_{k-j}} \left(\frac{a}{\lambda}\right)^j, \tag{2.8.1}$$

where λ is an arbitrary parameter. If $\{A_k\}$ is an arbitrary sequence, it follows that

$$\sum_{k=0}^{\infty} \frac{(a,b,c;q)_k}{(q,aq/b,aq/c;q)_k} A_k$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda bc/a;q)_k}{(qa^2/\lambda bc;q)_k} A_k \sum_{j=0}^{k} \frac{(\lambda;q)_j(1-\lambda q^{2j})\,(\lambda b/a,\lambda c/a,aq/bc;q)_j}{(q;q)_j(1-\lambda)(aq/b,aq/c,\lambda bc/a;q)_j}$$

$$\times \frac{(a;q)_{k+j}(a/\lambda;q)_{k-j}}{(\lambda q;q)_{k+j}(q;q)_{k-j}} \left(\frac{a}{\lambda}\right)^j$$

$$= \sum_{j=0}^{\infty} \frac{(\lambda;q)_j(1-\lambda q^{2j})(\lambda b/a,\lambda c/a,aq/bc;q)_j(a;q)_{2j}}{(q;q)_j(1-\lambda)(aq/b,aq/c,qa^2/\lambda bc;q)_j(\lambda q;q)_{2j}} \left(\frac{a}{\lambda}\right)^j$$

$$\times \sum_{k=0}^{\infty} \frac{(aq^{2j},a/\lambda,\lambda bcq^j/a;q)_k}{(q,\lambda q^{2j+1},a^2q^{j+1}/\lambda bc;q)_k} A_{j+k}, \tag{2.8.2}$$

provided the change in order of summation is justified (e.g., if all of the series terminate or are absolutely convergent).

It is clear that appropriate choices of λ and A_k will lead to transformation formulas for basic hypergeometric series which have at least a partial well-poised structure.

First, let us take $A_k = q^k(d,q^{-n};q)_k/(aq/d,a^2q^{-n}/\lambda^2;q)_k$ and $\lambda = qa^2/bcd$, so that the inner series on the right of (2.8.2) becomes a balanced and terminating ${}_3\phi_2$. Summing this ${}_3\phi_2$ series and simplifying the coefficients, we obtain Bailey's [1947a,c] formula

$$ {}_5\phi_4 \left[\begin{matrix} a,b,c,d,q^{-n} \\ aq/b,aq/c,aq/d,a^2q^{-n}/\lambda^2 \end{matrix} ;q,q \right]$$

$$= \frac{(\lambda q/a,\lambda^2 q/a;q)_n}{(\lambda q,\lambda^2 q/a^2;q)_n} \, {}_{12}\phi_{11} \left[\begin{matrix} \lambda, q\lambda^{\frac12}, -q\lambda^{\frac12}, b\lambda/a, c\lambda/a, d\lambda/a, \\ \lambda^{\frac12}, -\lambda^{\frac12}, aq/b, aq/c, aq/d, \end{matrix} \right.$$

$$\left. \begin{matrix} a^{\frac12}, -a^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}, \lambda^2 q^{n+1}/a, q^{-n} \\ \lambda q/a^{\frac12}, -\lambda q/a^{\frac12}, \lambda(q/a)^{\frac12}, -\lambda(q/a)^{\frac12}, aq^{-n}/\lambda, \lambda q^{n+1} \end{matrix} ;q,q \right], \tag{2.8.3}$$

where $\lambda = qa^2/bcd$.

Note that the ${}_5\phi_4$ series on the left is balanced and nearly-poised of the second kind, while the ${}_{12}\phi_{11}$ series on the right is balanced and VWP-balanced. Note also that a terminating nearly-poised series of the second kind can be expressed as a multiple of a nearly-poised series of the first kind by simply reversing the series.

By proceeding as in the proof of (2.8.3), one can obtain the following variation of (2.8.3)

$$ {}_5\phi_4 \left[\begin{matrix} q^{-n},b,c,d,e \\ q^{1-n}/b,q^{1-n}/c,q^{1-n}/d,eq^{-2n}/\lambda^2 \end{matrix} ;q,q \right]$$

$$= \frac{(\lambda^2 q^{n+1},\lambda q/e;q)_n}{(\lambda^2 q^{n+1}/e,\lambda q;q)_n} \, {}_{12}\phi_{11} \left[\begin{matrix} \lambda, q\lambda^{\frac12}, -q\lambda^{\frac12}, \lambda bq^n, \lambda cq^n, \lambda dq^n, \\ \lambda^{\frac12}, -\lambda^{\frac12}, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, \end{matrix} \right.$$

$$\left. \begin{matrix} q^{-n/2}, -q^{-n/2}, q^{(1-n)/2}, -q^{(1-n)/2}, e, \lambda^2 q^{n+1}/e \\ \lambda q^{1+n/2}, -\lambda q^{1+n/2}, \lambda q^{(1+n)/2}, -\lambda q^{(1+n)/2}, \lambda q/e, eq^{-n}/\lambda \end{matrix} ;q,q \right], \tag{2.8.4}$$

where $\lambda = q^{1-2n}/bcd$.

Next, let us choose $A_k = q^k(1-aq^{2k})(d,q^{-n};q)_k/(1-a)(aq/d, a^2q^{2-n}/\lambda^2;q)_k$ and $\lambda = qa^2/bcd$ in (2.8.2) so that the inner series on the right side takes the form

$$q^j \frac{1-aq^{2j}}{1-a} \frac{(d,q^{-n};q)_j}{(aq/d, a^2q^{2-n}/\lambda^2;q)_j} \times {}_5\phi_4 \left[\begin{matrix} aq^{2j}, q^{j+1}a^{\frac{1}{2}}, -q^{j+1}a^{\frac{1}{2}}, a/\lambda, q^{j-n} \\ q^j a^{\frac{1}{2}}, -q^j a^{\frac{1}{2}}, \lambda q^{2j+1}, a^2q^{j-n+2}/\lambda^2 \end{matrix} ; q,q \right].$$

This ${}_5\phi_4$ series is a special case of the ${}_5\phi_4$ series on the left side of (2.8.3); in fact, the ${}_{12}\phi_{11}$ series on the right side of (2.8.3) in this case reduces to a terminating ${}_8\phi_7$ series, which we can sum by Jackson's formula (2.6.2). Carrying out the straightforward calculations, we get Bailey's [1947c] second transformation formula

$$\begin{aligned} &{}_7\phi_6 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, a^2q^{2-n}/\lambda^2 \end{matrix} ; q,q \right] \\ &= \frac{(\lambda/aq, \lambda^2/aq;q)_n}{(\lambda q, \lambda^2/a^2q;q)_n} \frac{1-\lambda^2q^{2n-1}/a}{1-\lambda^2/aq} \\ &\times {}_{12}\phi_{11} \left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, b\lambda/a, c\lambda/a, d\lambda/a, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda(q/a)^{\frac{1}{2}}, -\lambda(q/a)^{\frac{1}{2}}, \end{matrix} \right. \\ &\qquad \left. \begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, \lambda^2q^{n-1}/a, q^{-n} \\ \lambda/a^{\frac{1}{2}}, -\lambda/a^{\frac{1}{2}}, aq^{2-n}/\lambda, \lambda q^{n+1} \end{matrix} ; q,q \right], \end{aligned} \tag{2.8.5}$$

where $\lambda = qa^2/bcd$.

2.9 Bailey's transformation formula for a terminating ${}_{10}\phi_9$ series

One of the most important transformation formulas for basic hypergeometric series is Bailey's [1929] formula transforming a terminating ${}_{10}\phi_9$ series, which is both balanced and VWP-balanced, into a series of the same type:

$$\begin{aligned} &{}_{10}\phi_9 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1} \end{matrix} ; q,q \right] \\ &= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f;q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q;q)_n} {}_{10}\phi_9 \left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \end{matrix} \right. \\ &\qquad \left. \begin{matrix} e, f, \lambda aq^{n+1}/ef, q^{-n}, \\ \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1} \end{matrix} ; q,q \right], \end{aligned} \tag{2.9.1}$$

where $\lambda = qa^2/bcd$.

To derive this formula, first observe that by (2.6.2)

$${}_8\phi_7 \left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, aq^m, q^{-m} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda q^{1-m}/a, \lambda q^{m+1} \end{matrix} ; q,q \right]$$

$$= \frac{(b,c,d,\lambda q;q)_m}{(aq/b,aq/c,aq/d,a/\lambda;q)_m}, \tag{2.9.2}$$

and hence the left side of (2.9.1) equals

$$\begin{aligned}
&\sum_{m=0}^{n} \frac{(a;q)_m(1-aq^{2m})(e,f,\lambda aq^{n+1}/ef,q^{-n};q)_m(a/\lambda;q)_m}{(q;q)_m(1-a)(aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1};q)_m(\lambda q;q)_m}\, q^m \\
&\quad\times \sum_{j=0}^{m} \frac{(\lambda;q)_j(1-\lambda q^{2j})(\lambda b/a,\lambda c/a,\lambda d/a,aq^m,q^{-m};q)_j}{(q;q)_j(1-\lambda)(aq/b,aq/c,aq/d,\lambda q^{1-m}/a,\lambda q^{m+1};q)_j}\, q^j \\
&= \sum_{m=0}^{n}\sum_{j=0}^{m} \frac{(a;q)_{m+j}(1-aq^{2m})(e,f,\lambda aq^{n+1}/ef,q^{-n};q)_m}{(q;q)_{m-j}(1-a)(aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1};q)_m}\, q^m \\
&\quad\times \frac{(a/\lambda;q)_{m-j}(\lambda;q)_j(1-\lambda q^{2j})(\lambda b/a,\lambda c/a,\lambda d/a;q)_j}{(\lambda q;q)_{m+j}(q;q)_j(1-\lambda)(aq/b,aq/c,aq/d;q)_j}\left(\frac{a}{\lambda}\right)^j \\
&= \sum_{j=0}^{n} \frac{(\lambda;q)_j(1-\lambda q^{2j})(\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{n+1}/ef,q^{-n};q)_j}{(q;q)_j(1-\lambda)(aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1};q)_j} \\
&\quad\times \left(\frac{aq}{\lambda}\right)^j \frac{(aq;q)_{2j}}{(\lambda q;q)_{2j}}\, {}_8W_7\left(aq^{2j};eq^j,fq^j,a/\lambda,\lambda aq^{n+j+1}/ef,q^{j-n};q,q\right),
\end{aligned} \tag{2.9.3}$$

where the ${}_8W_7$ series is defined as in §2.1. Summing the above ${}_8W_7$ series by means of (2.6.2) and simplifying the coefficients, we obtain (2.9.1). It is sometimes helpful to rewrite (2.9.1) in a somewhat more symmetrical form:

$$\begin{aligned}
&{}_{10}W_9\,(a;b,c,d,e,f,g,h;q,q) \\
&= \frac{(aq,aq/ef,aq/eg,aq/eh,aq/fg,aq/fh,aq/gh,aq/efgh;q)_\infty}{(aq/e,aq/f,aq/g,aq/h,aq/efg,aq/efh,aq/egh,aq/fgh;q)_\infty} \\
&\quad\times {}_{10}W_9\left(qa^2/bcd;aq/bc,aq/bd,aq/cd,e,f,g,h;q,q\right),
\end{aligned} \tag{2.9.4}$$

where at least one of the parameters e,f,g,h is of form $q^{-n}, n=0,1,2,\ldots$, and

$$q^2a^3 = bcdefgh. \tag{2.9.5}$$

2.10 Limiting cases of Bailey's ${}_{10}\phi_9$ transformation formula

A number of the known transformation formulas for basic hypergeometric series follow as limiting cases of the transformation formula (2.9.1). If we let b, c, or $d \to \infty$ in (2.9.1), we obtain Watson's formula (2.5.1). On the other hand, if we take the limit $n \to \infty$, we get the transformation formula for a nonterminating ${}_8\phi_7$ series

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a,qa^{\frac12},-qa^{\frac12},b,c,d,e,f \\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right] \\
&= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q)_\infty}{(aq/e,aq/f,\lambda q,\lambda q/ef;q)_\infty}
\end{aligned}$$

$$\times\ {}_8\phi_7\left[\begin{matrix}\lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, e, f\\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f\end{matrix}; q, \frac{aq}{ef}\right], \tag{2.10.1}$$

where $\lambda = qa^2/bcd$ and

$$\max\left(|aq/ef|,\ |\lambda q/ef|\right) < 1. \tag{2.10.2}$$

The convergence of the two series in (2.10.1) is ensured by the inequalities (2.10.2) which, of course, are not required if both series terminate. For example, if $f = q^{-n}, n = 0, 1, 2, \ldots$, then (2.10.1) becomes

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n}\\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}\end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right]\\ &= \frac{(aq, \lambda q/e; q)_n}{(aq/e, \lambda q; q)_n}\\ &\quad\times\ {}_8\phi_7\left[\begin{matrix}\lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, e, q^{-n}\\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q^{n+1}\end{matrix}; q, \frac{aq^{n+1}}{e}\right].\end{aligned} \tag{2.10.3}$$

This identity expresses one terminating VWP-balanced ${}_8\phi_7$ series in terms of another.

Using (2.5.1) we can now express (2.10.3) as a transformation formula between two terminating balanced ${}_4\phi_3$ series:

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{-n}, a, b, c\\ d, e, f\end{matrix}; q, q\right]\\ &= \frac{(e/a, f/a; q)_n}{(e, f; q)_n}\, a^n\ {}_4\phi_3\left[\begin{matrix}q^{-n}, a, d/b, d/c\\ d, aq^{1-n}/e, aq^{1-n}/f\end{matrix}; q, q\right],\end{aligned} \tag{2.10.4}$$

where $abc = defq^{n-1}$. This is a very useful formula which was first derived by Sears [1951c], and hence is called *Sears' ${}_4\phi_3$ transformation formula.* It is a q-analogue of Whipple's [1926b] formula

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-n, & a, & b, & c\\ & d, & e, & f\end{matrix}; 1\right]\\ &= \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\ {}_4F_3\left[\begin{matrix}-n, a, d-b, d-c\\ d, 1+a-e-n, 1+a-f-n\end{matrix}; 1\right],\end{aligned} \tag{2.10.5}$$

where $a + b + c + 1 = d + e + f + n$.

Use of (2.5.1) and (2.10.1) also enables us to express a terminating balanced ${}_4\phi_3$ series in terms of a nonterminating ${}_8\phi_7$ series. For example, if b, c, or d in (2.10.1) is of the form $q^{-n}, n = 0, 1, 2, \ldots$, then the series on the left side of (2.10.1) terminates, but that on the right side does not. In particular, setting $d = q^{-n}$, and then replacing e and f by d and e, respectively, we obtain

$$\begin{aligned}&{}_8W_7\left(a; b, c, d, e, q^{-n}; q, a^2q^{n+2}/bcde\right)\\&= \frac{(aq, aq/de, a^2q^{n+2}/bcd, a^2q^{n+2}/bce; q)_\infty}{(aq/d, aq/e, a^2q^{n+2}/bc, a^2q^{n+2}/bcde; q)_\infty}\\&\quad\times {}_8W_7\left(a^2q^{n+1}/bc; aq^{n+1}/b, aq^{n+1}/c, aq/bc, d, e; q, aq/de\right), \qquad (2.10.6)\end{aligned}$$

provided $|aq/de| < 1$ to ensure that the nonterminating series on the right side converges. Use of (2.5.1) then leads to the formula

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{-n}, a, b, c\\ d, e, f\end{matrix}; q, q\right] = \frac{(deq^n/a, deq^n/b, deq^n/c, deq^n/abc; q)_\infty}{(deq^n, deq^n/ab, deq^n/ac, deq^n/bc; q)_\infty}\\&\times {}_8\phi_7\left[\begin{matrix}deq^{n-1}, & q\left(deq^{n-1}\right)^{\frac12}, & -q\left(deq^{n-1}\right)^{\frac12}, \\ & \left(deq^{n-1}\right)^{\frac12}, & -\left(deq^{n-1}\right)^{\frac12},\end{matrix}\right.\\&\qquad\left.\begin{matrix}a, b, c, dq^n, eq^n\\ deq^n/a, deq^n/b, deq^n/c, e, d\end{matrix}; q, \frac{de}{abc}\right], \qquad (2.10.7)\end{aligned}$$

provided $def = q^{1-n}abc$ and $|de/abc| < 1$.

As another limiting case of (2.9.1) Bailey [1935, 8.5(3)] found a nonterminating extension of (2.5.1) that expresses a VWP-balanced ${}_8\phi_7$ series in terms of two balanced ${}_4\phi_3$ series. First iterate (2.9.1) to get

$$\begin{aligned}&{}_{10}W_9\left(a; b, c, d, e, f, a^3q^{n+2}/bcdef, q^{-n}; q, q\right)\\&= \frac{(aq, aq/de, aq/df, aq/ef; q)_n}{(aq/d, aq/e, aq/f, aq/def; q)_n}\\&\quad\times {}_{10}W_9\left(defq^{-n-1}/a; aq/bc, d, e, f, bdefq^{-n-1}/a^2, cdefq^{-n-1}/a^2, q^{-n}; q, q\right).\end{aligned} \qquad (2.10.8)$$

Clearly, the ${}_{10}W_9$ on the left side of (2.10.8) tends to the ${}_8\phi_7$ series on the left side of (2.10.1) as $n \to \infty$. However, the terms near both ends of the series on the right side of (2.10.8) are large compared to those in the middle for large n, which prevents us from taking the term-by-term limit directly. To circumvent this difficulty, Bailey chose n to be an odd integer, say $n = 2m + 1$ (this is not necessary, but it makes the analysis simpler), and divided the series on the right into two halves, each containing $m + 1$ terms, and then reversed the order of the second series. The procedure is schematized as follows:

$$\begin{aligned}\sum_{k=0}^{2m+1} \lambda_k &= \sum_{k=0}^{m} \lambda_k + \sum_{k=m+1}^{2m+1} \lambda_k\\&= \sum_{k=0}^{m} \lambda_k + \sum_{k=0}^{m} \lambda_{2m+1-k},\end{aligned} \qquad (2.10.9)$$

where $\{\lambda_k\}$ is an arbitrary sequence. Letting $m \to \infty$ (and hence $n \to \infty$), it follows from (2.10.8) that

$${}_8\phi_7\left[\begin{matrix}a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f\\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f\end{matrix}; q, \frac{a^2q^2}{bcdef}\right]$$

$$= \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty} \, {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix} ; q, q \right]$$
$$+ \frac{(aq, aq/bc, d, e, f, a^2q^2/bdef, \ a^2q^2/cdef; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/bcdef, def/aq; q)_\infty}$$
$$\times \, {}_4\phi_3 \left[\begin{matrix} aq/de, aq/df, aq/ef, a^2q^2/bcdef \\ a^2q^2/bdef, a^2q^2/cdef, aq^2/def \end{matrix} ; q, q \right], \tag{2.10.10}$$

where $|a^2q^2/bcdef| < 1$, if the ${}_8\phi_7$ series does not terminate. Note that if either b or c is of the form $q^{-n}, n = 0, 1, 2, \ldots$, then the ${}_8\phi_7$ series on the left side terminates but the series on the right side do not necessarily terminate. On the other hand if one of the numerator parameters (except $a^2q^2/bcdef$) in either of the two ${}_4\phi_3$ series in (2.10.10) is of the form q^{-n}, then the coefficient of the other ${}_4\phi_3$ series vanishes and we get either (2.5.1) or (2.10.7).

If $aq/bc, aq/de, aq/df$ or aq/ef equals 1, then (2.10.10) reduces to the ${}_6\phi_5$ summation formula (2.7.1). If, on the other hand, $aq/cd = 1$ then the ${}_8\phi_7$ series in (2.10.10) reduces to a ${}_6\phi_5$ which, via (2.7.1), leads to the summation formula

$$\frac{(aq, aq/be, aq/bf, aq/ef; q)_\infty}{(aq/b, aq/e, aq/f, aq/bef; q)_\infty}$$
$$= \frac{(aq, c/e, c/f, aq/ef; q)_\infty}{(c, aq/e, aq/f, c/ef; q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} aq/bc, e, f \\ aq/b, efq/c \end{matrix} ; q, q \right]$$
$$+ \frac{(aq, aq/ef, e, f, aq/bc, acq/bef; q)_\infty}{(aq/e, aq/f, ef/c, c, aq/b, aq/bef; q)_\infty}$$
$$\times \, {}_3\phi_2 \left[\begin{matrix} c/e, c/f, aq/bef \\ cq/ef, acq/bef \end{matrix} ; q, q \right]. \tag{2.10.11}$$

Solving for the first ${}_3\phi_2$ series on the right and relabelling the parameters we get the following nonterminating extension of the q-Saalschütz formula

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; q, q \right] = \frac{(q/e, f/a, f/b, f/c; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty}$$
$$- \frac{(q/e, a, b, c, qf/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, f; q)_\infty}$$
$$\times \, {}_3\phi_2 \left[\begin{matrix} aq/e, bq/e, cq/e \\ q^2/e, qf/e \end{matrix} ; q, q \right], \tag{2.10.12}$$

where $ef = abcq$. Sears [1951a, (5.2)] derived this formula by a different method. If a, b, or c is of the form $q^{-n}, n = 0, 1, 2, \ldots$, then (2.10.12) reduces to (1.7.2).

A special case of (2.10.12) which is worth mentioning is obtained by setting $c = 0, f = 0$, and then replacing e by c to get

$${}_2\phi_1(a, b; c; q, q) = \frac{(q/c, abq/c; q)_\infty}{(aq/c, bq/c; q)_\infty}$$
$$- \frac{(q/c, a, b; q)_\infty}{(c/q, aq/c, bq/c; q)_\infty} \, {}_2\phi_1(aq/c, bq/c; q^2/c; q, q). \tag{2.10.13}$$

If a or b is of the form $q^{-n}, n = 0, 1, 2, \ldots$, then (2.10.13) reduces to (1.5.3). In general, a ${}_2\phi_1(a, b; c; q, q)$ series does not have a sum which can be written as a ratio of infinite products. However, we can still express (2.10.13) as the summation formula for a single bilateral infinite series in the following way.

First, use Heine's transformation formula (1.4.1) to transform both ${}_2\phi_1$ series in (2.10.13):

$$\begin{aligned} {}_2\phi_1(a, b; c; q, q) &= \frac{(b, aq; q)_\infty}{(c, q; q)_\infty} \, {}_2\phi_1(c/b, q; aq; q, b) \\ &= \frac{(b, aq; q)_\infty}{(c, q; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b; q)_n}{(aq; q)_n} b^n, \end{aligned} \tag{2.10.14}$$

$${}_2\phi_1(aq/c, bq/c; q^2/c; q, q) = \frac{(aq/c, bq^2/c; q)_\infty}{(q^2/c, q; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a; q)_n}{(bq^2/c; q)_n} \left(\frac{aq}{c}\right)^n. \tag{2.10.15}$$

Next, note that

$$\sum_{n=0}^{\infty} \frac{(q/a; q)_n}{(bq^2/c; q)_n} \left(\frac{aq}{c}\right)^n = -\frac{c}{q} \frac{1 - bq/c}{1 - a} \sum_{n=1}^{\infty} \frac{(1/a; q)_n}{(bq/c; q)_n} \left(\frac{aq}{c}\right)^n. \tag{2.10.16}$$

Using (2.10.14) - (2.10.16) and the identity (1.2.28) in (2.10.13) we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(c/b; q)_n}{(aq; q)_n} b^n = \frac{(c, q/c, abq/c, q; q)_\infty}{(b, aq, aq/c, bq/c; q)_\infty}, \tag{2.10.17}$$

which is Ramanujan's sum (see Chapter 5 and Andrews and Askey [1978]). However, the conditions under which (2.10.17) is valid, namely, $|q| < 1, |b| < 1$ and $|aq/c| < 1$, are more restrictive then those for (2.10.13). Note that (2.10.17) tends to Jacobi's triple product identity (1.6.1) when $a = 0$ and $b \to 0$. We shall give an alternative derivation of this important sum in Chapter 5 where bilateral basic series are considered.

As was pointed out by Al-Salam and Verma [1982a], both (2.10.10) and (2.10.12) can be conveniently expressed as q-integrals. Thus (2.10.12) is equivalent to

$$\begin{aligned} &\int_a^b \frac{(qt/a, qt/b, ct; q)_\infty}{(dt, et, ft; q)_\infty} \, d_q t \\ &= b(1-q) \frac{(q, bq/a, a/b, c/d, c/e, c/f; q)_\infty}{(ad, ae, af, bd, be, bf; q)_\infty}, \end{aligned} \tag{2.10.18}$$

where $c = abdef$, while (2.10.10) is equivalent to

$$\begin{aligned} &\int_a^b \frac{(qt/a, qt/b, ct, dt; q)_\infty}{(et, ft, gt, ht; q)_\infty} \, d_q t \\ &= b(1-q) \frac{(q, bq/a, a/b, cd/eh, cd/fh, cd/gh, bc, bd; q)_\infty}{(ae, af, ag, be, bf, bg, bh, bcd/h; q)_\infty} \\ &\quad \times \, {}_8W_7\left(bcd/hq; be, bf, bg, c/h, d/h; q, ah\right), \end{aligned} \tag{2.10.19}$$

provided $cd = abefgh$ and $|ah| < 1$.

By substituting $c = abdef$ into (2.10.18), letting $f \to 0$ and then replacing a, d, e by $a, c/a, d/b$, respectively, we obtain Andrews and Askey's [1981] formula

$$\int_a^b \frac{(qt/a, qt/b; q)_\infty}{(ct/a, dt/b; q)_\infty}\, d_q t = \frac{b(1-q)(q, a/b, bq/a, cd; q)_\infty}{(c, d, bc/a, ad/b; q)_\infty}. \tag{2.10.20}$$

In view of (1.3.18), (2.10.20) is a q-extension of the beta-type integral

$$\int_{-c}^{d} (1+t/c)^{a-1}(1-t/d)^{b-1}\, dt = B(a,b)\frac{(c+d)^{a+b-1}}{c^{a-1}d^{b-1}}, \tag{2.10.21}$$

which follows from (1.11.8) by a change of variable.

The notational compactness of (2.10.18) and (2.10.19) is advantageous in many applications (see, e.g., the next section). In addition, the symmetry of the parameters in the q-integral on the left side of (2.10.19) implies the transformation formula (2.10.1).

2.11 Bailey's three-term transformation formula for VWP-balanced ${}_8\phi_7$ series

The q-integral representation (2.10.19) of an ${}_8\phi_7$ series can be put to advantage in deriving Bailey's [1936] three-term transformation formula for VWP-balanced ${}_8\phi_7$ series:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\
&= \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q/c, efq/c, be/a, bf/a; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} ef/c, q(ef/c)^{\frac12}, -q(ef/c)^{\frac12}, aq/bc, aq/cd, ef/a, e, f \\ (ef/c)^{\frac12}, -(ef/c)^{\frac12}, bef/a, def/a, aq/c, fq/c, eq/c \end{matrix}; q, \frac{bd}{a}\right] \\
&\quad + \frac{b}{a}\frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a; q)_\infty} \\
&\quad \times \frac{(aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/def, q/c, b^2q/a; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} b^2/a, qba^{-\frac12}, -qba^{-\frac12}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac12}, -ba^{-\frac12}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right],
\end{aligned} \tag{2.11.1}$$

where $|bd/a| < 1$ and $|a^2q^2/bcdef| < 1$.

To prove this formula, first observe that by (2.10.19)

$${}_8W_7\left(a; b, c, d, e, f; q, a^2q^2/bcdef\right)$$

$$= \frac{aq-def}{adefq(1-q)} \frac{(aq,d,e,f,aq/bc,aq/de,aq/df,aq/ef;q)_\infty}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,def/aq,aq/def;q)_\infty}$$
$$\times \int_{aq}^{def} \frac{(t/a,qt/def,aqt/bdef,aqt/cdef;q)_\infty}{(t/de,t/df,t/ef,aqt/bcdef;q)_\infty}\, d_qt. \tag{2.11.2}$$

Since

$$\int_{aq}^{bdef/a} f(t)\, d_qt + \int_{bdef/a}^{def} f(t)\, d_qt = \int_{aq}^{def} f(t)\, d_qt, \tag{2.11.3}$$

where

$$f(t) = \frac{(t/a,qt/def,aqt/bdef,aqt/cdef;q)_\infty}{(t/de,t/df,t/ef,aqt/bcdef;q)_\infty}, \tag{2.11.4}$$

and

$$\int_{aq}^{bdef/a} f(t)\, d_qt$$
$$= \frac{bdef(1-q)(q,bdef/a^2,a^2q/bdef,bq/d,bq/e,bq/f,bq/a,bq/c;q)_\infty}{a(aq/de,aq/df,aq/ef,bd/a,be/a,bf/a,q/c,b^2q/a;q)_\infty},$$
$$\times\, {}_8W_7\left(b^2/a;\ b,\ bc/a,\ bd/a,\ be/a,\ bf/a;\ q,\ a^2q^2/bcdef\right), \tag{2.11.5}$$

$$\int_{bdef/a}^{def} f(t)\, d_qt = \frac{def(1-q)(q,aq/b,aq/c,eq/c,fq/c,bef/a,def/a;q)_\infty}{(d,e,f,be/a,bf/a,aq/bc,q/c,efq/c;q)_\infty}$$
$$\times\, {}_8W_7\left(ef/c;aq/bc,aq/cd,ef/a,e,f;q,bd/a\right), \tag{2.11.6}$$

we immediately get (2.11.1) by using (2.11.5) and (2.11.6) in (2.11.3). The advantage of our use of the q-integral notation can be seen by comparing the above proof with that given in Bailey [1936].

The special case $qa^2 = bcdef$ is particularly important since the series on the left side of (2.11.1) and the second series on the right become balanced, while the first series on the right becomes a ${}_6\phi_5$ series with sum

$$\frac{(aq/ce,aq/cf,efq/c,q/c;q)_\infty}{(aq/c,eq/c,fq/c,aq/cef;q)_\infty},$$

provided $|aq/cef| < 1$. This gives Bailey's summation formula:

$$ {}_8\phi_7\left[\begin{matrix} a,qa^{\frac12},-qa^{\frac12},b,c,d,e,f \\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f \end{matrix};q,q\right]$$
$$-\frac{b}{a}\frac{(aq,c,d,e,f,bq/a,bq/c,bq/d,bq/e,bq/f;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a,b^2q/a;q)_\infty}$$
$$\times\, {}_8\phi_7\left[\begin{matrix} b^2/a,qba^{-\frac12},-qba^{-\frac12},b,bc/a,bd/a,be/a,bf/a \\ ba^{-\frac12},-ba^{-\frac12},bq/a,bq/c,bq/d,bq/e,bq/f \end{matrix};q,q\right]$$
$$= \frac{(aq,b/a,aq/cd,aq/ce,aq/cf,aq/de,aq/df,aq/ef;q)_\infty}{(aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a;q)_\infty}, \tag{2.11.7}$$

where $qa^2 = bcdef$, which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

$$\int_a^b \frac{(qt/a, qt/b, t/a^{\frac{1}{2}}, -t/a^{\frac{1}{2}}, qt/c, qt/d, qt/e, qt/f; q)_\infty}{(t, bt/a, qt/a^{\frac{1}{2}}, -qt/a^{\frac{1}{2}}, ct/a, dt/a, et/a, ft/a; q)_\infty} \, d_q t$$
$$= \frac{b(1-q)(q, a/b, bq/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_\infty}{(b, c, d, e, f, bc/a, bd/a, be/a, bf/a; q)_\infty}. \tag{2.11.8}$$

2.12 Bailey's four-term transformation formula for balanced ${}_{10}\phi_9$ series

Let us start by replacing $a, b, c, d, e,$ and f in (2.11.8) by $\lambda, bq^n, \lambda c/a, \lambda d/a, \lambda e/a$ and a/bq^n, respectively, to obtain

$$\int_\lambda^{bq^n} \frac{(qt/\lambda, tq^{1-n}/b, t/\lambda^{\frac{1}{2}}, -t/\lambda^{\frac{1}{2}}, aqt/c\lambda, aqt/d\lambda, aqt/e\lambda, btq^{n+1}/a; q)_\infty}{(t, btq^n/\lambda, qt/\lambda^{\frac{1}{2}}, -qt/\lambda^{\frac{1}{2}}, ct/a, dt/a, et/a, atq^{-n}/b\lambda; q)_\infty} \, d_q t$$
$$= \frac{b(1-q)(q, \lambda/b, bq/\lambda, bq/c, bq/d, bq/e, c, d, e; q)_\infty}{(a/\lambda, b, \lambda c/a, \lambda d/a, \lambda e/a, a/b, bc/a, bd/a, be/a; q)_\infty}$$
$$\times \frac{(b, bc/a, bd/a, be/a; q)_n}{(bq/a, bq/c, bq/d, bq/e; q)_n} \left(\frac{\lambda q}{a}\right)^n, \tag{2.12.1}$$

where $n = 0, 1, 2, \ldots$. Let f, g, h be arbitrary complex numbers such that $|a^3q^3/bcdefgh| < 1$. Set $\rho = a^3q^2/bcdefgh$, multiply both sides of (2.12.1) by

$$\frac{(b^2/a; q)_n(1 - b^2q^{2n}/a)(bf/a, bg/a, bh/a; q)_n}{(q; q)_n(1 - b^2/a)(bq/f, bq/g, bq/h; q)_n} \left(\frac{a\rho}{\lambda}\right)^n$$

and sum over n from 0 to ∞. Then the right side of (2.12.1) leads to

$$\frac{b(1-q)(q, \lambda/b, bq/\lambda, bq/c, bq/d, bq/e, c, d, e; q)_\infty}{(a/\lambda, b, \lambda c/a, \lambda d/a, \lambda e/a, a/b, bc/a, bd/a, be/a; q)_\infty}$$
$$\times \, {}_{10}W_9\left(b^2/a; b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q, \rho q\right). \tag{2.12.2}$$

The left side leads to two double sums, one from each of the two limits of the q-integral. From the upper limit, bq^n, we get

$$\frac{b(1-q)(q, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, b^2q/a; q)_\infty}{(b, a/\lambda, b^2q/\lambda, bc/a, bd/a, be/a; q)_\infty}$$
$$\times \sum_{n=0}^\infty \frac{(b^2/a; q)_n(1 - b^2q^{2n}/a)(bf/a, bg/a, bh/a; q)_n}{(q; q)_n(1 - b^2/a)(bq/f, bq/g, bq/h; q)_n} \left(\frac{a\rho}{\lambda}\right)^n$$
$$\times \sum_{j=0}^\infty \frac{(b^2/\lambda; q)_{2n+j}(1 - b^2q^{2n+2j}/\lambda)(b, bc/a, bd/a, be/a; q)_{n+j}(a/\lambda; q)_j}{(q; q)_j(1 - b^2/\lambda)(bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda; q)_{n+j}(b^2q/a; q)_{2n+j}} q^j$$
$$= \frac{b(1-q)(q, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, b^2q/a; q)_\infty}{(b, a/\lambda, b^2q/\lambda, bc/a, bd/a, be/a; q)_\infty}$$
$$\times \sum_{m=0}^\infty \frac{(b^2/\lambda; q)_m(1 - b^2q^{2m}/\lambda)(b, bc/a, bd/a, be/a, a/\lambda; q)_m}{(q; q)_m(1 - b^2/\lambda)(bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, b^2q/a; q)_m} q^m$$
$$\times \, {}_8W_7\left(b^2/a; bf/a, bg/a, bh/a, b^2q^m/\lambda, q^{-m}; q, \rho q\right). \tag{2.12.3}$$

Let us now assume that $\rho = 1$, that is,

$$a^3q^2 = bcdefgh. \tag{2.12.4}$$

Then, by Jackson's formula (2.6.2), the ${}_8\phi_7$ series in (2.12.3) has the sum

$$\frac{(b^2q/a, aq/fg, aq/fh, aq/gh; q)_m}{(bq/f, bq/g, bq/h, aq/fgh; q)_m}$$

and the expression in (2.12.3) simplifies to

$$\frac{b(1-q)(q, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, b^2q/a; q)_\infty}{(b, a/\lambda, b^2q/\lambda, bc/a, bd/a, be/a; q)_\infty} \times {}_{10}W_9\left(b^2/\lambda; b, bc/a, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda; q, q\right), \tag{2.12.5}$$

since, by (2.12.4), $aq/fg = bh/\lambda$, $aq/fh = bg/\lambda$, and $aq/gh = bf/\lambda$.

We now turn to the double sum that corresponds to the lower limit, λ, in the q-integral (2.12.1). This leads to
the series

$$\begin{aligned}
&\frac{-\lambda(1-q)(q, aq/c, aq/d, aq/e, \lambda q/b, b\lambda q/a; q)_\infty}{(b, \lambda q, a/b, c\lambda/a, d\lambda/a, e\lambda/a; q)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(b^2/a; q)_n (1 - b^2q^{2n}/a)(b, bf/a, bg/a, bh/a, b/\lambda; q)_n}{(q;q)_n(1-b^2/a)(bq/a, bq/f, bq/g, bq/h, b\lambda q/a; q)_n} q^n \\
&\times {}_8W_7\left(\lambda; bq^n, c\lambda/a, d\lambda/a, e\lambda/a, aq^{-n}/b; q, q\right) \\
=& \frac{-\lambda(1-q)(q, aq/c, aq/d, aq/e, \lambda q/b, b\lambda q/a; q)_\infty}{(b, \lambda q, a/b, c\lambda/a, d\lambda/a, e\lambda/a; q)_\infty} \\
&\times \sum_{m=0}^{\infty} \frac{(\lambda;q)_m(1-\lambda q^{2m})(b, c\lambda/a, d\lambda/a, e\lambda/a, a/b; q)_m}{(q;q)_m(1-\lambda)(\lambda q/b, aq/c, aq/d, aq/e, b\lambda q/a; q)_m} q^m \\
&\times {}_8W_7\left(b^2/a; bq^m, bq^{-m}/\lambda, bf/a, bg/a, bh/a; q, q\right).
\end{aligned} \tag{2.12.6}$$

The last ${}_8\phi_7$ series in (2.12.6) is balanced and nonterminating, so we may use (2.11.7) to get

$$\begin{aligned}
&{}_8W_7\left(b^2/a; bq^m, bq^{-m}/\lambda, bf/a, bg/a, bh/a; q, q\right) \\
&= \frac{(b^2q/a, \lambda q^{m+1}/f, \lambda q^{m+1}/g, \lambda g^{m+1}/h, aq/fg, aq/fh, aq/gh, aq^m/b; q)_\infty}{(b\lambda q^{m+1}/a, bq/f, bq/g, bq/h, a/\lambda, fq^m, gq^m, hq^m; q)_\infty} \\
&\quad + \frac{aq^m}{b} \frac{(b^2q/a, bq^{-m}/\lambda, bf/a, bg/a, bh/a, aq^{m+1}/b, aq^{m+1}/f; q)_\infty}{(bq^{1-m}/a, b\lambda q^{m+1}/a, bq/f, bq/g, bq/h, a/\lambda, fq^m, gq^m; q)_\infty} \\
&\quad \times \frac{(aq^{m+1}/g, aq^{m+1}/h, \lambda q^{2m+1}; q)_\infty}{(hq^m, aq^{2m+1}; q)_\infty} \\
&\quad \times {}_8W_7\left(aq^{2m}; bq^m, fq^m, gq^m, hq^m, a/\lambda; q, q\right).
\end{aligned} \tag{2.12.7}$$

Use of this breaks up the double series in (2.12.6) into two parts:

$$\frac{-\lambda(1-q)(q, b^2q/a, \lambda q/b, \lambda q/f, \lambda q/g, \lambda q/h, bf/\lambda, bg/\lambda; q)_\infty}{(b, f, g, h, bq/f, bq/g, bq/h, \lambda q; q)_\infty}$$

$$\times \frac{(bh/\lambda, aq/c, aq/d, aq/e; q)_\infty}{(a/\lambda, c\lambda/a, d\lambda/a, e\lambda/a; q)_\infty}$$
$$\times\ {}_{10}W_9\left(\lambda; b, c\lambda/a, d\lambda/a, e\lambda/a, f, g, h; q, q\right)$$
$$- \frac{a\lambda(1-q)(q, b/\lambda, \lambda q/b, b^2q/a, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g; q)_\infty}{(b, a/b, aq, c\lambda/a, d\lambda/a, e\lambda/a, bq/a, bq/f, bq/g, bq/h; q)_\infty}$$
$$\times \frac{(aq/h, bf/a, bg/a, bh/a; q)_\infty}{(a/\lambda, f, g, h; q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(a;q)_n(1-aq^{2n})(b, f, g, h, a/\lambda; q)_n}{(q;q)_n(1-a)(aq/b, aq/f, aq/g, aq/h, \lambda q; q)_n} q^n$$
$$\times\ {}_8W_7\left(\lambda; c\lambda/a, d\lambda/a, e\lambda/a, aq^n, q^{-n}; q, q\right). \tag{2.12.8}$$

Summing the last ${}_8\phi_7$ series by (2.6.2) we find that the sum over n in (2.12.8) equals ${}_{10}W_9\left(a; b, c, d, e, f, g, h; q, q\right)$ which is, of course, balanced by virtue of (2.12.4). Equating the expression in (2.12.2) with the sum of those in (2.12.5) and (2.12.8), and simplifying the coefficients, we finally obtain Bailey's [1947b] four-term transformation formula

$$
\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f, g, h \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h \end{matrix}; q, q\right] \\
&+ \frac{(aq, b/a, c, d, e, f, g, h, bq/c, bq/d; q)_\infty}{(b^2q/a, a/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, bc/a, bd/a; q)_\infty} \\
&\times \frac{(bq/e, bq/f, bq/g, bq/h; q)_\infty}{(be/a, bf/a, bg/a, bh/a; q)_\infty} \\
&\times\ {}_{10}\phi_9\left[\begin{matrix} b^2/a, qba^{-\frac12}, -qba^{-\frac12}, b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a \\ ba^{-\frac12}, -ba^{-\frac12}, bq/a, bq/c, bq/d, bq/e, bq/f, bq/g, bq/h \end{matrix}; q, q\right] \\
&= \frac{(aq, b/a, \lambda q/f, \lambda q/g, \lambda q/h, bf/\lambda, bg/\lambda, bh/\lambda; q)_\infty}{(\lambda q, b/\lambda, aq/f, aq/g, aq/h, bf/a, bg/a, bh/a; q)_\infty} \\
&\times\ {}_{10}\phi_9\left[\begin{matrix} \lambda, q\lambda^{\frac12}, -q\lambda^{\frac12}, b, \lambda c/a, \lambda d/a, \lambda e/a, f, g, h \\ \lambda^{\frac12}, -\lambda^{\frac12}, \lambda q/b, aq/c, aq/d, aq/e, \lambda q/f, \lambda q/g, \lambda q/h \end{matrix}; q, q\right] \\
&+ \frac{(aq, b/a, f, g, h, bq/f, bq/g, bq/h, \lambda c/a, \lambda d/a; q)_\infty}{(b^2q/\lambda, \lambda/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, bc/a, bd/a; q)_\infty} \\
&\times \frac{(\lambda e/a, abq/\lambda c, abq/\lambda d, abq/\lambda e; q)_\infty}{(be/a, bf/a, bg/a, bh/a; q)_\infty} \\
&\times\ {}_{10}\phi_9\left[\begin{matrix} b^2/\lambda, qb\lambda^{-\frac12}, -qb\lambda^{-\frac12}, b, bc/a, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda \\ b\lambda^{-\frac12}, -b\lambda^{-\frac12}, bq/\lambda, abq/c\lambda, abq/d\lambda, abq/e\lambda, bq/f, bq/g, bq/h \end{matrix}; q, q\right].
\end{aligned}
\tag{2.12.9}
$$

In terms of the q-integrals this can be written in a more compact form:

$$\int_a^b \frac{(qt/a, qt/b, ta^{-\frac12}, -ta^{-\frac12}, qt/c, qt/d, qt/e, qt/f, qt/g, qt/h; q)_\infty}{(t, bt/a, qta^{-\frac12}, -qta^{-\frac12}, ct/a, dt/a, et/a, ft/a, gt/a, ht/a; q)_\infty} d_qt$$
$$= \frac{a}{\lambda} \frac{(b/a, aq/b, \lambda c/a, \lambda d/a, \lambda e/a, bf/\lambda, bg/\lambda, bh/\lambda; q)_\infty}{(b/\lambda, \lambda q/b, c, d, e, bf/a, bg/a, bh/a; q)_\infty}$$

$$\times \int_\lambda^b \frac{(qt/\lambda, qt/b, t\lambda^{-\frac{1}{2}}, -t\lambda^{-\frac{1}{2}}, aqt/c\lambda, aqt/d\lambda, aqt/e\lambda, qt/f, qt/g, qt/h; q)_\infty}{(t, bt/\lambda, qt\lambda^{-\frac{1}{2}}, -qt\lambda^{-\frac{1}{2}}, ct/a, dt/a, et/a, ft/\lambda, gt/\lambda, ht/\lambda; q)_\infty}\, d_q t, \tag{2.12.10}$$

where $\lambda = qa^2/cde$ and $a^3q^2 = bcdefgh$.

Exercises

2.1 Show that

$$ {}_3\phi_2 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}} \end{matrix} ; q, t \right] = \left(1 - aqt^2\right) \frac{\left(atq^2; q\right)_\infty}{(t;q)_\infty}, \quad |t| < 1.$$

2.2 Show that, for $\max(|t|, |aq|) < 1$,

$$ {}_4\phi_3 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b \end{matrix} ; q, t \right] = \frac{(aq, bt; q)_\infty}{(t, aq/b; q)_\infty}\ {}_2\phi_1\left(b^{-1}, t; bqt; q, aq\right).$$

(See Gasper and Rahman [1983a].

2.3 Give an alternate proof of the ${}_6\phi_5$ summation formula (2.4.2) by first using (2.2.4) to derive a terminating form of the q-Dixon formula (2.7.2) and then using it along with the q-Saalschütz formula (1.7.2).

2.4 Prove Sears' identity (2.10.4) by using (1.4.3) and the coefficients in the power series expansion of the product

$${}_2\phi_1(a, b; c; q, z)\ {}_2\phi_1(d, e; abde/c; q, abz/c).$$

2.5 Prove that

$$\sum_{k=0}^{n} \frac{(a;q)_k(1-aq^{2k})(b, c, a/bc; q)_k}{(q;q)_k(1-a)(aq/b, aq/c, bcq; q)_k}\, q^k = \frac{(aq, bq, cq, aq/bc; q)_n}{(q, aq/b, aq/c, bcq; q)_n}$$

for $n = 0, 1, \ldots$.

2.6 Show that

$$ {}_4\phi_3 \left[\begin{matrix} q^{-n}, b, c, -q^{1-n}/bc \\ q^{1-n}/b, q^{1-n}/c, -bc \end{matrix} ; q, q \right] = \begin{cases} 0, & n = 2m+1, \\ \dfrac{(q, b^2, c^2; q^2)_m\ (bc;q)_{2m}}{(b, c; q)_{2m}\ (b^2c^2; q^2)_m}, & n = 2m, \end{cases}$$

where $n, m = 0, 1, 2, \ldots$. (Bailey [1941], Carlitz [1969a])

2.7 Derive Jackson's terminating q-analogue of Dixon's sum:

$$ {}_3\phi_2 \left[\begin{matrix} q^{-2n}, & b, & c \\ & q^{1-2n}/b, & q^{1-2n}/c \end{matrix} ; q, \frac{q^{2-n}}{bc} \right] = \frac{(b, c; q)_n (q, bc; q)_{2n}}{(q, bc; q)_n (b, c; q)_{2n}},$$

where $n = 0, 1, 2, \ldots$. (See Jackson [1921, 1941], Bailey [1941], and Carlitz [1969a])

2.8 Show that

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b^{\frac{1}{2}}, -b^{\frac{1}{2}} \\ q^{\frac{1-n}{2}}a^{\frac{1}{2}}, -q^{\frac{1-n}{2}}a^{\frac{1}{2}}, b \end{matrix}; q, q\right]
= \left\{\begin{matrix} 0, & \text{n = 2m+1}, \\ \dfrac{(q, bq/a; q^2)_m}{(bq, q/a; q^2)_m}, & \text{n = 2m}, \end{matrix}\right\}
$$

where $n, m = 0, 1, \ldots$.
(Andrews [1976a])

2.9 Prove that

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, b, -b, aq/c^2 \\ aq/c, -aq/c, b^2 \end{matrix}; q, q\right] \\
&\quad + \frac{(q/b^2, a, b, -b, aq/c^2, aq^2/b^2c, -aq^2/b^2c; q)_\infty}{(b^2/q, q/b, -q/b, aq/b^2, aq/c, -aq/c, aq^2/b^2c^2; q)_\infty} \\
&\quad \times {}_4\phi_3\left[\begin{matrix} q/b, -q/b, aq/b^2, aq^2/b^2c^2 \\ aq^2/b^2c, -aq^2/b^2c, q^2/b^2 \end{matrix}; q, q\right] \\
&= \frac{(q/b^2, -q; q)_\infty (a^2q^2, aq^2/b^2, aq^2/c^2, a^2q^2/b^2c^2; q^2)_\infty}{(aq/b^2, -aq; q)_\infty (q^2/b^2, a^2q^2/c^2, aq^2, aq^2/b^2c^2; q^2)_\infty}.
\end{aligned}
$$

2.10 The *q-Racah polynomials*, which were introduced by Askey and Wilson [1979], are defined by

$$
W_n(x; a, b, c, N; q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N} \\ aq, bcq, q^{-N} \end{matrix}; q, q\right],
$$

where $n = 0, 1, 2, \ldots, N$. Show that

$$
W_n(x; a, b, c, N; q) = \frac{(aq/c, bq; q)_n}{(aq, bcq; q)_n} c^n \, W_n(N - x; b, a, c^{-1}, N; q).
$$

2.11 The *Askey-Wilson polynomials* are defined in Askey and Wilson [1985] by

$$
\begin{aligned}
&p_n(x; a, b, c, d \mid q) \\
&= a^{-n}(ab, ac, ad; q)_n \, {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right],
\end{aligned}
$$

where $x = \cos\theta$. Show that
(i) $p_n(x; a, b, c, d \mid q) = p_n(x; b, a, c, d \mid q)$,
(ii) $p_n(-x; a, b, c, d \mid q) = (-1)^n p_n(x; -a, -b, -c, -d \mid q)$.

2.12 Show that

$$
\begin{aligned}
&{}_{10}W_9(a; b^{\frac{1}{2}}, -b^{\frac{1}{2}}, (bq)^{\frac{1}{2}}, -(bq)^{\frac{1}{2}}, a/b, a^2q^{n+1}/b, q^{-n}; q, q) \\
&= \frac{(aq, a^2q/b^2; q)_n}{(aq/b, a^2q/b; q)_n}, \quad n = 0, 1, 2, \ldots .
\end{aligned}
$$

2.13 If $\lambda = qa^2/bcd$ and $|q\lambda/a| < 1$, prove that

$$\text{(i)}\quad {}_4\phi_3\left[\begin{matrix} a, b, c, d \\ aq/b, aq/c, aq/d \end{matrix}; q, \frac{q\lambda^2}{a^2}\right] = \frac{(\lambda q/a, \lambda^2 q/a; q)_\infty}{(\lambda q, \lambda^2 q/a^2; q)_\infty}$$
$$\times\, {}_{10}W_9\left(\lambda; a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a; q, q\lambda/a\right),$$

$$\text{(ii)}\quad {}_4\phi_3\left[\begin{matrix} a, b, c, d \\ aq/b, aq/c, aq/d \end{matrix}; q, \frac{-\lambda q}{a}\right]$$
$$= \frac{(aq, -q, \lambda q a^{-\frac{1}{2}}, -\lambda q a^{-\frac{1}{2}}; q)_\infty}{(\lambda q, -\lambda q/a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_\infty}$$
$$\times\, {}_8W_7\left(\lambda; a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a; q, -q\right).$$

2.14 (i) Show that

$${}_4\phi_3\left[\begin{matrix} a, qa^{\frac{1}{2}}, b, q^{-n} \\ a^{\frac{1}{2}}, aq/b, b^2q^{1-n} \end{matrix}; q, q\right] = \frac{(ab^{-2}, b^{-1}, -qb^{-1}a^{\frac{1}{2}}; q)_n}{(b^{-2}, aq/b, -b^{-1}a^{\frac{1}{2}}; q)_n},$$

which is a q-analogue of Bailey [1935, 4.5(1.3)].

(ii) Using (i) in the formula (2.8.2) prove the following q-analogue of Bailey [1935, 4.5(4)]:

$${}_6\phi_5\left[\begin{matrix} a, qa^{\frac{1}{2}}, b, c, d, q^{-n} \\ a^{\frac{1}{2}}, aq/b, aq/c, aq/d, a^2q^{1-n}/\lambda^2 \end{matrix}; q, q\right]$$
$$= \frac{(\lambda/a, \lambda^2/a, -\lambda q a^{-\frac{1}{2}}; q)_n}{(\lambda q, \lambda^2/a^2, -\lambda a^{-\frac{1}{2}}; q)_n}$$
$$\times\, {}_{12}\phi_{11}\left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \lambda b/a, \lambda c/a, \lambda d/a, \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, aq/b, aq/c, aq/d, \end{matrix}\right.$$
$$\left.\begin{matrix} qa^{\frac{1}{2}}, a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, \lambda^2 q^n/a, q^{-n} \\ \lambda a^{-\frac{1}{2}}, -\lambda q a^{-\frac{1}{2}}, \lambda(q/a)^{\frac{1}{2}}, -\lambda(q/a)^{\frac{1}{2}}, aq^{1-n}/\lambda, \lambda q^{n+1} \end{matrix}; q, q\right],$$

where $\lambda = qa^2/bcd$.

This formula is equivalent to Jain's [1982, (4.6)] transformation formula.

2.15 By taking suitable q-integrals of the function

$$f(t) = \frac{(qt/b, qt/c, aqt/bc, tq^2/bcdef; q)_\infty}{(at, qt/bcd, qt/bce, qt/bcf; q)_\infty},$$

prove Bailey's [1936, (4.6)] identity

$$
\begin{aligned}
& a^{-1}\frac{(aq/d, aq/e, aq/f, q/ad, q/ae, q/af;q)_\infty}{(qa^2, ab, ac, b/a, c/a;q)_\infty} \\
&\quad \times\ {}_8W_7\left(a^2;ab,ac,ad,ae,af;q,q^2/abcdef\right) \\
&\quad + b^{-1}\frac{(bq/d, bq/e, bq/f, q/bd, q/be, q/bf;q)_\infty}{(qb^2, ba, bc, a/b, c/b;q)_\infty} \\
&\quad \times\ {}_8W_7\left(b^2;ba,bc,bd,be,bf;q,q^2/abcdef\right) \\
&\quad + c^{-1}\frac{(cq/d, cq/e, cq/f, q/cd, q/ce, q/cf;q)_\infty}{(qc^2, ca, cb, a/c, b/c;q)_\infty} \\
&\quad \times\ {}_8W_7\left(c^2;ca,cb,cd,ce,cf;q,q^2/abcdef\right) \\
& = 0,
\end{aligned}
$$

provided $|q^2/abcdef| < 1$.

2.16 (i) Let $S(\lambda,\mu,\nu,\rho) = (\lambda, q/\lambda, \mu, q/\mu, \nu, q/\nu, \rho, q/\rho; q)_\infty$. Using Ex. 2.15, prove that

$$
\begin{aligned}
& S(x\lambda, x/\lambda, \mu\nu,\ \mu/\nu) - S(x\nu, x/\nu, \lambda\mu, \mu/\lambda) \\
& = \frac{\mu}{\lambda} S(x\mu, x/\mu, \lambda\nu, \lambda/\nu),
\end{aligned}
$$

where x, λ, μ, ν are non-zero complex numbers. (Sears [1951c,d], Bailey [1936])

(ii) Deduce that

$$
\begin{aligned}
& (q^{\frac12}, q^{\frac12}; q)_\infty\ S(\lambda q^{\frac14}, \lambda q^{\frac34}, -\mu q^{\frac14}, -\mu q^{\frac34}) \\
& = (q^{\frac12}\lambda\mu, q^{\frac12}/\lambda\mu, q^{\frac12}\lambda/\mu, q^{\frac12}\mu/\lambda; q)_\infty \\
& \quad - q^{\frac14}(\lambda+\lambda^{-1}-\mu-\mu^{-1})(q\lambda\mu, q/\lambda\mu, q\lambda/\mu, q\mu/\lambda; q)_\infty.
\end{aligned}
$$

(Ismail and Rahman [2002b])

2.17 Show that

(i)
$$
\begin{aligned}
& {}_8\phi_7\left[\begin{matrix}\lambda, q\lambda^{\frac12}, -q\lambda^{\frac12}, a, b, c, -c, \lambda q/c^2 \\ \lambda^{\frac12}, -\lambda^{\frac12}, \lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, c^2\end{matrix}; q, -\frac{\lambda q}{ab}\right] \\
& = \frac{(\lambda q, c^2/\lambda; q)_\infty (aq, bq, c^2q/a, c^2q/b; q^2)_\infty}{(\lambda q/a, \lambda q/b; q)_\infty (q, abq, c^2q, c^2q/ab; q^2)_\infty},
\end{aligned}
$$

where $\lambda = -c(ab/q)^{\frac12}$ and $|\lambda q/ab| < 1$, and

(ii)
$$
\begin{aligned}
& {}_8\phi_7\left[\begin{matrix}-c, q(-c)^{\frac12}, -q(-c)^{\frac12}, a, q/a, c, -d, -q/d \\ (-c)^{\frac12}, -(-c)^{\frac12}, -cq/a, -ac, -q, cq/d, cd\end{matrix}; q, c\right] \\
& = \frac{(-c, -cq; q)_\infty (acd, acq/d, cdq/a, cq^2/ad; q^2)_\infty}{(cd, cq/d, -ac, -cq/a; q)_\infty}, \quad |c| < 1.
\end{aligned}
$$

Verify that (i) is a q-analogue of Watson's summation formula (Bailey [1935, 3.3(1)]) while (ii) is a q-analogue of Whipple's formula (Bailey [1935, 3.4(1)]). (See Jain and Verma [1985] and Gasper and Rahman [1986]).

2.18 In the ${}_6\phi_5$ summation formula (2.7.1) let $b, c, d \to \infty$. Then set $a = 1$ to prove Euler's [1748] identity

$$1 + \sum_{n=1}^{\infty} (-1)^n \left\{ q^{n(3n-1)/2} + q^{n(3n+1)/2} \right\} = (q;q)_\infty .$$

2.19 Show that

$$\begin{aligned} &{}_{10}W_9\left(a; b, c, d, e, f, g, q^{-n}; q, q\right) \\ &= \frac{(aq, aq/ce, aq/de, aq/ef, aq/eg, b; q)_n}{(aq/c, aq/d, aq/e, aq/f, aq/g, b/e; q)_n}\, e^n \\ &\quad \times {}_{10}W_9\left(eq^{-n}/b; e, aq/bc, aq/bd, aq/bf, aq/bg, eq^{-n}/a, q^{-n}; q, q\right), \end{aligned}$$

where $a^3 q^{n+2} = bcdefg$ and $n = 0, 1, 2, \ldots$.

2.20 Prove that

$$\begin{aligned} &{}_{10}W_9\left(a; b, c, d, e, f, g, q^{-n}; q, a^3 q^{n+3}/bcdefg\right) \\ &= \frac{(aq, aq/fg; q)_n}{(aq/f, aq/g; q)_n} \sum_{j=0}^{n} \frac{(q^{-n}, f, g, aq/de; q)_j q^j}{(q, aq/d, aq/e, fgq^{-n}/a; q)_j} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{-j}, d, e, aq/bc \\ aq/b, aq/c, deq^{-j}/a \end{matrix}; q, q\right], \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

2.21 Show that

$$\begin{aligned} &{}_{10}W_9\left(a; b, c, d, e, f, g, h; q, a^3 q^3/bcdefgh\right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda; q)_n (1-\lambda q^{2n})(\lambda b/a, \lambda c/a, \lambda d/a, e, f, g, h; q)_n (aq; q)_{2n}}{(q;q)_n (1-\lambda)(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_n (\lambda q; q)_{2n}} \\ &\quad \times \left(\frac{a^2 q^2}{efgh}\right)^n {}_8W_7\left(aq^{2n}; a/\lambda, eq^n, fq^n, gq^n, hq^n; q, a^3 q^3/bcdefgh\right), \end{aligned}$$

where $\lambda = qa^2/bcd$ and $|a^3 q^3/bcdefgh| < 1$.

2.22 Prove that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a;q)_n (1-aq^{2n})(b, c, d, e; q)_n}{(q;q)_n (1-a)(aq/b, aq/c, aq/d, aq/e; q)_n} \left(-\frac{a^2 q^2}{bcde}\right)^n q^{\binom{n}{2}} \\ &= \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} aq/bc, d, e \\ aq/b, aq/c \end{matrix}; q, \frac{aq}{de}\right], \quad |aq/de| < 1. \end{aligned}$$

Deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a;q)_n (1-aq^{2n})(d, e; q)_n}{(q;q)_n (1-a)(aq/d, aq/e; q)_n} \left(-\frac{aq}{de}\right)^n q^{\binom{n}{2}} \\ &= \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty}. \end{aligned}$$

2.23 Prove that

$$\sum_{j=0}^{n} \frac{(ab, ac, ad; q)_j}{(abcd, aqz, aq/z; q)_j}\, q^j$$

$$= \frac{(1-z/a)(1-abcz)}{(1-bz)(1-cz)}\, {}_8W_7\,(abcz; ab, ac, bc, qz/d, q; q, dz)$$

$$- \frac{(ab, ac, ad; q)_{n+1}}{(abcd, aqz, aq/z; q)_{n+1}} \frac{(1-aq^{n+1}/z)(1-abczq^{n+1})}{(1-a/z)(1-abcz)}$$

$$\times\, {}_8W_7\left(abczq^{n+1}; abq^{n+1}, acq^{n+1}, bc, qz/d, q; q, dz\right).$$

2.24 Show that

$$ {}_5\phi_4 \left[\begin{matrix} a, b, c, d, e \\ aq/b, aq/c, aq/d, f \end{matrix}; q, q\right] = \frac{(\lambda q/a, \lambda q/e, q\lambda^2/a, q/f; q)_\infty}{(\lambda q, aq/f, eq/f, aq/\lambda f; q)_\infty}$$

$$\times\ {}_{12}W_{11}\left(\lambda; a^{\frac12}, -a^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}, \lambda b/a, \lambda c/a, \lambda d/a, e, aq/f; q, q\right)$$

$$- \frac{(a, e, a/\lambda, q/f, \lambda q^2/f; q)_\infty}{(f/q, \lambda q, aq/f, aq/\lambda f, eq/f; q)_\infty}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\lambda; q)_j (1-\lambda q^{2j})(\lambda b/a, \lambda c/a, \lambda d/a; q)_j}{(q; q)_j (1-\lambda)(aq/b, aq/c, aq/d; q)_j}$$

$$\times\ \frac{(f/q, aq/f; q)_j\ (aq^{j+1}/f, aq^{1-j}/\lambda f, eq/f; q)_k}{(\lambda q^2/f, \lambda f/a; q)_j\ (q, q^{2-j}/f, \lambda q^{2+j}/f; q)_k}\, q^{j+k},$$

where $\lambda = qa^2/bcd$ and $f = ea^2/\lambda^2$. Note that this reduces to (2.8.3) when $e = q^{-n},\ n = 0, 1, 2, \ldots$.

2.25 By interchanging the order of summation in the double sum in Ex. 2.24 and using Bailey's summation formula (2.11.7), prove Jain and Verma's [1982, (7.1)] transformation formula

$$ {}_5\phi_4 \left[\begin{matrix} a, b, c, d, e \\ aq/b, aq/c, aq/d, f \end{matrix}; q, q\right]$$

$$+ \frac{(a, b, c, d, e, q/f, aq^2/bf, aq^2/cf, aq^2/df; q)_\infty}{(aq/b, aq/c, aq/d, f/q, aq/f, bq/f, cq/f, dq/f, eq/f; q)_\infty}$$

$$\times\ {}_5\phi_4 \left[\begin{matrix} eq/f, aq/f, bq/f, cq/f, dq/f \\ q^2/f, aq^2/bf, aq^2/cf, aq^2/df \end{matrix}; q, q\right]$$

$$= \frac{(\lambda q/a, \lambda q/e, q\lambda^2/a, q/f; q)_\infty}{(\lambda q, aq/f, eq/f, aq/\lambda f; q)_\infty}$$

$$\times\ {}_{12}W_{11}\left(\lambda; a^{\frac12}, -a^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}, \lambda b/a, \lambda c/a, \lambda d/a, e, aq/f; q, q\right)$$

$$+ \frac{(a, e, \lambda b/a, \lambda c/a, \lambda d/a, q/f, a^2q^2/\lambda bf, a^2q^2/\lambda cf, a^2q^2/\lambda df, aq^3/f^2; q)_\infty}{(aq/b, aq/c, aq/d, aq/f, bq/f, cq/f, dq/f, eq/f, \lambda f/aq, a^2q^3/\lambda f^2; q)_\infty}$$

$$\times\ {}_{12}W_{11}\big(a^2q^2/\lambda f^2; qa^{\frac32}/\lambda f, -qa^{\frac32}/\lambda f, (qa)^{\frac32}/\lambda f, -(qa)^{\frac32}/\lambda f,$$

$$\lambda q/a, aq/f, bq/f, cq/f, dq/f; q, q\big),$$

where the parameters are related in the same way as in Ex. 2.24. Note that this is a nonterminating extension of (2.8.3) and that the first ${}_5\phi_4$ series on the left is a nearly-poised series of the second kind while the second ${}_5\phi_4$ series is a nearly-poised series of the first kind.

2.26 If $a = q^{-n}, n = 0, 1, 2, \ldots$, prove that

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix}; q, \frac{aqx}{bc}\right] = \frac{(ax;q)_\infty}{(x;q)_\infty}
\times\ {}_5\phi_4\left[\begin{matrix} a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, aq/bc \\ aq/b, aq/c, ax, q/x \end{matrix}; q, q\right].
$$

(Sears [1951a, (4.1)], Carlitz [1969a, (2.4)])

2.27 Show that

$$
{}_{r+3}\phi_{r+2}\left[\begin{matrix} q^{-n}, c, ab/c, a_1, \ldots, a_r \\ a, b, b_1, \ldots, b_r \end{matrix}; q, z\right] = \frac{(c, ab/c;q)_n}{(a,b;q)_n}
\times \sum_{k=0}^{n} \frac{(q^{-n}, c/a, c/b;q)_k}{(q, c, ca^{-1}b^{-1}q^{1-n};q)_k}\, q^k\ {}_{r+2}\phi_{r+1}\left[\begin{matrix} q^{k-n}, c, a_1, \ldots, a_r \\ cq^k, b_1, \ldots, b_r \end{matrix}; q, z\right].
$$

2.28 Show that

$$
{}_{r+3}\phi_{r+2}\left[\begin{matrix} a, b_1, \ldots, b_r, c, q^{-n} \\ aq/b_1, \ldots, aq/b_r, aq/c, aq^{n+1} \end{matrix}; q, qz\right] = \frac{(aq/cz, q^{1-n}/c;q)_n}{(aq/c, q^{1-n}/cz;q)_n}
\times \sum_{k=0}^{n} \frac{(q^{-n}/cz, 1/z, q^{-n}/a, q^{-n};q)_k}{(q, aq/cz, q/cz, q^{1-n}/c;q)_k} \frac{(1-q^{2k-n}/cz)}{(1-q^{-n}/cz)} \left(\frac{aq^{n+1}}{c}\right)^k
\times {}_{r+3}\phi_{r+2}\left[\begin{matrix} a, b_1, \ldots, b_r, czq^{-k}, q^{k-n} \\ aq/b_1, \ldots, aq/b_r, aq^{k+1}/cz, aq^{n+1-k} \end{matrix}; q, q\right].
$$

2.29 Show that

$$
{}_{r+2}\phi_{r+1}\left[\begin{matrix} q^{-n}, cdq^{n+1}, a_1, \ldots, a_r \\ cq, b_1, \ldots, b_r \end{matrix}; q, z\right]
= \sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/2}(aq, cdq^{n+1}, q^{-n};q)_k}{(q, cq, abq^{k+1};q)_k}
$$

$$
\times\ {}_3\phi_2\left[\begin{matrix} q^{k-n}, cdq^{n+k+1}, aq^{k+1} \\ cq^{k+1}, abq^{2k+2} \end{matrix}; q, q\right] {}_{r+2}\phi_{r+1}\left[\begin{matrix} q^{-k}, abq^{k+1}, a_1, \ldots, a_r \\ aq, b_1, \ldots, b_r \end{matrix}; q, z\right].
$$

2.30 Iterate (2.12.9) to prove Bailey's [1947b, (8.1)] transformation formula:

$$
{}_{10}W_9(a; b, c, d, e, f, g, h; q, q)
+ \frac{(aq, c, d, e, f, g, h, b/a, bq/c, bq/d, bq/e;q)_\infty}{(qb^2/a, bc/a, bd/a, be/a, bf/a, bg/a, bh/a, a/b, aq/c, aq/d, aq/e;q)_\infty}
\times \frac{(bq/f, bq/g, bq/h;q)_\infty}{(aq/f, aq/g, ag/h;q)_\infty}\ {}_{10}W_9(b^2/a; b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q, q)
$$

$$= \frac{(aq, b/a, g, bq/g, aq/ch, aq/dh, aq/eh, aq/fh, bch/a, bdh/a; q)_\infty}{(bhq/g, bh/a, g/h, aq/h, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a; q)_\infty}$$
$$\times \frac{(beh/a, bfh/a; q)_\infty}{(be/a, bf/a; q)_\infty} {}_{10}W_9(bh/g; b, aq/cg, aq/dg, aq/eg, aq/fg, bh/a, h; q, q)$$
$$+ \frac{(aq, b/a, h, bq/h, aq/cg, aq/dg, aq/eg, aq/fg, bcg/a, bdg/a; q)_\infty}{(bgq/h, bg/a, h/g, aq/g, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a; q)_\infty}$$
$$\times \frac{(beg/a, bfg/a; q)_\infty}{(be/a, bf/a; q)_\infty} {}_{10}W_9(bg/h; b, aq/ch, aq/dh, aq/eh, aq/fh, bg/a, g; q, q),$$

where $a^3q^2 = bcdefgh$.

2.31 By using the q-Dixon formula (2.7.2) prove that the constant term in the Laurent expansion of

$$(x_1/x_2, x_1/x_3; q)_{a_1} \ (x_2/x_3, qx_2/x_1; q)_{a_2} \ (qx_3/x_1, qx_3/x_2; q)_{a_3}$$

is

$$(q;q)_{a_1+a_2+a_3}/(q;q)_{a_1}(q;q)_{a_2}(q;q)_{a_3}.$$

where a_1, a_2 and a_3 are nonnegative integers.
(Andrews [1975a])

2.32 Use (2.10.18), the q-binomial theorem, and the generating function in Ex. 1.29 to derive the formula

$$C_n(\cos\theta; \beta|q) = \frac{2i\sin\theta}{1-q} \frac{(\beta, \beta, \beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}{(q, \beta^2, e^{2i\theta}, e^{-2i\theta}; q)_\infty}$$
$$\times \frac{(\beta^2;q)_n}{(q;q)_n} \int_{e^{i\theta}}^{e^{-i\theta}} \frac{(qte^{i\theta}, qte^{-i\theta}; q)_\infty}{(\beta te^{i\theta}, \beta te^{-i\theta}; q)_\infty} t^n \, d_q t, \quad 0 < \theta < \pi.$$

(Rahman and Verma [1986a])

2.33 (i) Prove that

$${}_{6+2k}W_{5+2k}(a; b, a/b, d, e_1, \ldots, e_k, aq^{n_1+1}/e_1, \ldots, aq^{n_k+1}/e_k; q, q^{1-N}/d)$$
$$= \frac{(q, aq, aq/bd, bq/d; q)_\infty}{(bq, aq/b, aq/d, q/d; q)_\infty} \prod_{j=1}^{k} \frac{(aq/be_j, bq/e_j; q)_{n_j}}{(aq/e_j, q/e_j; q)_{n_j}}, \quad k = 1, 2, \ldots,$$

where $n_1, \ldots, n_k$ are nonnegative integers, $N = n_1 + \cdots + n_k$, and $|q^{1-N}/d| < 1$ when the series does not terminate. (Gasper [1997])

(ii) Deduce that

$$\frac{(be^{i\theta}, be^{-i\theta}; p)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}; q)_\infty}$$
$$= \sum_{k=0}^{\infty} \frac{1-a^2q^{2k}}{1-a^2} \frac{(a^2q)_k}{(q;q)_k} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q, qa^2; q)_\infty} \frac{(ae^{i\theta}, ae^{-i\theta}; q)_k (abq^k, bq^{-k}/a; p)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}; q)_k}$$

when $0 < p < q$, or $p = q$ and $|b| < |a|$.
(Ismail and Stanton [2003a]).

2.34 Derive the formulas

(i)

$$ {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)} \int_0^1 t^{\lambda-1} \frac{(tq, tzq^{\alpha+\beta-\mu}; q)_\infty}{(tq^{\gamma-\lambda}, tz; q)_\infty} \times {}_2\phi_1(q^{\mu-\alpha}, q^{\mu-\beta}; q^\lambda; q, tz)\, {}_3\phi_2(q^{\alpha+\beta-\mu}, q^{\mu-\lambda}, t^{-1}; q^{\gamma-\lambda}, q/tz; q, q) d_q t, $$

(ii)

$$ {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)} \int_0^1 t^{\lambda-1} \frac{(tq, tzq^{\sigma}; q)_\infty}{(tq^{\gamma-\lambda}, tz; q)_\infty} \times {}_2\phi_1(q^{\alpha-\sigma}, q^{\beta}; q^\lambda; q, tzq^\sigma)\, {}_3\phi_2(q^{\sigma}, q^{\beta-\lambda}, t^{-1}; q^{\gamma-\lambda}, q/tz; q, q) d_q t, $$

(iii)

$$ {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) = \frac{\Gamma_q(\gamma)\Gamma_q(\mu)}{\Gamma_q(\lambda)\Gamma_q(\nu)\Gamma_q(\gamma+\mu-\lambda-\nu)} \int_0^1 t^{\lambda-1} \frac{(tq; q)_\infty}{(tq^{\gamma+\mu-\lambda-\nu}; q)_\infty} \times {}_3\phi_1(q^{\gamma-\nu}, q^{\mu-\nu}, t^{-1}; q^{\gamma+\mu-\lambda-\nu}; q, tq^{\nu-\lambda})\, {}_3\phi_2(q^{\alpha}, q^{\beta}, q^{\mu}; q^\lambda, q^\nu; q, tz) d_q t, $$

where Re $\gamma >$ Re $\lambda > 0$ in (i) and (ii), and Re $(\lambda, \nu, \gamma+\mu-\lambda-\nu) > 0$ in (iii). These formulas are q-integral analogues of Erdélyi's [1939, equations (17), (11), and (20), respectively] fractional integral representations for ${}_2F_1$ series. (Gasper [2000])

2.35 Derive the following *discrete extensions* of the formulas in Ex. 2.34:

(i)

$$ {}_3\phi_2(\alpha, \beta, q^{-n}; \gamma, \delta; q, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\lambda, \mu\delta/\alpha\beta; q)_n (\gamma/\lambda; q)_{n-k}}{(\gamma; q)_n (\delta; q)_k} \lambda^n \left(\frac{\alpha\beta}{\lambda\mu}\right)^k \times {}_3\phi_2(\mu/\alpha, \mu/\beta, q^{-k}; \lambda, \delta\mu/\alpha\beta; q, q)\, {}_3\phi_2(\alpha\beta/\mu, \mu/\lambda, q^{k-n}; \gamma/\lambda, \delta q^k; q, q), $$

(ii)

$$ {}_3\phi_2(\alpha, \beta, q^{-n}; \gamma, \delta; q, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\lambda, \delta/\sigma; q)_k (\gamma/\lambda; q)_{n-k}}{(\gamma; q)_n (\delta; q)_k} \lambda^n \left(\frac{\sigma}{\lambda}\right)^k \times {}_3\phi_2(\alpha/\sigma, \beta, q^{-k}; \lambda, \delta/\sigma; q, q)\, {}_3\phi_2(\sigma, \beta/\lambda, q^{k-n}; \gamma/\lambda, \delta q^k; q, q), $$

(iii)

$$ {}_3\phi_2(\alpha, \beta, q^{-n}; \gamma, \delta; q, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\nu; q)_n (\lambda; q)_k (\gamma\mu/\lambda\nu; q)_{n-k}}{(\gamma, \mu; q)_n} \lambda^{n-k} \times {}_3\phi_2(\mu/\gamma, \gamma/\nu, q^{k-n}; \gamma\mu/\lambda\nu, q^{1-n}/\nu; q, q^{1-k}/\lambda)\, {}_4\phi_3(\alpha, \beta, \mu, q^{-k}; \lambda, \nu, \delta; q, q). $$

(Gasper [2000])

2.36 Extend the formulas in Ex. 2.35 to:

(i)

$$ {}_4\phi_3(\alpha, \beta, \nu q^n, q^{-n}; \gamma, \delta, \rho; q, q) $$

$$= \frac{(\gamma/\lambda, \nu q/\gamma; q)_n}{(\gamma, \lambda\nu q/\gamma; q)_n}\lambda^n$$
$$\times \sum_{k=0}^{n} \frac{(\lambda\nu/\gamma, \lambda, \delta\mu/\alpha\beta, \mu\rho/\alpha\beta, \nu q^n, q^{-n}; q)_k (\lambda\nu q/\gamma; q)_{2k}}{(q, \nu q/\gamma, \rho, \delta, \lambda q^{1-n}/\gamma, \lambda\nu q^{n+1}/\gamma; q)_k (\lambda\nu/\gamma; q)_{2k}} \left(\frac{\delta\sigma}{\mu\nu}\right)^k$$
$$\times {}_4\phi_3(\mu/\alpha, \mu/\beta, \lambda\nu q^k/\gamma, q^{-k}; \lambda, \delta\mu/\alpha\beta, \mu\rho/\alpha\beta; q, q)$$
$$\times {}_4\phi_3(\alpha\beta/\mu, \mu/\lambda, \nu q^{k+n}, q^{k-n}; \gamma/\lambda, \delta q^k, \rho q^k; q, q),$$

(ii)

$${}_4\phi_3(\alpha, \beta, \nu q^n, q^{-n}; \gamma, \delta, \rho; q, q)$$
$$= \frac{(\gamma/\lambda, \nu q/\gamma; q)_n}{(\gamma, \lambda\nu q/\gamma; q)_n}\lambda^n$$
$$\times \sum_{k=0}^{n} \frac{(\lambda\nu/\gamma, \lambda, \rho/\alpha, \delta/\sigma, \nu q^n, q^{-n}; q)_k (\lambda\nu q/\gamma; q)_{2k}}{(q, \nu q/\gamma, \delta, \rho\sigma/\alpha, \lambda q^{1-n}/\gamma, \nu\lambda q^{n+1}/\gamma; q)_k (\lambda\nu/\gamma; q)_{2k}} \left(\frac{\sigma q}{\gamma}\right)^k$$
$$\times {}_4\phi_3(\alpha/\sigma, \beta, \lambda\nu q^k/\gamma, q^{-k}; \lambda, \delta/\sigma, \rho; q, q)$$
$$\times {}_4\phi_3(\sigma, \beta/\lambda, \nu q^{n+k}, q^{k-n}, \gamma/\lambda, \delta q^k, \sigma\rho q^k/\alpha; q, q),$$

(iii)

$${}_{r+2}\phi_{s+1}\left[\begin{matrix} q^{-n}, \nu, a_1, a_2, \ldots, a_r \\ \gamma, b_1, b_2, \ldots, b_s \end{matrix}; q, z\right]$$
$$= \frac{(\lambda, \nu; q)_n}{(\gamma, \mu; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, \gamma\mu/\lambda\nu; q)_{n-k}}{(q, q^{1-n}/\lambda; q)_{n-k}} q^{n-k}$$
$$\times {}_3\phi_2\left[\begin{matrix} \mu/\nu, \gamma/\nu, q^{k-n} \\ \gamma\mu/\lambda\nu, q^{1-n}/\nu \end{matrix}; q, q^{1-k}/\lambda\right] {}_{r+2}\phi_{s+1}\left[\begin{matrix} q^{-k}, \mu, a_1, \ldots, a_r \\ \lambda, b_1, b_2, \ldots, b_s \end{matrix}; q, z\right],$$

where $\alpha\beta\nu q = \delta\gamma\rho$ in (i) and (ii). (Gasper [2000])

Notes

§2.5 Some applications of Watson's transformation formula (2.5.1) to mock theta functions are presented in Watson [1936, 1937].

§2.7 For additional proofs of the Rogers–Ramanujan identities, identities of Rogers–Ramanujan type, applications to combinatorics, Lie algebras, statistical mechanics, etc., see Adiga *et al.* [1985], Alladi [1997], Alladi, Andrews and Berkovich [2003], Alladi and Berkovich [2002], Andrews [1970b, 1974b,c, 1976, 1976b, 1979b, 1981a, 1984b,d, 1986, 1987a,b, 1997, 2001], Andrews, Askey, Berndt *et al.* [1988], Andrews and Baxter [1986, 1987], Andrews, Baxter, Bressoud *et al.* [1987], Andrews, Baxter, and Forrester [1984], Andrews, Schilling and Warnaar [1999], Bailey [1947a, 1949, 1951], Baxter [1980–1988], Baxter and Andrews [1986], Baxter and Pearce [1983, 1984], Berkovich and McCoy [1998], Berkovich, McCoy and Schilling [1998], Berkovich and Paule [2001a,b], Berkovich and Warnaar [2003], Berndt [1985–2001], Borwein and Borwein [1988], Bressoud [1980a, 1981a,b, 1983a], Bressoud, Ismail and Stanton [2000], Burge [1993], Dobbie [1962], Dyson [1988], Fine [1988], Foda and Quano [1995], Garrett, Ismail and Stanton [1999], Garsia and Milne [1981], Garvan [1988], Lepowsky [1982], Lepowsky and Wilson [1982], McCoy [1999], Misra [1988], Paule [1985a], Ramanujan [1919], Rogers and Ramanujan [1919], Schilling and Warnaar [2000], Schur [1917], Sills [2003a,b,c], Slater [1952a], Stanton [2001a,b], Warnaar [1999–2002a, 2003a–2003e], and Watson [1931].

§2.9 Agarwal [1953e] showed that Bailey's transformation (2.9.1) gives a transformation formula for truncated ${}_8\phi_7$ series, where the sum of the first N terms of an infinite series is called a truncated series.

§2.10 Many additional transformation formulas for hypergeometric series are derived in Whipple [1926a,b].

§2.11 Additional transformation formulas for ${}_8\phi_7$ series are derived in Agarwal [1953c].

Ex. 2.6 Also see the summation formulas for very-well-poised series in Joshi and Verma [1979].

Ex. 2.31 This exercise is the $n = 3$ case of the Zeilberger and Bressoud [1985] theorem that if $x_1, \ldots, x_n, q$ are commuting indeterminates and $a_1, \ldots, a_n$ are nonnegative integers, then the constant term in the Laurent expansion of

$$\prod_{1\le i<j\le n} (x_i/x_j; q)_{a_i} (qx_j/x_i; q)_{a_j}$$

is equal to the q-multinomial coefficient

$$\frac{(q;q)_{a_1+\cdots+a_n}}{(q;q)_{a_1}\cdots(q;q)_{a_n}}.$$

This was called the Andrews' q-Dyson conjecture because Andrews [1975a] had conjectured it as a q-analogue of a previously proved conjecture of Dyson [1962] that the constant term in the Laurent expansion of

$$\prod_{1\le i\ne j\le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}$$

is equal to the multinomial coefficient

$$\frac{(a_1 + \cdots + a_n)!}{a_1! \cdots a_n!}.$$

The $n = 4$ case of the Andrews' q-Dyson conjecture was proved independently by Kadell [1985b]. Additional constant term results are derived in Baker and Forrester [1998, 1999], Bressoud [1989], Bressoud and Goulden [1985], Cherednik [1995], Cooper [1997a,b], Evans, Ismail and Stanton [1982], Forrester [1990], Kadell [1994, 1997, 2000], Kaneko [1996–2001], Macdonald [1972–1998b], Morris [1982], Opdam [1989], Stanton [1986b, 1989], Stembridge [1988], and Zeilberger [1987–1994].

3

ADDITIONAL SUMMATION, TRANSFORMATION, AND EXPANSION FORMULAS

3.1 Introduction

In this chapter we shall use the summation and transformation formulas of Chapters 1 and 2 to deduce additional transformation formulas for basic hypergeometric series which are useful in many applications. In §3.2 and §3.3 we shall obtain q-analogues of some of Thomae's [1879] ${}_3F_2$ transformation formulas, typical among which are

$$ {}_3F_2\left[\begin{matrix}-n, a, b\\ c, d\end{matrix}; 1\right] = \frac{(d-b)_n}{(d)_n}\, {}_3F_2\left[\begin{matrix}-n, c-a, b\\ c, 1+b-d-n\end{matrix}; 1\right], \tag{3.1.1}$$

$n = 0, 1, 2, \ldots,$

$$ {}_3F_2\left[\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right] = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\, {}_3F_2\left[\begin{matrix}d-a, e-a, s\\ s+b, s+c\end{matrix}; 1\right], \tag{3.1.2}$$

$s = d + e - a - b - c$, and

$$ {}_3F_2\left[\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right] = \frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)\Gamma(1+b-a)\Gamma(c)} \times {}_3F_2\left[\begin{matrix}b, b-d+1, b-e+1\\ 1+b-c, 1+b-a\end{matrix}; 1\right] + \text{ idem } (b; c), \tag{3.1.3}$$

where the symbol "idem $(b; c)$" after an expression means that the preceding expression is repeated with b and c interchanged.

The main topic of this chapter, however, will be the q-analogues of a large class of transformations known as quadratic transformations. Two functions $f(z)$ and $g(w)$ are said to satisfy a *quadratic transformation* if z and w identically satisfy a quadratic equation and $f(z) = g(w)$. Among the important examples of quadratic transformation formulas are

$$ (1+z)^a\, {}_2F_1(a, b; 1+a-b; -z) = {}_2F_1\Big(\frac{a}{2}, \frac{a+1}{2} - b; 1+a-b; \frac{4z}{(1+z)^2}\Big), \tag{3.1.4}$$

$$ (1-z)^a\, {}_2F_1(a, b; 2b; 2z) = {}_2F_1\Big(\frac{a}{2}, \frac{a+1}{2}; b+\frac{1}{2}; \frac{z^2}{(1-z)^2}\Big), \tag{3.1.5}$$

$$ (1-z)^a\, {}_2F_1(2a, a+b; 2a+2b; z) = {}_2F_1\Big(a, b; a+b+\frac{1}{2}; \frac{z^2}{4(z-1)}\Big), \tag{3.1.6}$$

$$ {}_2F_1(2a,2b;a+b+\frac{1}{2};z) = {}_2F_1(a,b;a+b+\frac{1}{2};4z(1-z)), \tag{3.1.7} $$

$$ (1-z)^a\ {}_3F_2\left[\begin{matrix} a,b,c \\ 1+a-b,1+a-c \end{matrix};z\right] $$

$$ = {}_3F_2\left[\begin{matrix} \frac{1}{2}a,\frac{1}{2}(a+1),1+a-b-c \\ 1+a-b,1+a-c \end{matrix};-\frac{4z}{(1-z)^2}\right], \tag{3.1.8} $$

$$ \frac{(1-z)^{a+1}}{1+z}\ {}_4F_3\left[\begin{matrix} a,1+\frac{1}{2}a,b,c \\ \frac{1}{2}a,1+a-b,1+a-c \end{matrix};z\right] $$

$$ = {}_3F_2\left[\begin{matrix} \frac{1}{2}(a+1),1+\frac{1}{2}a,1+a-b-c \\ 1+a-b,1+a-c \end{matrix};-\frac{4z}{(1-z)^2}\right]. \tag{3.1.9} $$

The above definition of a quadratic transformation cannot be directly applied to basic hypergeometric series. For example, the $a=q^{-n}$ case of the identity in Ex. 2.26 is a q-analogue of the $a=-n$ case of (3.1.8), but it does not fit into the above definition of a quadratic transformation. So we shall just say that a transformation between basic hypergeometric series is "*quadratic*" if it is a q-analogue of a quadratic transformation for hypergeometric series.

It will be seen that one important feature of the quadratic transformations derived for basic hypergeometric series in the following sections is that the series obtained from an ${}_r\phi_s(a_1,\ldots,a_r;\ b_1,\ldots,b_s;q,z)$ series by a quadratic transformation will have squares or square roots of at least one of $a_1,\ldots,a_r$, $b_1,\ldots,b_s,q,z$ and possibly a square or square root of q as its base.

3.2 Two-term transformation formulas for ${}_3\phi_2$ series

In general, a convergent ${}_3\phi_2(a,b,c;d,e;q,z)$ series cannot be expressed as a multiple of another ${}_3\phi_2$ or of any other ${}_r\phi_s$ series. It is natural to expect that for such a transformation to exist there has to be some relationship among the parameters and the argument z. Sears [1951a,c] found that in the cases $z=q$ and $z=de/abc$ there is a whole family of transformation formulas for ${}_3\phi_2$ series, analogous to Thomae's [1879] transformation formulas for ${}_3F_2$ series. For the sake of convenience we will say that a basic hypergeometric series ${}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z)$ is of *type* I if $z=q$, and of *type* II if z is the product of the denominator parameters divided by the product of the numerator parameters. Note that a series is of both types if it is balanced.

In this section we shall consider transformations between two ${}_3\phi_2$ series. Such formulas may be obtained in a very straightforward manner as special and limiting cases of Sears' identity (2.10.4) which, for our present purposes, is rewritten in the form

$$ {}_4\phi_3\left[\begin{matrix} q^{-n},a,b,c, \\ d,e,abcq^{1-n}/de \end{matrix};q,q\right] $$

$$ = \frac{(e/a,de/bc;q)_n}{(e,de/abc;q)_n}\ {}_4\phi_3\left[\begin{matrix} q^{-n},a,d/b,d/c, \\ d,de/bc,aq^{1-n}/e \end{matrix};q,q\right], \tag{3.2.1} $$

with $n = 0, 1, 2, \ldots$.

Keeping n fixed and choosing special or limiting values of one of the other parameters leads to transformation formulas for terminating $_3\phi_2$ series. Let us consider this class of formulas first.

Case (i) Letting $c \to 0$ in (3.2.1) we get

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ d, e \end{matrix}; q, q\right] = \frac{(e/a;q)_n}{(e;q)_n} a^n \, {}_3\phi_2\left[\begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix}; q, \frac{bq}{e}\right]. \tag{3.2.2} $$

Note that the series on the left is of type I and that on the right is of type II. Formula (3.2.2) is a q-analogue of (3.1.1).

Case (ii) Letting $a \to 0$ in (3.2.1) gives

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q\right] = \frac{(de/bc;q)_n}{(e;q)_n} \left(\frac{bc}{d}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n}, d/b, d/c \\ d, de/bc \end{matrix}; q, q\right]. \tag{3.2.3} $$

If we let $c \to 0$ in (3.2.3) we obtain

$$ {}_2\phi_1\left(q^{-n}, d/b; d; q, bq/e\right) = (-1)^n q^{-\binom{n}{2}} (e;q)_n e^{-n} \, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, 0 \\ d, e \end{matrix}; q, q\right], $$

which may be written in the form (Ex. 1.15(i))

$$ {}_2\phi_1(a, b; c; q, z) = \frac{(abz/c;q)_\infty}{(bz/c;q)_\infty} \, {}_3\phi_2\left[\begin{matrix} a, c/b, 0 \\ c, cq/bz \end{matrix}; q, q\right], \tag{3.2.4} $$

where $a = q^{-n}, \ n = 0, 1, 2, \ldots$.

Case (iii) Let $c \to \infty$ in (3.2.1). This gives Sears' [1951c, (4.5)] formula

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ d, e \end{matrix}; q, \frac{deq^n}{ab}\right] = \frac{(e/a;q)_n}{(e;q)_n} \, {}_3\phi_2\left[\begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix}; q, q\right]. \tag{3.2.5} $$

Note that there is no essential difference between (3.2.2) and (3.2.5) since one can be obtained from the other by a change of parameters.

Case (iv) Replacing a by aq^n in (3.2.1) and simplifying, we get

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ d, e, abcq/de \end{matrix}; q, q\right] \\ &= \frac{(aq/e, de/bc;q)_n}{(e, abcq/de;q)_n} \left(\frac{bc}{d}\right)^n {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, d/b, d/c \\ d, de/bc, aq/e \end{matrix}; q, q\right]. \end{aligned} $$

Set $d = \lambda c$ and then let $c \to \infty$. In the resulting formula we replace λ, e and $abq/\lambda e$ by c, d and e, respectively, to get

$$ \begin{aligned} &{}_3\phi_2\left[\begin{matrix} q^{-n}, aq^n, b \\ d, e \end{matrix}; q, \frac{de}{ab}\right] \\ &= \frac{(aq/d, aq/e;q)_n}{(d, e;q)_n} \left(\frac{de}{aq}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n}, aq^n, abq/de \\ aq/d, aq/e \end{matrix}; q, \frac{q}{b}\right] \end{aligned} \tag{3.2.6} $$

which is a transformation formula between two terminating ${}_3\phi_2$ series of type II.

Let us now consider the class of transformation formulas that connect two nonterminating ${}_3\phi_2$ series.

Case (v) In (3.2.1) let us take $n \to \infty$. A straightforward term-by-term limiting process gives the formula

$$ {}_3\phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a}\right]. \quad (3.2.7) $$

Apart from the general requirement that no zero shall appear in the denominators of the two ${}_3\phi_2$ series, the parameters must be restricted by the convergence conditions: $|de/abc| < 1$ and $|e/a| < 1$. This formula is a q-analogue of the Kummer-Thomae-Whipple formula

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; 1\right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}\, {}_3F_2\left[\begin{matrix} a, d-b, d-c \\ d, d+e-b-c \end{matrix}; 1\right], \quad (3.2.8) $$

where Re $(e-a) > 0$ and Re $(d+e-a-b-c) > 0$.

Case (vi) Iterating (3.2.1) once gives

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right] $$
$$ = \frac{(b, de/ab, de/bc; q)_n}{(d, e, de/abc; q)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, d/b, e/b, de/abc \\ de/ab, de/bc, q^{1-n}/b \end{matrix}; q, q\right]. \quad (3.2.9) $$

Let us assume that $\max(|b|, |de/abc|) < 1$. Then, taking the limit $n \to \infty$, we obtain Hall's [1936] formula

$$ {}_3\phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right] $$
$$ = \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b\right]. \quad (3.2.10) $$

Note that this is a q-analogue of formula (3.1.2).

Before leaving this section it is worth mentioning that by taking the limit $n \to \infty$ in Watson's formula (2.5.1), we get another transformation formula:

$$ {}_3\phi_2\left[\begin{matrix} aq/bc, d, e \\ aq/b, aq/c \end{matrix}; q, \frac{aq}{de}\right] = \frac{(aq/d, aq/e; q)_\infty}{(aq, aq/de; q)_\infty} $$
$$ \times \sum_{k=0}^{\infty} \frac{(a;q)_k (1-aq^{2k})(b, c, d, e; q)_k}{(q;q)_k (1-a)(aq/b, aq/c, aq/d, aq/e; q)_k} q^{\binom{k}{2}} \left(-\frac{a^2q^2}{bcde}\right)^k, \quad (3.2.11) $$

provided $|aq/de| < 1$. This is a q-analogue of the formula

$$ {}_3F_2\left[\begin{matrix} 1+a-b-c, d, e \\ 1+a-b, 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+a)\Gamma(1+a-d-e)}{\Gamma(1+a-d)\Gamma(1+a-e)} $$
$$ \times\, {}_6F_5\left[\begin{matrix} a, 1+\frac{1}{2}a, b, c, d, e \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix}; -1\right], \quad (3.2.12) $$

where Re $(1+a-d-e)>0$; see Bailey [1935, 4.4(2)].

3.3 Three-term transformation formulas for ${}_3\phi_2$ series

In (2.10.10) let us replace a,b,c,d,e,f by $Aq^N, Bq^N, C, D, E, Fq^N$, respectively, and then let $N\to\infty$. In the resulting formula replace $C,D,E,Aq/B$ and Aq/F by a,b,c,d and e, respectively, to obtain

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,\frac{de}{abc}\right] = \frac{(e/b,e/c;q)_\infty}{(e,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} d/a,b,c \\ d,bcq/e \end{matrix};q,q\right] + \frac{(d/a,b,c,de/bc;q)_\infty}{(d,e,bc/e,de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} e/b,e/c,de/abc \\ de/bc,eq/bc \end{matrix};q,q\right], \tag{3.3.1} $$

where $|de/abc|<1$, and bc/e is not an integer power of q. This expresses a ${}_3\phi_2$ series of type II in terms of a ${}_3\phi_2$ series of type I. As a special case of (3.3.1), let $a=q^{-n}$ with $n=0,1,2,\ldots$. Then

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},b,c \\ d,e \end{matrix};q,\frac{deq^n}{bc}\right] = \frac{(e/b,e/c;q)_\infty}{(e,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} b,c,dq^n \\ bcq/e,d \end{matrix};q,q\right] + \frac{(b,c;q)_\infty}{(e,bc/e;q)_\infty}\frac{(de/bc;q)_n}{(d;q)_n}\,{}_3\phi_2\left[\begin{matrix} e/b,e/c,deq^n/bc \\ eq/bc,de/bc \end{matrix};q,q\right]. \tag{3.3.2} $$

Setting $n=0$ in (3.3.2) gives the summation formula (2.10.13).

We shall now obtain a transformation formula involving three ${}_3\phi_2$ series of type II. We start by replacing a,b,c,d,e,f in (2.11.1) by $Aq^N, Bq^N, C, D, E, Fq^N$, respectively, and then taking the limit $N\to\infty$. In the resulting formula we replace $C,D,E,Aq/B,Aq/F$ by a,b,c,d,e, respectively, and obtain

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,\frac{de}{abc}\right] = \frac{(e/b,e/c,cq/a,q/d;q)_\infty}{(e,cq/d,q/a,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} c,d/a,cq/e \\ cq/a,bcq/e \end{matrix};q,\frac{bq}{d}\right] - \frac{(q/d,eq/d,b,c,d/a,de/bcq,bcq^2/de;q)_\infty}{(d/q,e,bq/d,cq/d,q/a,e/bc,bcq/e;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} aq/d,bq/d,cq/d \\ q^2/d,eq/d \end{matrix};q,\frac{de}{abc}\right], \tag{3.3.3} $$

provided $|bq/d|<1, |de/abc|<1$ and none of the denominator parameters on either side produces a zero factor. If $|q|<|de/abc|<1$, then

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,\frac{de}{abc}\right] = \frac{(e/b,e/c,q/d,bq/a,cq/a,abcq/de;q)_\infty}{(e,e/bc,q/a,bq/d,cq/d,bcq/e;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} q/a,d/a,e/a \\ bq/a,cq/a \end{matrix};q,\frac{abcq}{de}\right] $$

$$-\frac{(b,c,q/d,d/a,eq/d,de/bcq,bcq^2/de;q)_\infty}{(e,e/bc,q/a,bq/d,cq/d,bcq/e,d/q;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq/d,bq/d,cq/d\\ q^2/d,eq/d\end{matrix};q,\frac{de}{abc}\right], \tag{3.3.4}$$

by observing that from (3.2.7)

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}c,d/a,cq/e\\ cq/a,bcq/e\end{matrix};q,\frac{bq}{d}\right]\\ &=\frac{(abcq/de,bq/a;q)_\infty}{(bcq/e,bq/d;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}q/a,d/a,e/a\\ bq/a,cq/a\end{matrix};q,\frac{abcq}{de}\right].\end{aligned}$$

If we set $e=\lambda c$ in (3.3.3), let $c\to 0$ and then replace d and λ by c and abz/c, respectively, where $|z|<1, |bq/c|<1$, then we obtain

$$\begin{aligned}{}_2\phi_1(a,b;c;q,z)&=\frac{(abz/c,q/c;q)_\infty}{(az/c,q/a;q)_\infty}\,{}_2\phi_1(c/a,cq/abz;cq/az;q,bq/c)\\ &-\frac{(b,q/c,c/a,az/q,q^2/az;q)_\infty}{(c/q,bq/c,q/a,az/c,cq/az;q)_\infty}\,{}_2\phi_1(aq/c,bq/c;q^2/c;q,z).\end{aligned} \tag{3.3.5}$$

Sears' [1951c, p. 173] four-term transformation formulas involving ${}_3\phi_2$ series of types I and II can also be derived by a combination of the formulas obtained in this and the previous section. Some of these transformation formulas also arise as special cases of the more general formulas that we shall obtain in the next chapter by using contour integrals.

3.4 Transformation formulas for well-poised ${}_3\phi_2$ and very-well-poised ${}_5\phi_4$ series with arbitrary arguments

Gasper and Rahman [1986] found the following formula connecting a well-poised ${}_3\phi_2$ series with two balanced ${}_5\phi_4$ series:

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}a,b,c\\ aq/b,aq/c\end{matrix};q,\frac{aqx}{bc}\right]\\ &=\frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_5\phi_4\left[\begin{matrix}a^{\frac12},-a^{\frac12},(aq)^{\frac12},-(aq)^{\frac12},aq/bc\\ aq/b,aq/c,ax,q/x\end{matrix};q,q\right]\\ &\quad+\frac{(a,aq/bc,aqx/b,aqx/c;q)_\infty}{(aq/b,aq/c,aqx/bc,x^{-1};q)_\infty}\\ &\quad\times{}_5\phi_4\left[\begin{matrix}xa^{\frac12},-xa^{\frac12},x(aq)^{\frac12},-x(aq)^{\frac12},aqx/bc\\ aqx/b,aqx/c,xq,ax^2\end{matrix};q,q\right].\end{aligned} \tag{3.4.1}$$

Convergence of the ${}_3\phi_2$ series on the left requires that $|aqx/bc|<1$. It is also essential to assume that x does not equal $q^{\pm j}, j=0,1,2,\ldots,$ because of the factors $(x;q)_\infty$ and $(x^{-1};q)_\infty$ appearing in the denominators on the right side of (3.4.1). Note that if either a or aq/bc is 1 or a negative integer power of q, then the coefficient of the second $_5\phi_4$ series on the right vanishes, so that (3.4.1) reduces to the Sears-Carlitz formula (Ex. 2.26). An important application of (3.4.1) is given in §8.8.

To prove (3.4.1) we replace d by dq^n in (2.8.3) and then let $n \to \infty$. This gives

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix}; q, \frac{d}{a}\right] = \frac{(bcd/aq;q)_\infty}{(bcd/qa^2;q)_\infty}
\times \lim_{n\to\infty} {}_{12}W_{11}\Big(a^2q^{1-n}/bcd;\ a^{\frac12},\ -a^{\frac12},\ (aq)^{\frac12},\ -(aq)^{\frac12}, aq^{1-n}/bc,
aq^{1-n}/bd,\ aq^{1-n}/cd,\ a^3q^{3-n}/b^2c^2d^2,\ q^{-n}; q, q\Big). \tag{3.4.2}
$$

To take the limit on the right side of (3.4.2) it suffices to proceed as in (2.10.9) to obtain

$$
\begin{aligned}
&\lim_{n\to\infty} {}_{12}W_{11}(\quad) \\
&= {}_5\phi_4\left[\begin{matrix} a^{\frac12}, -a^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}, aq/bc \\ aq/b, aq/c, bcd/aq, a^2q^2/bcd \end{matrix}; q, q\right] \\
&\quad - \frac{bcd}{qa^2}\frac{(bcd/a^2, bd/a, cd/a, aq/bc, a;q)_\infty}{(d/a, aq/b, aq/c, bcd/a, a^2q^2/bcd;q)_\infty} \\
&\quad \times {}_5\phi_4\left[\begin{matrix} d/a, bcd/qa^{\frac32}, -bcd/qa^{\frac32}, bcd/q^{\frac12}a^{\frac32}, -bcd/q^{\frac12}a^{\frac32} \\ bd/a, cd/a, bcd/a^2, b^2c^2d^2/qa^2 \end{matrix}; q, q\right].
\end{aligned} \tag{3.4.3}
$$

Using this in (3.4.2) and replacing d by qxa^2/bc, we get (3.4.1).

If we now replace d by dq^n in (2.8.5) and then let $n \to \infty$, we obtain the transformation formula

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c \end{matrix}; q, \frac{x(aq)^{\frac12}}{bc}\right] \\
&= \frac{(1-x^2)(xq(aq)^{\frac12};q)_\infty}{(x(aq)^{-\frac12};q)_\infty} \\
&\quad \times {}_5\phi_4\left[\begin{matrix} (aq)^{\frac12}, -(aq)^{\frac12}, qa^{\frac12}, -qa^{\frac12}, aq/bc \\ aq/b, aq/c, xq(aq)^{\frac12}, q(aq)^{\frac12}/x \end{matrix}; q, q\right] \\
&\quad + \frac{(aq, aq/bc, x(aq)^{\frac12}/b, x(aq)^{\frac12}/c;q)_\infty}{(aq/b, aq/c, x(aq)^{\frac12}/bc, (aq)^{\frac12}/x;q)_\infty} \\
&\quad \times {}_5\phi_4\left[\begin{matrix} x, -x, xq^{\frac12}, -xq^{\frac12}, x(aq)^{\frac12}/bc \\ x(aq)^{\frac12}/b, x(aq)^{\frac12}/c, x(q/a)^{\frac12}, qx^2 \end{matrix}; q, q\right].
\end{aligned} \tag{3.4.4}
$$

In terms of q-integrals formulas (3.4.1) and (3.4.4) are equivalent to

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix}; q, \frac{aqx}{bc}\right]
$$

$$= \frac{(a, aq/bc; q)_\infty}{s(1-q)(q, aq/b, aq/c, q/x, x; q)_\infty}$$

$$\times \int_{sx}^{s} \frac{(qu/xs, qu/s, aqu/bs, aqu/cs, axu/s; q)_\infty}{(ua^{\frac{1}{2}}/s, -ua^{\frac{1}{2}}/s, u(aq)^{\frac{1}{2}}/s, -u(aq)^{\frac{1}{2}}/s, aqu/bcs; q)_\infty} d_q u, \tag{3.4.5}$$

and

$$_5\phi_4\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c \end{matrix}; q, \frac{x(aq)^{\frac{1}{2}}}{bc}\right]$$

$$= \frac{(1-x^2)(aq, aq/bc; q)_\infty}{s(1-q)(q, aq/b, aq/c, x(aq)^{-\frac{1}{2}}, q(aq)^{\frac{1}{2}}/x; q)_\infty}$$

$$\times \int_{sx(aq)^{-\frac{1}{2}}}^{s} \frac{(uq(aq)^{\frac{1}{2}}/sx, qu/s, aqu/bs, aqu/cs, uxq(aq)^{\frac{1}{2}}/s; q)_\infty}{(u(aq)^{\frac{1}{2}}/s, -u(aq)^{\frac{1}{2}}/s, uqa^{\frac{1}{2}}/s, -uqa^{\frac{1}{2}}/s, aqu/bsc; q)_\infty} d_q u, \tag{3.4.6}$$

respectively, where $s \neq 0$ is an arbitrary parameter.

If we now set $c = (aq)^{\frac{1}{2}}$ in (3.4.5), replace x by $x/b(aq)^{\frac{1}{2}}$, and use (2.10.19), then we get

$$_2\phi_1(a, b; aq/b; q, qx/b^2) = \frac{(xq/b, aqx^2/b^2; q)_\infty}{(aqx/b, qx^2/b^2; q)_\infty}$$

$$\times\, _8\phi_7\left[\begin{matrix} ax/b, q(ax/b)^{\frac{1}{2}}, -q(ax/b)^{\frac{1}{2}}, x, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}} \\ (ax/b)^{\frac{1}{2}}, -(ax/b)^{\frac{1}{2}}, aq/b, xqa^{\frac{1}{2}}/b, -xqa^{\frac{1}{2}}/b, x(aq)^{\frac{1}{2}}/b, -x(aq)^{\frac{1}{2}}/b \end{matrix}; q, \frac{qx}{b^2}\right] \tag{3.4.7}$$

provided $|qx/b^2| < 1$ when the two series do not terminate.

Similarly, setting $c = (aq)^{\frac{1}{2}}$ and replacing x by x/bq in (3.4.6) we obtain

$$_4\phi_3\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b \end{matrix}; q, \frac{x}{qb^2}\right] = \frac{(ax^2/b^2, x/qb; q)_\infty}{(aqx/b, x^2/qb^2; q)_\infty}$$

$$\times\, _8\phi_7\left[\begin{matrix} ax/b, q(ax/b)^{\frac{1}{2}}, -q(ax/b)^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, x \\ (ax/b)^{\frac{1}{2}}, -(ax/b)^{\frac{1}{2}}, x(aq)^{\frac{1}{2}}/b, -x(aq)^{\frac{1}{2}}/b, xa^{\frac{1}{2}}/b, -xa^{\frac{1}{2}}/b, aq/b \end{matrix}; q, \frac{x}{qb^2}\right], \tag{3.4.8}$$

provided $|x/qb^2| < 1$ when the series do not terminate.

3.5 Transformations of series with base q^2 to series with base q

If in Sears' summation formula (2.10.12) we set $b=-c,\ e=-q$, replace a by $aq^r, r=0,1,2,\ldots$, multiply both sides by

$$\frac{(x^2,y^2;q^2)_r}{(-q;q)_r(x^2y^2b^2;q^2)_r}b^{2r}q^r$$

and then sum over r from 0 to ∞, we get

$$\begin{aligned}
&\frac{(-1,-q,ab,-ab,b^2;q)_\infty}{(a,b,-a,-b,b,-b;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}a^2,x^2,y^2\\a^2b^2,x^2y^2b^2\end{matrix};q^2,qb^2\right]\\
&=\frac{(-q,ab^2;q)_\infty}{(a,b,-b;q)_\infty}\sum_{j=0}^{\infty}\frac{(a,b,-b;q)_j}{(q,-q,ab^2;q)_j}q^j\,{}_3\phi_2\left[\begin{matrix}q^{-2j},x^2,y^2\\x^2y^2b^2,q^{2-2j}/b^2\end{matrix};q^2,q^2\right]\\
&\quad+\frac{(-q,-ab^2;q)_\infty}{(-a,-b,b;q)_\infty}\sum_{j=0}^{\infty}\frac{(-a,-b,b;q)_j}{(q,-q,-ab^2;q)_j}q^j\,{}_3\phi_2\left[\begin{matrix}q^{-2j},x^2,y^2\\x^2y^2b^2,q^{2-2j}/b^2\end{matrix};q^2,q^2\right]
\end{aligned}\tag{3.5.1}$$

assuming that $|qb^2|<1$ when the series on the left is nonterminating.

Since the two ${}_3\phi_2$ series on the right side can be summed by the q-Saalschütz formula (1.7.2) with the base q replaced by q^2, it follows from (3.5.1) that

$$\begin{aligned}
&\frac{(a^2b^2;q^2)_\infty}{(b^2;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix}a^2,x^2,y^2\\a^2b^2,x^2y^2b^2\end{matrix};q^2,qb^2\right]\\
&=\frac{(-a,ab^2;q)_\infty}{(-1,b^2;q)_\infty}\,{}_5\phi_4\left[\begin{matrix}a,bx,-bx,by,-by\\-q,ab^2,bxy,-bxy\end{matrix};q,q\right]\\
&\quad+\frac{(a,-ab^2;q)_\infty}{(-1,b^2;q)_\infty}\,{}_5\phi_4\left[\begin{matrix}-a,-bx,bx,-by,by\\-q,-ab^2,-bxy,bxy\end{matrix};q,q\right].
\end{aligned}\tag{3.5.2}$$

Note that one of the terms on the right side of (3.5.2) drops out when $a=\pm q^{-n}, n=0,1,2,\ldots$. Setting $y=ab$ and using (2.10.10) gives

$$\begin{aligned}
{}_2\phi_1(a^2,x^2;a^2x^2b^4;q^2,qb^2)\ &=\frac{(b^2,a^2b^2x^2;q^2)_\infty(ab^2,b^2x^2;q)_\infty}{(a^2b^2,b^2x^2;q^2)_\infty(b^2,ab^2x^2;q)_\infty}\\
&\times\ {}_8W_7(ab^2x^2/q;a,x,-x,bx,-bx;q,ab^2),
\end{aligned}\tag{3.5.3}$$

where $|qb^2|<1$ and $|ab^2|<1$ when the series do not terminate. By applying Heine's transformation formula (1.4.1) twice to the ${}_2\phi_1$ series above and replacing b by $q^{\frac{1}{2}}/b$ we find that

$$\begin{aligned}
&{}_2\phi_1(a^2,b^2;a^2q^2/b^2;q^2,x^2q^2/b^4)\\
&=\frac{(qa^2x^2/b^2,q^2a^2x^2/b^4;q^2)_\infty(aq/b^2,qx^2/b^2;q)_\infty}{(qx^2/b^2,q^2x^2/b^4;q^2)_\infty(qa^2/b^2,aqx^2/b^2;q)_\infty}
\end{aligned}$$

$$\times\ {}_8\phi_7\left[\begin{matrix}\frac{ax^2}{b^2}, q\left(\frac{ax^2}{b^2}\right)^{\frac12}, -q\left(\frac{ax^2}{b^2}\right)^{\frac12}, a, x, -x, \frac{xq^{\frac12}}{b}, \frac{-xq^{\frac12}}{b}\\ \left(\frac{ax^2}{b^2}\right)^{\frac12}, -\left(\frac{ax^2}{b^2}\right)^{\frac12}, \frac{qx^2}{b^2}, \frac{aqx}{b^2}, \frac{-aqx}{b^2}, \frac{axq^{\frac12}}{b}, \frac{-axq^{\frac12}}{b}\end{matrix}; q, \frac{aq}{b^2}\right], \tag{3.5.4}$$

provided $|aq/b^2| < 1$ and $|xq/b^2| < 1$ when the series do not terminate. This formula was derived by Gasper and Rahman [1986], and a terminating version of it was given earlier by Verma [1980]. Application of the transformation formula (2.10.1) to the ${}_8\phi_7$ series on the right of (3.5.4) yields an equivalent formula

$$\begin{aligned}&{}_2\phi_1(a^2, b^2; a^2q^2/b^2; q^2, x^2q^2/b^4)\\ &= \frac{(qa^2x^2/b^2, q^2a^2x^2/b^4; q^2)_\infty(-xq^{\frac12}/b, -axq^{\frac32}/b^3; q)_\infty}{(qx^2/b^2, q^2x^2/b^4; q^2)_\infty(-axq^{\frac12}/b, -a^2xq^{\frac32}/b^3; q)_\infty}\\ &\times\ {}_8\phi_7\left[\begin{matrix}-xa^2q^{\frac12}/b^3, & q(-xa^2q^{\frac12}/b^3)^{\frac12}, & -q(-xa^2q^{\frac12}/b^3)^{\frac12}, & a,\\ & (-xa^2q^{\frac12}/b^3)^{\frac12}, & -(-xa^2q^{\frac12}/b^3)^{\frac12}, & -axq^{\frac32}/b^3,\end{matrix}\right.\\ &\qquad\left.\begin{matrix}aq^{\frac12}/b, & -aq^{\frac12}/b, & -aq/b^2, & -xq^{\frac12}/b\\ -aqx/b^2, & aqx/b^2, & axq^{\frac12}/b, & qa^2/b^2\end{matrix}; q, -\frac{xq^{\frac12}}{b}\right],\end{aligned} \tag{3.5.5}$$

where $|xq/b^2| < 1$ and $|xq^{\frac12}/b| < 1$ when the series do not terminate. It is clear that formula (3.5.5) is a q-analogue of the quadratic transformation formula

$${}_2F_1(a, b; 1+a-b; x^2) = (1-x)^{-2a}\ {}_2F_1(a, a+\frac12-b; 2a+1-2b; \frac{-4x}{(1-x)^2}). \tag{3.5.6}$$

We shall now prove the following transformation formula due to Jain and Verma [1982]:

$$\begin{aligned}&{}_{10}\phi_9\left[\begin{matrix}a, q^2a^{\frac12}, -q^2a^{\frac12}, b, c, cq, d, dq, e, eq\\ a^{\frac12}, -a^{\frac12}, aq^2/b, aq^2/c, aq/c, aq^2/d, aq/d, aq^2/e, aq/e\end{matrix}; q^2, \frac{a^3q^3}{bc^2d^2e^2}\right]\\ &= \frac{(aq, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/c, aq/d, aq/e, aq/cde; q)_\infty}\ {}_5\phi_4\left[\begin{matrix}(aq/b)^{\frac12}, -(aq/b)^{\frac12}, c, d, e\\ (aq)^{\frac12}, -(aq)^{\frac12}, aq/b, cde/a\end{matrix}; q, q\right]\\ &+ \frac{(aq^2, a^3q^3/c^2d^2e^2; q^2)_\infty(c, d, e, a^2q^2/bcde; q)_\infty}{(aq^2/b, a^3q^3/bc^2d^2e^2; q^2)_\infty(aq/c, aq/d, aq/e, cde/aq; q)_\infty}\\ &\times\ {}_5\phi_4\left[\begin{matrix}(a^3q^3/bc^2d^2e^2)^{\frac12}, -(a^3q^3/bc^2d^2e^2)^{\frac12}, aq/cd, aq/ce, aq/de\\ (a^3q^3/c^2d^2e^2)^{\frac12}, -(a^3q^3/c^2d^2e^2)^{\frac12}, a^2q^2/bcde, aq^2/cde\end{matrix}; q, q\right],\end{aligned} \tag{3.5.7}$$

with the usual understanding that if the ${}_{10}\phi_9$ series on the left does not terminate then the convergence condition $|a^3q^3/bc^2d^2e^2| < 1$ must be assumed to hold.

First we rewrite (2.10.12) in the form

$$\frac{(aq^{4n+1}, aq/cd, aq/ce, aq/de; q)_\infty}{(cq^{2n}, dq^{2n}, eq^{2n}, aq^{1-2n}/cde; q)_\infty} \sum_{r=0}^{\infty} \frac{(cq^{2n}, dq^{2n}, eq^{2n}; q)_r}{(q, aq^{4n+1}, cdeq^{2n}/a; q)_r} q^r$$
$$+ \frac{(a^2q^{2n+2}/cde; q)_\infty}{(cdeq^{2n-1}/a; q)_\infty} \sum_{r=0}^{\infty} \frac{(aq/cd, aq/ce, aq/de; q)_r}{(q, a^2q^{2n+2}/cde, aq^{2-2n}/cde; q)_r} q^r$$
$$= \frac{(aq/c, aq/d, aq/e; q)_\infty}{(c, d, e; q)_\infty} \frac{(c, d, e; q)_{2n}}{(aq/c, aq/d, aq/e; q)_{2n}}, \tag{3.5.8}$$

where n is a nonnegative integer. Using (1.2.39) and (1.2.40), multiplying both sides of (3.5.8) by

$$\frac{(a, b; q^2)_n(1 - aq^{4n})}{(q^2, aq^2/b; q^2)_n(1 - a)} \left(\frac{a^3q^3}{bc^2d^2e^2}\right)^n,$$

and summing over n from 0 to ∞, we get

$${}_{10}W_9(a; b, c, cq, d, dq, e, eq; q^2, a^3q^3/bc^2d^2e^2)$$
$$= \frac{(aq, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/c, aq/d, aq/e, aq/cde; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q^2)_n(1 - aq^{4n})}{(q^2, aq^2/b; q^2)_n(1 - a)}$$
$$\times \frac{(c, d, e; q)_{2n}}{(cde/a; q)_{2n}(aq; q)_{4n}} q^{n(2n-1)} \left(\frac{aq^3}{b}\right)^n {}_3\phi_2\left[\begin{matrix} cq^{2n}, dq^{2n}, eq^{2n} \\ aq^{4n+1}, cdeq^{2n}/a \end{matrix}; q, q\right]$$
$$+ \frac{(c, d, e, a^2q^2/cde; q)_\infty}{(aq/c, aq/d, aq/e, cde/aq; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q^2)_n(1 - aq^{4n})}{(q^2, aq^2/b; q^2)_n(1 - a)}$$
$$\times \frac{(cde/aq; q)_{2n}}{(a^2q^2/cde; q)_{2n}} \left(\frac{a^3q^3}{bc^2d^2e^2}\right)^n {}_3\phi_2\left[\begin{matrix} aq/cd, aq/ce, aq/de \\ a^2q^{2n+2}/cde, aq^{2-2n}/cde \end{matrix}; q, q\right]. \tag{3.5.9}$$

The first double series on the right side of (3.5.9) easily transforms to

$$\sum_{m=0}^{\infty} \frac{(c, d, e; q)_m}{(q, aq, cde/a; q)_m} q^m \, {}_6W_5\left(a; b, q^{1-m}, q^{-m}; q^2, aq^{2m+1}/b\right),$$

which, by (2.4.2), equals

$${}_5\phi_4\left[\begin{matrix} c, d, e, (aq/b)^{\frac{1}{2}}, -(aq/b)^{\frac{1}{2}} \\ aq/b, cde/a, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}} \end{matrix}; q, q\right].$$

Similarly we can express the second double series on the right side of (3.5.9) as a single balanced ${}_5\phi_4$ series. Combining the two we get (3.5.7).

The special case of (3.5.7) that results from setting $e = (aq)^{\frac{1}{2}}$ is particularly interesting because both ${}_5\phi_4$ series on the right side become balanced ${}_4\phi_3$

series which, via (2.10.10), combine into a single ${}_8\phi_7$ series with base q. Thus we have the formula

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, q^2a^{\frac12}, -q^2a^{\frac12}, b, c, cq, d, dq \\ a^{\frac12}, -a^{\frac12}, aq^2/b, aq^2/c, aq/c, aq^2/d, aq/d \end{matrix}; q^2, \frac{a^2q^2}{bc^2d^2}\right] \\
&= \frac{(aq, aq/bc, aq/cd, -aq/cd, aq/db^{\frac12}, -aq/db^{\frac12}; q)_\infty}{(aq/b, aq/c, aq/d, -aq/d, aq/cdb^{\frac12}, -aq/cdb^{\frac12}; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} -a/d, q(-a/d)^{\frac12}, -q(-a/d)^{\frac12}, c, b^{\frac12}, -b^{\frac12}, (aq)^{\frac12}/d, -(aq)^{\frac12}/d \\ (-a/d)^{\frac12}, -(-a/d)^{\frac12}, -aq/cd, -aq/db^{\frac12}, aq/db^{\frac12}, -(aq)^{\frac12}, (aq)^{\frac12} \end{matrix}; q, \frac{aq}{bc}\right],
\end{aligned}
\tag{3.5.10}
$$

where $|a^2q^2/bc^2d^2| < 1$ and $|aq/bc| < 1$ when the series do not terminate.

3.6 Bibasic summation formulas

Our main objective in this section is to derive summation formulas containing two independent bases. Let us start by observing that when $d = a/bc$ Jackson's ${}_8\phi_7$ summation formula (2.6.2) reduces to the following sum of a truncated series

$$\sum_{k=0}^{n} \frac{1-aq^{2k}}{1-a} \frac{(a, b, c, a/bc; q)_k}{(q, aq/b, aq/c, bcq; q)_k} q^k = \frac{(aq, bq, cq, aq/bc; q)_n}{(q, aq/b, aq/c, bcq; q)_n}, \tag{3.6.1}$$

where $n = 0, 1, \ldots$. Notice that this series telescopes, for if we set $\sigma_{-1} = 0$ and

$$\sigma_k = \frac{(aq, bq, cq, aq/bc; q)_k}{(q, aq/b, aq/c, bcq; q)_k} \tag{3.6.2}$$

for $k = 0, 1, \ldots$, and apply the difference operator Δ defined by $\Delta u_k = u_k - u_{k-1}$ to σ_k, then we get

$$\Delta\sigma_k = \frac{(1-aq^{2k})(a, b, c, a/bc; q)_k}{(1-a)(q, aq/b, aq/c, bcq; q)_k} q^k, \tag{3.6.3}$$

which gives (3.6.1), since

$$\sum_{k=0}^{n} \Delta u_k = u_n - u_{-1} \tag{3.6.4}$$

for any sequence $\{u_k\}$.

These observations and the bibasic extension

$$\tau_k = \frac{(ap, bp; p)_k (cq, aq/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k} \tag{3.6.5}$$

of σ_k were used in Gasper [1989a] to show that

$$\Delta\tau_k = \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k} q^k, \tag{3.6.6}$$

which, by (3.6.4), gave the indefinite bibasic summation formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k = \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n} \tag{3.6.7}$$

for $n=0,1,\ldots$. Notice that the part of the series on the left side of (3.6.7) containing the q-shifted factorials is split-poised in the sense that $aq = b(aq/b)$ and $c(ap/c) = (a/bc)(bcp) = ap$, while the expression on the right side is balanced and well-poised since

$$(ap)(bp)(cq)(aq/bc) = q(aq/b)(ap/c)(bcp)$$

and

$$(ap)q = (bp)(aq/b) = (cq)(ap/c) = (aq/bc)(bcp).$$

The $b \to 0$ case of (3.6.7)

$$\sum_{k=0}^{n} \frac{1-ap^kq^k}{1-a} \frac{(a;p)_k(c;q)_k}{(q;q)_k(ap/c;p)_k} c^{-k} = \frac{(ap;p)_n(cq;q)_n}{(q;q)_n(ap/c;p)_n} c^{-n} \tag{3.6.8}$$

is due to Gosper.

To derive a useful extension of (3.6.7), Gasper and Rahman [1990] set

$$s_k = \frac{(ap,bp;p)_k(cq,ad^2q/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} \tag{3.6.9}$$

for $k=0,\pm1,\pm2,\ldots$, and observed that

$$\begin{aligned} \Delta s_k &= s_k - s_{k-1} \\ &= \frac{(ap,bp;p)_{k-1}(cq,ad^2q/bc;q)_{k-1}}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} \\ &\quad \times \big\{(1-ap^k)(1-bp^k)(1-cq^k)(1-ad^2q^k/bc) \\ &\qquad -(1-dq^k)(1-adq^k/b)(1-adp^k/c)(1-bcp^k/d)\big\} \\ &= \frac{d(1-c/d)(1-ad/bc)(1-adp^kq^k)(1-bp^k/dq^k)}{(1-a)(1-b)(1-c)(1-ad^2/bc)} \\ &\quad \times \frac{(a,b;p)_k(c,ad^2/bc;q)_kq^k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k}. \end{aligned} \tag{3.6.10}$$

Since (3.6.4) extends to

$$\sum_{k=-m}^{n} \Delta u_k = u_n - u_{-m-1}, \tag{3.6.11}$$

where we employed the standard convention of defining

$$\sum_{k=m}^{n} a_k = \begin{cases} a_m + a_{m+1} + \cdots + a_n, & m \le n, \\ 0, & m = n+1, \\ -(a_{n+1}+a_{n+2}+\cdots+a_{m-1}), & m \ge n+2, \end{cases} \tag{3.6.12}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, it follows from (3.6.10) that (3.6.7) extends to the indefinite bibasic summation formula

$$\sum_{k=-m}^{n} \frac{(1-adp^k q^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k (c, ad^2/bc;q)_k}{(dq, adq/b;q)_k (adp/c, bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\times \left\{ \frac{(ap,bp;p)_n (cq, ad^2q/bc;q)_n}{(dq, adq/b;q)_n (adp/c, bcp/d;p)_n} - \frac{(c/ad, d/bc;p)_{m+1} (1/d, b/ad;q)_{m+1}}{(1/c, bc/ad^2;q)_{m+1} (1/a, 1/b;p)_{m+1}} \right\} \tag{3.6.13}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, by applying the identity (1.2.28). Observe that (3.6.7) is the case $d = 1$ of (3.6.13) and that the right side of (3.6.9) is balanced and well-poised since

$$(ap)(bp)(cq)(ad^2q/bc) = (dq)(adq/b)(adp/c)(bcp/d)$$

and

$$(ap)(dq) = (bp)(adq/b) = (cq)(adp/c) = (ad^2q/bc)(bcp/d).$$

It is these observations and the factorization that occurred in (3.6.10) which motivated the choice of s_k in (3.6.9).

If $|p| < 1$ and $|q| < 1$, then by letting n or m tend to infinity in (3.6.13) we find that (3.6.13) also holds with n or m replaced by ∞. In particular, this yields the following evaluation of a bilateral bibasic series

$$\sum_{k=-\infty}^{\infty} \frac{(1-adp^k q^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k (c, ad^2/bc;q)_k}{(dq, adq/b;q)_k (adp/c, bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\times \left\{ \frac{(ap,bp;p)_\infty (cq, ad^2q/bc;q)_\infty}{(dq, adq/b;q)_\infty (adp/c, bcp/d;p)_\infty} - \frac{(c/ad, d/bc;p)_\infty (1/d, b/ad;q)_\infty}{(1/c, bc/ad^2;q)_\infty (1/a, 1/b;p)_\infty} \right\}, \tag{3.6.14}$$

where $|p| < 1$ and $|q| < 1$.

In §3.8 we shall use the $m = 0$ case of (3.6.13) in the form

$$\sum_{k=0}^{n} \frac{(1-adp^k q^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k (c, ad^2/bc;q)_k}{(dq, adq/b;q)_k (adp/c, bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\times \frac{(ap,bp;p)_n (cq, ad^2q/bc;q)_n}{(dq, adq/b;q)_n (adp/c, bcp/d;p)_n}$$
$$- \frac{(1-d)(1-ad/b)(1-ad/c)(1-bc/d)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}. \tag{3.6.15}$$

There is no loss in generality since, by setting $k = j - m$ in (3.6.13), it is seen that (3.6.13) is equivalent to (3.6.15) with n, a, b, c, d replaced by $n +$

$m, ap^{-m}, bp^{-m}, cq^{-m}, dq^{-m}$, respectively. We shall also use the special case $c = q^{-n}$ of (3.6.15) in the form

$$\sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(q^{-n}, ad^2q^n/b;q)_k}{(dq, adq/b;q)_k(adpq^n, bp/dq^n;p)_k} q^k$$
$$= \frac{(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)}{(1-ad)(1-d/b)(1-dq^n)(1-adq^n/b)}, \tag{3.6.16}$$

where $n = 0, 1, \ldots$. The $d \to 1$ limit case of (3.6.16)

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(q^{-n}, aq^n/b;q)_k}{(q, aq/b;q)_k(apq^n, bpq^{-n};p)_k} q^k = \delta_{n,0}, \tag{3.6.17}$$

where $\delta_{n,m}$ is the Kronecker delta function and $n = 0, 1, \ldots$, was derived independently by Bressoud [1988], Gasper [1989a], and Krattenthaler [1996].

If we replace n, a, b and k in (3.6.17) by $n-m, ap^mq^m, bp^mq^{-m}$ and $j-m$, respectively, we obtain the orthogonality relation

$$\sum_{j=m}^{n} a_{nj}b_{jm} = \delta_{n,m} \tag{3.6.18}$$

with

$$a_{nj} = \frac{(-1)^{n+j}(1-ap^jq^j)(1-bp^jq^{-j})(apq^n, bpq^{-n};p)_{n-1}}{(q;q)_{n-j}(apq^n, bpq^{-n};p)_j(bq^{1-2n}/a;q)_{n-j}}, \tag{3.6.19}$$

$$b_{jm} = \frac{(ap^mq^m, bp^mq^{-m};p)_{j-m}}{(q, aq^{1+2m}/b;q)_{j-m}} \left(-\frac{a}{b}q^{1+2m}\right)^{j-m} q^{2\binom{j-m}{2}}. \tag{3.6.20}$$

This shows that the triangular matrix $A = (a_{nj})$ is inverse to the triangular matrix $B = (b_{jm})$. Since inverse matrices commute, by computing the jk^{th} term of BA, we obtain the orthogonality relation

$$\sum_{n=0}^{j-k} \frac{(1-ap^kq^k)(1-bp^kq^{-k})(ap^{k+1}q^{k+n}, bp^{k+1}q^{-k-n};p)_{j-k-1}}{(q;q)_n(q;q)_{j-k-n}(aq^{2k+n}/b;q)_{j-k-1}}$$
$$\times \left(1-\frac{a}{b}q^{2k+2n}\right)(-1)^n q^{n(j-k-1)+\binom{j-k-n}{2}} = \delta_{j,k}, \tag{3.6.21}$$

which, by replacing j, n, a, b by $n+k, k, ap^{-k-1}q^{-k}, bp^{-k-1}q^k$, respectively, yields the bibasic summation formula

$$\left(1-\frac{a}{p}\right)\left(1-\frac{b}{p}\right) \sum_{k=0}^{n} \frac{(aq^k, bq^{-k};p)_{n-1}(1-aq^{2k}/b)}{(q;q)_k(q;q)_{n-k}(aq^k/b;q)_{n+1}} (-1)^k q^{\binom{k}{2}} = \delta_{n,0} \tag{3.6.22}$$

for $n = 0, 1, \ldots$. The $b \to 0$ limit case of (3.6.22) was derived in Al-Salam and Verma [1984] by using the fact that the n^{th} q-difference of a polynomial in q of degree less than n is equal to zero. For applications to q-analogues of Lagrange inversion, see Gessel and Stanton [1983, 1986] and Gasper [1989a]. Formulas (3.6.17) and (3.6.22) will be used in §3.7 to derive some useful general expansion formulas.

3.7 Bibasic expansion formulas

One of the most important general expansion formulas for hypergeometric series is the Fields and Wimp [1961] expansion

$$
{}_{r+t}F_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; xw\right] = \sum_{n=0}^{\infty} \frac{(a_R)_n(\alpha)_n(\beta)_n}{(b_S)_n(\gamma+n)_n} \frac{(-x)^n}{n!}
\times {}_{r+2}F_{s+1}\left[\begin{matrix} n+\alpha, n+\beta, n+a_R \\ 1+2n+\gamma, n+b_S \end{matrix}; x\right]
\times {}_{t+2}F_{u+2}\left[\begin{matrix} -n, n+\gamma, c_T \\ \alpha, \beta, d_U \end{matrix}; w\right], \tag{3.7.1}
$$

where we employed the contracted notation of representing $a_1, \ldots, a_r$ by a_R, $(a_1)_n \cdots (a_r)_n$ by $(a_R)_n$, and $n + a_1, \ldots, n + a_r$ by $n + a_R$. In (3.7.1), as elsewhere, either the parameters and variables are assumed to be such that the (multiple) series converge absolutely or the series are considered to be formal power series in the variables x and w. Special cases of (3.7.1) were employed, e.g., in Gasper [1975a] to prove the nonnegativity of certain sums (kernels) of Jacobi polynomials and to give additional proofs of the Askey and Gasper [1976] inequalities that de Branges [1985] used at the last step in his proof of the Bieberbach conjecture.

Verma [1972] showed that (3.7.1) is a special case of the expansion

$$
\sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(\gamma+n)_n} \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k}(\beta)_{n+k}}{k!(\gamma+2n+1)_k} B_{n+k} x^k
\times \sum_{j=0}^{n} \frac{(-n)_j(n+\gamma)_j}{j!(\alpha)_j(\beta)_j} A_j w^j \tag{3.7.2}
$$

and derived the q-analogue

$$
\sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q;q)_n}
= \sum_{n=0}^{\infty} \frac{(-x)^n}{(q, \gamma q^n; q)_n} q^{\binom{n}{2}} \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k}}{(q, \gamma q^{2n+1}; q)_k} B_{n+k} x^k
\times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n; q)_j}{(q, \alpha, \beta; q)_j} A_j (wq)^j. \tag{3.7.3}
$$

To derive a bibasic extension of (3.7.3) we first observe that, by (3.6.17),

$$
\sum_{j=0}^{m} \frac{(1-\gamma p^{r+j} q^{r+j})(1-\sigma p^{r+j} q^{-r-j})}{(1-\gamma p^r q^r)(1-\sigma p^r q^{-r})} \frac{(\gamma p^r q^r, \sigma p^r q^{-r}; p)_j}{(q, \gamma\sigma^{-1} q^{2r+1}; q)_j}
\times \frac{(q^{-m}, \gamma\sigma^{-1} q^{2r+m}; q)_j}{(\gamma p^{r+1} q^{r+m}, \sigma p^{r+1} q^{-r-m}; p)_j} q^j = \delta_{m,0} \tag{3.7.4}
$$

for $m = 0, 1, \ldots$. Hence, if $C_{r,m}$ are complex numbers such that $C_{r,0} = 1$ for $r = 0, 1, \ldots$, then

$$B_r x^r = \sum_{m=0}^{\infty} \frac{1-\gamma\sigma^{-1}q^{2r+2m}}{1-\gamma\sigma^{-1}q^{2r}} \frac{(\gamma\sigma^{-1}q^{2r};q)_m(\gamma pq^r, \sigma pq^{-r};p)_r}{(q;q)_m(\gamma pq^{r+m}, \sigma pq^{-r-m};p)_r}$$

$$\times q^{-mr} B_{r+m} C_{r,m} x^{r+m} \delta_{m,0}$$

$$= \sum_{k=0}^{\infty}\sum_{n=r}^{\infty} \frac{(1-\gamma p^n q^n)(1-\sigma p^n q^{-n})(1-\gamma\sigma^{-1}q^{2n+2k})}{(q;q)_k(q;q)_n(\gamma pq^{n+k}, \sigma pq^{-n-k};p)_n}$$

$$\times (\gamma\sigma^{-1}q^{n+r+1};q)_{n+k-r-1}(\gamma pq^r, \sigma pq^{-r};p)_{n-1}(q^{-n};q)_r$$

$$\times (-1)^n B_{n+k} C_{r,n+k-r} x^{n+k} q^{n(1+r-n-k)+\binom{n}{2}} \tag{3.7.5}$$

by setting $j = n - r$ and $m = n + k - r$. Then by multiplying both sides of (3.7.5) by $A_r w^r/(q;q)_r$ and summing from $r = 0$ to ∞ we obtain Gasper's [1989a] bibasic expansion formula

$$\sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(1-\gamma p^n q^n)(1-\sigma p^n q^{-n})}{(q;q)_n}(-x)^n q^{n+\binom{n}{2}}$$

$$\times \sum_{k=0}^{\infty} \frac{1-\gamma\sigma^{-1}q^{2n+2k}}{(q;q)_k(\gamma pq^{n+k}, \sigma pq^{-n-k};p)_n} B_{n+k} x^k$$

$$\times \sum_{j=0}^{n} \frac{(q^{-n};q)_j(\gamma\sigma^{-1}q^{n+j+1};q)_{n+k-j-1}}{(q;q)_j}$$

$$\times (\gamma pq^j, \sigma pq^{-j};p)_{n-1} A_j C_{j,n+k-j} w^j q^{n(j-n-k)}, \tag{3.7.6}$$

where $C_{j,0} = 1$, for $j = 0, 1, \ldots$.

Note that if $p = q$ and $C_{j,m} \equiv 1$, then (3.7.6) reduces to an expansion which is equivalent to

$$\sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(\sigma, \gamma q^{n+1}/\sigma, \alpha, \beta;q)_n}{(q, \gamma q^n;q)_n}\left(\frac{x}{\sigma}\right)^n$$

$$\times \sum_{k=0}^{\infty} \frac{(\gamma q^{2n}/\sigma, q^{n+1}\sqrt{\gamma/\sigma}, -q^{n+1}\sqrt{\gamma/\sigma}, 1/\sigma, \alpha q^n, \beta q^n;q)_k}{(q, q^n\sqrt{\gamma/\sigma}, -q^n\sqrt{\gamma/\sigma}, \gamma q^{2n+1};q)_k} B_{n+k} x^k$$

$$\times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n;q)_j}{(q, \gamma q^{n+1}/\sigma, q^{1-n}/\sigma, \alpha, \beta;q)_j} A_j (wq)^j. \tag{3.7.7}$$

Verma's expansion (3.7.3) is the $\sigma \to \infty$ limit case of (3.7.7). For basic hypergeometric series, (3.7.7) gives the following q-extension of (3.7.1)

$${}_{r+t}\phi_{s+u}\left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix}; q, xw\right]$$

$$= \sum_{j=0}^{\infty} \frac{(c_T, e_K, \sigma, \gamma q^{j+1}/\sigma; q)_j}{(q, d_U, f_M, \gamma q^j; q)_j} \left(\frac{x}{\sigma}\right)^j [(-1)^j q^{\binom{j}{2}}]^{u+m-t-k}$$

$$\times {}_{t+k+4}\phi_{u+m+3} \left[\begin{matrix} \gamma q^{2j}/\sigma, q^{j+1}\sqrt{\gamma/\sigma}, -q^{j+1}\sqrt{\gamma/\sigma}, \sigma^{-1}, \\ q^j\sqrt{\gamma/\sigma}, -q^j\sqrt{\gamma/\sigma}, \gamma q^{2j+1}, d_U q^j, \end{matrix} \right.$$

$$\left. \begin{matrix} c_T q^j, e_K q^j \\ f_M q^j \end{matrix} ; q, x q^{j(u+m-t-k)} \right]$$

$$\times {}_{r+m+2}\phi_{s+k+2} \left[\begin{matrix} q^{-j}, \gamma q^j, a_R, f_M \\ \gamma q^{j+1}/\sigma, q^{1-j}/\sigma, b_S, e_K \end{matrix} ; q, wq \right], \tag{3.7.8}$$

where we used a contracted notation analogous to that used in (3.7.1).

Note that by letting $\sigma \to \infty$ in (3.7.8) and setting $m = 2, f_1 = f_2 = 0$ we get the expansion

$$_{r+t}\phi_{s+u} \left[\begin{matrix} a_R, c_T \\ b_S, d_U \end{matrix} ; q, xw \right]$$

$$= \sum_{j=0}^{\infty} \frac{(c_T, e_K; q)_j}{(q, d_U, \gamma q^j; q)_j} x^j [(-1)^j q^{\binom{j}{2}}]^{u+3-t-k}$$

$$\times {}_{t+k}\phi_{u+1} \left[\begin{matrix} c_T q^j, e_K q^j \\ \gamma q^{2j+1}, d_U q^j \end{matrix} ; q, x q^{j(u+2-t-k)} \right]$$

$$\times {}_{r+2}\phi_{s+k} \left[\begin{matrix} q^{-j}, \gamma q^j, a_R \\ b_S, e_K \end{matrix} ; q, wq \right], \tag{3.7.9}$$

which is equivalent to Verma's [1966] q-extension of the Fields and Wimp expansion (3.7.1). Other types of expansions are given in Fields and Ismail [1975].

Al-Salam and Verma [1984] used the $b \to 0$ limit case of the summation formula (3.6.22) to show that Euler's transformation formula

$$\sum_{n=0}^{\infty} a_n b_n x^n = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} f^{(k)}(x) \Delta^k a_0, \tag{3.7.10}$$

where

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

and

$$\Delta^k a_0 = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{k-j},$$

has the bibasic extension

$$\sum_{n=0}^{\infty} A_n B_n (xw)^n = \sum_{k=0}^{\infty} (apq^k; p)_{k-1} x^k \sum_{n=0}^{k} \frac{(1 - ap^n q^n) w^n A_n}{(q;q)_{k-n} (apq^k; p)_n}$$

$$\times \sum_{j=0}^{\infty} \frac{(ap^k q^k;p)_j}{(q;q)_j} B_{j+k}(-x)^j q^{\binom{j}{2}}. \tag{3.7.11}$$

The $p = q$ case of (3.7.11) is due to Jackson [1910a].

In order to employ (3.6.22) to extend (3.7.10), replace n in (3.6.22) by j, multiply both sides by $B_{n+j}x^{n+j}(a/b)^j q^{j^2}$, sum from $j = 0$ to ∞, change the order of summation and then replace k by $k - n$ and j by $j + k - n$ to obtain

$$\begin{aligned} B_n x^n = & \left(1-\frac{a}{p}\right)\left(1-\frac{b}{p}\right) \sum_{k=n}^{\infty} \frac{1-aq^{2k-2n}/b}{(q;q)_{k-n}} x^k \\ & \times \sum_{j=0}^{\infty} \frac{(aq^{k-n}, bq^{n-k};p)_{j+k-n-1}}{(q;q)_j (aq^{k-n}/b;q)_{j+k-n+1}} \left(-\frac{a}{b}\right)^{j+k-n} \\ & \times (-x)^j B_{j+k} q^{(k-n)(j+k-n-1)+\binom{j}{2}+\binom{j+k-n+1}{2}}. \end{aligned} \tag{3.7.12}$$

Next we replace a by $ap^{n+1}q^n$, b by $bp^{n+1}q^{-n}$, multiply both sides by $A_n w^n$ and then sum from $n = 0$ to ∞ to get

$$\begin{aligned} \sum_{n=0}^{\infty} A_n B_n (xw)^n = & \sum_{k=0}^{\infty} \frac{(apq^k, bpq^{-k};p)_{k-1}}{(aq^k/b;q)_k} x^k \\ & \times \sum_{n=0}^{k} \frac{(1-ap^nq^n)(1-bp^nq^{-n})(aq^k/b;q)_n}{(q;q)_{k-n}(apq^k, bpq^{-k};p)_n} A_n w^n \\ & \times \sum_{j=0}^{\infty} \frac{(ap^kq^k, bp^kq^{-k};p)_j}{(q;q)_j (aq^{2k+1}/b;q)_j} \left(-\frac{a}{b}q^{2n}\right)^{j+k-n} \\ & \times B_{j+k}(-x)^j q^{(k-n)(j+k-n-1)+\binom{j}{2}+\binom{j+k-n+1}{2}}. \end{aligned} \tag{3.7.13}$$

This formula tends directly to (3.7.11) as $b \to 0$. By replacing A_n, B_n, x, w by suitable multiples, we may change (3.7.13) to an equivalent form which tends to (3.7.11) as $b \to \infty$. In addition, by replacing A_n, B_n, x, w by $A_n q^{2\binom{n}{2}}$, $B_n q^{-2\binom{n}{2}}, bx/a, aw/b$, respectively, we can write (3.7.13) in the simpler looking equivalent form

$$\begin{aligned} \sum_{n=0}^{\infty} A_n B_n x^n w^n = & \sum_{k=0}^{\infty} \frac{(apq^k, bpq^{-k};p)_{k-1}}{(q, aq^k/b;q)_k} (-x)^k q^{\binom{k+1}{2}} \\ & \times \sum_{n=0}^{k} \frac{(1-ap^nq^n)(1-bp^nq^{-n})(q^{-k}, aq^k/b;q)_n}{(apq^k, bpq^{-k};p)_n} A_n w^n \\ & \times \sum_{j=0}^{\infty} \frac{(ap^kq^k, bp^kq^{-k};p)_j}{(q, aq^{2k+1}/b;q)_j} B_{j+k} x^j q^j. \end{aligned} \tag{3.7.14}$$

As in the derivation of (3.7.6), one may extend (3.7.14) by replacing B_{j+k} by $B_{j+k}C_{n,j+k-n}$ with $C_{n,0} = 1$ for $n = 0, 1, \ldots$. Multivariable expansions, which are really special cases of (3.7.6) and (3.7.14), may be obtained by replacing A_n and B_n in (3.7.6) and (3.7.14) by multiple power series, see, e.g. Gasper [1989a], Ex. 3.22 and, in the hypergeometric limit case, Luke [1969]. For a multivariable special case of the Al-Salam and Verma expansion (3.7.11), see Srivastava [1984].

3.8 Quadratic, cubic, and quartic summation and transformation formulas

By setting $p = q^j$ or $q = p^j$, $j = 2, 3, \ldots$, in the bibasic summation formulas of §3.7 and using summation and transformation formulas for basic hypergeometric series, one can derive families of quadratic, cubic, etc. summation, transformation and expansion formulas. To illustrate this we shall derive a quadratic transformation formula containing five arbitrary parameters by starting with the $q = p^2$ case of (3.6.16)

$$\sum_{k=0}^{n} \frac{(1-adp^{3k})(1-b/dp^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(p^{-2n}, ad^2p^{2n}/b;p^2)_k}{(dp^2, adp^2/b;p^2)_k(adp^{2n+1}, bp^{1-2n}/d;p)_k} p^{2k}$$
$$= \frac{(1-d)(1-ad/b)(1-adp^{2n})(1-dp^{2n}/b)}{(1-ad)(1-d/b)(1-dp^{2n})(1-adp^{2n}/b)}, \tag{3.8.1}$$

where $n = 0, 1, \ldots$.

Change p to q and d to c in (3.8.1), multiply both sides by

$$\frac{(ac^2/b;q^2)_n(c/b;q)_{2n}}{(q^2;q^2)_n(acq;q)_{2n}} C_n$$

and sum over n to get

$$\sum_{n=0}^{\infty} \frac{(ac^2/b;q^2)_n(cq/b;q)_{2n}(1-c)(1-ac/b)}{(q^2;q^2)_n(ac;q)_{2n}(1-cq^{2n})(1-acq^{2n}/b)} C_n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)}$$
$$\times \frac{(a,b;q)_k(ac^2/b;q^2)_{n+k}(c/b;q)_{2n-k}}{(cq^2, acq^2/b;q^2)_k(q^2;q^2)_{n-k}(acq;q)_{2n+k}} \left(\frac{cq}{b}\right)^k q^{\binom{k}{2}} C_n$$
$$= \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b;q)_k(ac^2/b;q^2)_{2k}(c/b;q)_k}{(cq^2, acq^2/b;q^2)_k(acq;q)_{3k}} \left(\frac{cq}{b}\right)^k q^{\binom{k}{2}}$$
$$\times \sum_{m=0}^{\infty} \frac{(ac^2q^{4k}/b;q^2)_m(cq^k/b;q)_{2m}}{(q^2;q^2)_m(acq^{3k+1};q)_{2m}} C_{k+m}. \tag{3.8.2}$$

Setting

$$C_n = \frac{(1-ac^2q^{4n}/b)(d,e,f;q^2)_n(a^2q^3/def)^n}{(1-ac^2/b)(ac^2q^2/bd, ac^2q^2/be, ac^2q^2/bf;q^2)_n},$$

it follows from (3.8.2) that

$${}_{10}W_9(ac^2/b; ac/b, c, cq/b, cq^2/b, d, e, f; q^2, a^2c^2q^3/def)$$

$$= \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b,c/b;q)_k (ac^2q^2/b;q^2)_{2k}}{(cq^2, acq^2/b;q^2)_k (acq;q)_{3k}}$$
$$\times \frac{(d,e,f;q^2)_k (a^2c^3q^4/bdef)^k}{(ac^2q^2/bd, ac^2q^2/be, ac^2q^2/bf;q^2)_k} q^{\binom{k}{2}}$$
$$\times {}_8W_7(ac^2q^{4k}/b; cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k}, fq^{2k}; q^2, a^2c^2q^3/def). \quad (3.8.3)$$

If we now assume that

$$a^2c^2q = def, \quad (3.8.4)$$

then we can apply (2.11.7) to get

$${}_8W_7(ac^2q^{4k}/b; cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k}, fq^{2k}; q^2, q^2)$$
$$= \frac{(ac^2q^{4k+2}/b, bf/ac^2q^{2k}, abq^{2k+1}, acq^{k+2}/d;q^2)_\infty}{(acq^{3k+2}, acq^{3k+1}, ac^2q^{2k+2}/bd, ac^2q^{2k+2}/be;q^2)_\infty}$$
$$\times \frac{(acq^{k+2}/e, acq^{k+1}/d, acq^{k+1}/e, ac^2q^2/bde;q^2)_\infty}{(bef/ac^2, bdf/ac^2, f/acq^k, f/acq^{k-1};q^2)_\infty}$$
$$+ \frac{bfq^{-2k}(ac^2q^{4k+2}/b, cq^k/b, cq^{k+1}/b, dq^{2k}, eq^{2k};q^2)_\infty}{ac^2(ac^2q^{2k+2}/bf, acq^{3k+2}, acq^{3k+1}, ac^2q^{2k+2}/bd;q^2)_\infty}$$
$$\times \frac{(fq^2/e, fq^2/d, bfq^{2-2k}/ac^2, bfq^{k+1}/c, bfq^{k+2}/c;q^2)_\infty}{(ac^2q^{2k+2}/be, bef/ac^2, bdf/ac^2, bf^2q^2/ac^2, f/acq^k, f/acq^{k-1};q^2)_\infty}$$
$$\times {}_8W_7(bf^2/ac^2; fq^{2k}, bef/ac^2, bdf/ac^2, f/acq^k, f/acq^{k-1}; q^2, q^2), \quad (3.8.5)$$

which, combined with (3.8.3), gives

$${}_{10}W_9(ac^2/b; c, d, e, f, ac/b, cq/b, cq^2/b; q^2, q^2)$$
$$= \frac{(ac^2q^2/b, ac^2q^2/bde, abq, bf/ac^2;q^2)_\infty (acq/d, acq/e;q)_\infty}{(ac^2q^2/bd, ac^2q^2/be, bdf/ac^2, bef/ac^2;q^2)_\infty (acq, f/ac;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)(a,b,c/b;q)_k (d,e,f;q^2)_k}{(1-ac)(1-b/c)(cq^2, acq^2/b, abq;q^2)_k (acq/d, acq/e, acq/f;q)_k} q^{2k}$$
$$+ \frac{bf(ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2;q^2)_\infty}{ac^2(ac^2q^2/bf, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, bf^2q^2/ac^2;q^2)_\infty}$$
$$\times \frac{(bfq/c, c/b;q)_\infty}{(acq, f/ac;q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(1-acq^{3k})(1-b/cq^k)}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (ac^2/bf, f;q^2)_k}{(cq^2, acq^2/b;q^2)_k (bfq/c, acq/f;q)_k} q^{2k}$$
$$\times {}_8W_7(bf^2/ac^2; fq^{2k}, bdf/ac^2, bef/ac^2, f/acq^k, f/acq^{k-1}; q^2, q^2). \quad (3.8.6)$$

The last sum over k in (3.8.6) is

$$\sum_{j=0}^{\infty} \frac{(bf^2/ac^2, f, bdf/ac^2, bef/ac^2, f/ac, fq/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, bfq^2/ac^2, fq^2/d, fq^2/e, bfq^2/c, bfq/c; q^2)_j (1 - bf^2/ac^2)} q^{2j}$$
$$\times \sum_{k=0}^{\infty} \frac{(1 - acq^{3k})(1 - b/cq^k)}{(1 - ac)(1 - b/c)} \frac{(a, b; q)_k (ac^2/bfq^{2j}, fq^{2j}; q^2)_k}{(cq^2, acq^2/b; q^2)_k (bfq^{2j+1}/c, acq^{1-2j}/f; q)_k} q^{2k}$$
$$= \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, f, bdf/ac^2, bef/ac^2, f/ac, fq/ac; q^2)_j (1 - bf^2q^{4j}/ac^2)}{(q^2, bfq^2/ac^2, fq^2/d, fq^2/e, bfq^2/c, bfq/c; q^2)_j (1 - bf^2/ac^2)} q^{2j}$$
$$\times \frac{(1 - c)(1 - ac/b)(1 - ac/fq^{2j})(1 - bfq^{2j}/c)}{c(1 - ac)(1 - b/c)(1 - fq^{2j}/c)(1 - ac/bfq^{2j})}$$
$$\times \left\{ \frac{(a, b; q)_\infty (fq^{2j}, ac^2/bfq^{2j}; q^2)_\infty}{(c, ac/b; q^2)_\infty (ac/fq^{2j}, bfq^{2j}/c; q)_\infty} - 1 \right\}$$
$$= \frac{(1 - c)(1 - ac/b)(1 - f/ac)(1 - bf/c)}{(1 - c/b)(1 - ac)(1 - f/c)(1 - bf/ac)}$$
$$\times {}_{10}W_9(bf^2/ac^2; f, bdf/ac^2, bef/ac^2, bf/ac, f/c, fq/ac, fq^2/ac; q^2, q^2)$$
$$+ \frac{f}{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - bf/ac)}$$
$$\times \frac{(a, b; q)_\infty (f, ac^2/bf; q^2)_\infty}{(cq^2, acq^2/b; q^2)_\infty (acq/f, bfq/c; q)_\infty}$$
$$\times \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, bdf/ac^2, bef/ac^2, f/c, bf/ac; q^2)_j}{(q^2, fq^2/d, fq^2/e, bfq^2/ac, fq^2/c; q^2)_j} \frac{(1 - bf^2q^{4j}/ac^2)}{(1 - bf^2/ac^2)} \left(-\frac{fq^2}{ab}\right)^j q^{j^2} \tag{3.8.7}$$

by the $n = \infty$ case of (3.6.15). Thus,

$${}_{10}W_9(ac^2/b; c, d, e, f, ac/b, cq/b, cq^2/b; q^2, q^2) - \frac{bf(1 - c)(1 - ac/b)}{ac^2(1 - f/c)(1 - bf/ac)}$$
$$\times \frac{(ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2; q^2)_\infty}{(bf^2q^2/ac^2, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, ac^2q^2/bf; q^2)_\infty}$$
$$\times \frac{(bf/c, cq/b; q)_\infty}{(ac, fq/ac; q)_\infty} {}_{10}W_9\left(\frac{bf^2}{ac^2}; f, \frac{bdf}{ac^2}, \frac{bef}{ac^2}, \frac{f}{c}, \frac{bf}{ac}, \frac{fq}{ac}, \frac{fq^2}{ac^2}; q^2, q^2\right)$$
$$= \frac{(acq/d, acq/e; q)_\infty (ac^2q^2/b, abq, bf/ac^2, ac^2q^2/bde; q^2)_\infty}{(acq, f/ac; q)_\infty (ac^2q^2/bd, ac^2q^2/be, bdf/ac^2, bef/ac^2; q^2)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{1 - acq^{3k}}{1 - ac} \frac{(a, b, cq/b; q)_k (d, e, f; q^2)_k}{(cq^2, acq^2/b, abq; q)_k (acq/d, acq/e, acq/f; q)_k} q^k$$
$$+ \frac{bf^2(a, b, cq/b; q)_\infty}{a^2c^3(1 - f/c)(1 - bf/ac)(ac, f/ac, acq/f; q)_\infty}$$

$$\times \frac{(f, ac^2/bf, ac^2q^2/b, d, e, fq^2/d, fq^2/e, bfq^2/ac^2; q^2)_\infty}{(cq^2, acq^2/b, ac^2q^2/bf, ac^2q^2/be, ac^2q^2/bd, bdf/ac^2, bef/ac^2, bf^2q^2/ac^2; q^2)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(bf^2/ac^2, bdf/ac^2, bef/ac^2, f/c, bf/ac; q^2)_j}{(q^2, fq^2/d, fq^2/e, bfq^2/ac, fq^2/c; q^2)_j}$$

$$\times \frac{(1 - bf^2q^{4j}/ac^2)}{(1 - bf^2/ac^2)} \left(-\frac{fq^2}{ab}\right)^j q^{j^2} \tag{3.8.8}$$

when (3.8.4) holds.

Now observe that since

$$\sum_{j=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f; q)_j}{(q, \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f; q)_j} \left(-\frac{a^2q^2}{cdef}\right)^j q^{\binom{j}{2}}$$

$$= \frac{(aq, aq/ef; q)_\infty}{(aq/e, aq/f; q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} aq/cd, e, f \\ aq/c, aq/d \end{matrix} ; q, \frac{aq}{ef} \right] \tag{3.8.9}$$

by the $n \to \infty$ limit case of (2.5.1), the sum over j in (3.8.8) equals

$$\frac{(q^2, bf^2q^2/ac^2; q^2)_\infty}{(bfq^2/ac, fq^2/c; q^2)_\infty} \, {}_3\phi_2 \left[\begin{matrix} f/c, bf/ac, ac^2q^2/bde \\ fq^2/d, fq^2/e \end{matrix} ; q^2, q^2 \right]. \tag{3.8.10}$$

Hence, by setting $e = a^2c^2q/df$ in (3.8.8) we obtain the Gasper and Rahman [1990] quadratic transformation formula

$$\begin{aligned}
&{}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, d, a^2c^2q/df; q^2, q^2) \\
&\quad + \frac{(ac^2q^2/b, bf/ac^2, ac/b, c, cq/b, cq^2/b, bfq^2/ac; q^2)_\infty}{(bf^2q^2/ac^2, ac^2/bf, ac^2q^2/bd, dfq/ab, bdf/ac^2, abq/d, cq^2; q^2)_\infty} \\
&\quad \times \frac{(fq^2/c, bf/c, bfq/c, fq^2/d, df^2q/a^2c^2, d, a^2c^2q/df; q^2)_\infty}{(acq^2/b, f/c, bf/ac, ac, acq, fq/ac, fq^2/ac; q^2)_\infty} \\
&\quad \times {}_{10}W_9(bf^2/ac^2; f, bdf/ac^2, abq/d, f/c, bf/ac, fq/ac, fq^2/ac; q^2, q^2) \\
&\quad - \frac{(a, b, cq/b; q)_\infty}{(ac, ac/f, fq/ac; q)_\infty} \\
&\quad \times \frac{(f, d, a^2c^2q/df, bf/ac^2, ac^2q^2/b, fq^2/d, df^2q/a^2c^2, q^2; q^2)_\infty}{(bf/ac, f/c, cq^2, acq^2/b, ac^2q^2/bd, dfq/ab, bdf/ac^2, abq/d; q^2)_\infty} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} f/c, bf/ac, fq/ab \\ fq^2/d, df^2q/a^2c^2 \end{matrix} ; q^2, q^2 \right] \\
&= \frac{(acq/d, df/ac; q)_\infty (ac^2q^2/b, abq, bf/ac^2, fq/ab; q^2)_\infty}{(acq, f/ac; q)_\infty (ac^2q^2/bd, dfq/ab, bdf/ac^2, abq/d; q^2)_\infty} \\
&\quad \times \sum_{k=0}^{\infty} \frac{1 - acq^{3k}}{1 - ac} \frac{(a, b, cq/b; q)_k (d, f, a^2c^2q/df; q^2)_k q^k}{(cq^2, acq^2/b, abq; q^2)_k (acq/d, acq/f, df/ac; q)_k}.
\end{aligned} \tag{3.8.11}$$

Note that the first two terms on the left side of (3.8.11) containing the ${}_{10}W_9$ series can be transformed to another pair of ${}_{10}W_9$ series by applying the four-term transformation formula (2.12.9). Since the ${}_3\phi_2$ series in (3.8.11) is balanced it can be summed by (1.7.2) whenever it terminates. When $c = 1$ formula (3.8.11) reduces to the quadratic summation formula

$$\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a,b,q/b;q)_k (d,f,a^2q/df;q^2)_k}{(q^2,aq^2/b,abq;q^2)_k (aq/d,aq/f,df/a;q)_k} q^k$$
$$+ \frac{(aq,f/a,b,q/b;q)_\infty (d,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty (aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}$$
$$\times {}_3\phi_2 \left[\begin{matrix} f, bf/a, fq/ab \\ fq^2/d, df^2q/a^2 \end{matrix} ; q^2, q^2 \right]$$
$$= \frac{(aq,f/a;q)_\infty (aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty (aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}. \qquad (3.8.12)$$

By multiplying both sides of (3.8.11) by $(f/ac;q)_\infty$ and then setting $f = ac$ we obtain Rahman's [1993] quadratic transformation formula

$$\sum_{k=0}^{\infty} \frac{(a;q^2)_k (1-aq^{3k})(d,aq/d;q^2)_k (b,c,aq/bc;q)_k}{(q;q)_k (1-a)(aq/d,d;q)_k (aq^2/b,aq^2/c,bcq;q^2)_k} q^k$$
$$= \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty} {}_3\phi_2 \left[\begin{matrix} b, c, aq/bc \\ dq, aq^2/d \end{matrix} ; q^2, q^2 \right], \qquad (3.8.13)$$

provided d or aq/d is not of the form q^{-2n}, n a nonnegative integer. Also, the case $d = q^{-2n}$ of (3.8.11) gives

$$\sum_{k=0}^{n} \frac{1-acq^{3k}}{1-ac} \frac{(a,b,cq/b;q)_k (f,a^2c^2q^{2n+1}/f,q^{-2n};q^2)_k}{(cq^2,acq^2/b,abq;q^2)_k (acq/f,f/acq^{2n},acq^{2n+1};q)_k} q^k$$
$$= \frac{(acq;q)_{2n} (ac^2q^2/bf,abq/f;q^2)_n}{(acq/f;q)_{2n} (abq,ac^2q^2/b;q^2)_n}$$
$$\times {}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, q^2) \quad (3.8.14)$$

and the case $b = cq^{2n+1}$ gives

$$\sum_{k=0}^{2n} \frac{1-acq^{3k}}{1-ac} \frac{(d,f,a^2c^2q/df;q^2)_k (a,cq^{2n+1},q^{-2n};q)_k}{(acq/d,acq/f,df/ac;q)_k (cq^2,aq^{1-2n},acq^{2n+2};q^2)_k} q^k$$
$$= \frac{(acq^2,dq/ac,fq/ac,acq^2/df;q^2)_n}{(q/ac,acq^2/d,acq^2/f,dfq/ac;q^2)_n}$$
$$\times {}_{10}W_9(acq^{-2n-1}; c, d, f, a^2c^2q/df, aq^{-2n-1}, q^{1-2n}, q^{-2n}; q^2, q^2)$$
$$(3.8.15)$$

for $n = 0, 1, \ldots$.

Similarly, the special case

$$\sum_{k=0}^{n} \frac{(1-acq^{4k})(1-b/cq^{2k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k (q^{-3n},ac^2q^{3n}/b;q^3)_k}{(cq^3,acq^3/b;q^3)_k (acq^{3n+1},b/cq^{3n-1};q)_k} q^{3k}$$

$$= \frac{(1-c)(1-ac/b)(1-acq^{3n})(1-cq^{3n}/b)}{(1-ac)(1-c/b)(1-cq^{3n})(1-acq^{3n}/b)}, \quad n=0,1,2,\ldots, \tag{3.8.16}$$

of (3.6.16) is used in Gasper and Rahman [1990] to show that Gosper's sum (see Gessel and Stanton [1982])

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} a, a+1/2, b, 1-b, c, (2a+1)/3-c, a/2+1 \\ 1/2, (2a-b+3)/3, (2a+b+2)/3, 3c, 2a+1-3c, a/2 \end{matrix}; 1\right] \\
&= \frac{2}{\sqrt{3}} \frac{\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)\Gamma\left(\frac{2a+b+2}{3}\right)}{\Gamma\left(\frac{2a+2}{3}\right)\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{3c+b+1}{3}\right)\Gamma\left(\frac{3c+2-b}{3}\right)} \\
&\quad \times \frac{\Gamma\left(\frac{2+2a-3c}{3}\right)\Gamma\left(\frac{3+2a-3c}{3}\right)\sin\frac{\pi}{3}(b+1)}{\Gamma\left(\frac{2+2a+b-3c}{3}\right)\Gamma\left(\frac{3+2a-b-3c}{3}\right)}
\end{aligned} \tag{3.8.17}$$

has a q-analogue of the form

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, q/a; q)_k (ac; q)_{2k} (d, acq/d; q^3)_k}{(cq^3, a^2cq^2; q^3)_k (q; q)_{2k} (acq/d, d; q)_k} q^k \\
&= \frac{(acq^2, acq^3, d/ac, dq/ac, adq, aq, q^2/a, dq^2/a; q^3)_\infty}{(q, q^2, dq, dq^2, a^2cq^2, cq^3, dq/a^2c, d/c; q^3)_\infty} \\
&\quad + \frac{d(a, q/a, acq; q)_\infty (q^3, d, acq/d, d^2q^2/ac; q^3)_\infty}{ac(q, d, acq/d; q)_\infty (cq^3, a^2cq^2, d/c, dq/a^2c; q^3)_\infty} \\
&\quad \times {}_2\phi_1\left[\begin{matrix} d/c, dq/a^2c \\ d^2q^2/ac \end{matrix}; q^3, q^3\right]
\end{aligned} \tag{3.8.18}$$

and to derive the extension

$$\begin{aligned}
&{}_{10}W_9(ac^2/b; d, c, a^2bc/d, ac/b, cq/b, cq^2/b, cq^3/b; q^3, q^3) \\
&+ \frac{(1-c)(1-ac/b)}{(1-d/c)(1-bd/ac)} \frac{(cq/b, bd/c; q)_\infty (ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, dq/ac; q)_\infty (ac^2/bd, cdq^3/ab^2, ab^2/c, bd^2q^3/ac^2; q^3)_\infty} \\
&\times {}_{10}W_9(bd^2/ac^2; d, bd/ac, ab^2/c, d/c, dq/ac, dq^2/ac, dq^3/ac; q^3, q^3) \\
&- \frac{(a, b, cq/b; q)_\infty (q^3, d, ac^2q^3/b, a^2bc/d, d^2q^3/a^2bc, bd/ac^2; q^3)_\infty}{(ac, dq/ac, ac/d; q)_\infty (cq^3, acq^3/b, d/c, bd/ac, cdq^3/ab^2, ab^2/c; q^3)_\infty} \\
&\times {}_2\phi_1\left[\begin{matrix} d/c, bd/ac \\ d^2q^3/a^2bc \end{matrix}; q^3, q^3\right] \\
&= \frac{(ab, dq/ab; q)_\infty (bd/ac^2, ac^2q^3/b; q^3)_\infty}{(acq, d/ac; q)_\infty (ab^2/c, cdq^3/ab^2; q^3)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{1-acq^{4k}}{1-ac} \frac{(a, b; q)_k (cq/b; q)_{2k} (d, a^2bc/d; q^3)_k}{(cq^3, acq^3/b; q^3)_k (ab; q)_{2k} (acq/d, dq/ab; q)_k} q^k
\end{aligned} \tag{3.8.19}$$

and some other cubic transformation formulas.

The special case

$$\sum_{k=0}^{n} \frac{(1-acq^{5k})(1-b/cq^{3k})}{(1-ac)(1-b/c)} \frac{(a,b;q)_k(q^{-4n},ac^2q^{4n}/b;q^4)_k}{(cq^4,acq^4/b;q^4)_k(acq^{4n+1},b/cq^{4n-1};q)_k} q^{4k}$$
$$= \frac{(1-c)(1-ac/b)(1-acq^{4n})(1-cq^{4n}/b)}{(1-ac)(1-c/b)(1-cq^{4n})(1-acq^{4n}/b)}, \quad n=0,1,2,\ldots, \tag{3.8.20}$$

of (3.6.16) was used in Gasper and Rahman [1990] to derive the quartic transformation formula

$$\begin{aligned}
&{}_{10}W_9(ac^2/b;a^2b^2/q^2,ac/b,c,cq/b,cq^2/b,cq^3/b,cq^4/b;q^4,q^4)\\
&+\frac{(1-c)(1-ac/b)(a^2b^3/cq^2,cq/b;q)_\infty(ac^2q^4/b,ab^3/c^2q^2;q^4)_\infty}{(1-a^2b^2/cq^2)(1-ab^3/cq^2)(ab^2/cq,ac;q)_\infty(a^3b^5/c^2,c^2q^2/ab^3;q^4)_\infty}\\
&\times {}_{10}W_9\left(\frac{a^3b^5}{c^2q^4};\frac{a^2b^2}{q^2},\frac{a^2b^2}{cq^2},\frac{ab^3}{cq^2},\frac{ab^2}{cq},\frac{ab^2}{c},\frac{ab^2q}{c},\frac{ab^2q^2}{c};q^4,q^4\right)\\
&+\frac{ab^2(a,b,cq/b;q)_\infty(ac^2q^4/b,ab^3/c^2q^2,a^2b^2/q^2;q^4)_\infty}{cq^2(1-a^2b^2/cq^2)(ac,ab^2/cq^2,cq^3/ab^2;q)_\infty(ab^3/cq^2,cq^4,acq^4/b;q^4)_\infty}\\
&\times {}_1\phi_1\left[\begin{matrix}a^2b^2/cq^2\\ a^2b^2q^2/c\end{matrix};q^4,\frac{ab^3q^2}{c}\right]\\
&=\frac{(ab;q)_\infty(ab/q;q^2)_\infty(ac^2q^4/b,ab^3/c^2q^2;q^4)_\infty}{(acq,ab^2/cq^2;q)_\infty}\\
&\times\sum_{k=0}^{\infty}\frac{1-acq^{5k}}{1-ac}\frac{(a,b;q)_k(cq/b,cq^2/b,cq^3/b;q^3)_k(a^2b^2/q^2;q^4)_k}{(cq^4,acq^4/b;q^4)_k(abq,ab,ab/q;q^2)_k(cq^3/ab^2;q)_k}q^k.
\end{aligned} \tag{3.8.21}$$

When $b=q^2/a$ the sum of the two ${}_{10}W_9$ series in (3.8.21) reduces to a sum of two ${}_8W_7$ series, which can be summed by (2.11.7) to obtain the quartic summation formula

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1-acq^{5k}}{1-ac}\frac{(a,q^2/a;q)_k(ac/q,ac,acq;q^3)_k(q^2;q^4)_k}{(cq^4,a^2cq^2;q^4)_k(q^3,q^2,q;q^2)_k(ac/q;q)_k}q^k\\
&\quad-\frac{q^2(a,q^2/a,acq;q)_\infty(q^2;q^4)_\infty}{ac(1-q^2/c)(q^2,ac;q)_\infty(q;q^2)_\infty(cq^4,a^2cq^2,q^4/a^2c;q^4)_\infty}\\
&\quad\times{}_1\phi_1\left[\begin{matrix}q^2/c\\ q^6/c\end{matrix};q^4,\frac{q^8}{a^2c}\right]\\
&=\frac{(acq^2,q^2/ac,aq,q^3/a;q^2)_\infty}{(q,q^3;q^2)_\infty(cq^4,q^2/c,a^2cq^2,q^4/a^2c;q^4)_\infty}.
\end{aligned} \tag{3.8.22}$$

Additional quadratic, cubic and quartic summations and transformation formulas are given in the exercises.

3.9 Multibasic hypergeometric series

In view of the observation in §1.2 that a series $\sum_{n=0}^{\infty} v_n$ is a basic hypergeometric series in base q if $v_0 = 1$ and v_{n+1}/v_n is a rational function of q^n, a series $\sum_{n=0}^{\infty} v_n$ will be called a bibasic hypergeometric series in bases p and q if v_{n+1}/v_n is a rational function of p^n and q^n, and p and q are independent. More generally, we shall call a series $\sum_{n=0}^{\infty} v_n$ a multibasic (or m-basic) hypergeometric series in bases $q_1, \ldots, q_m$ if v_{n+1}/v_n is a rational function of $q_1^n, \ldots, q_m^n$, and $q_1, \ldots, q_m$ are independent. Similarly a bilateral series $\sum_{n=-\infty}^{\infty} v_n$ will be called a bilateral multibasic (or m-basic) hypergeometric series in bases $q_1, \ldots, q_m$ if v_{n+1}/v_n is a rational function of $q_1^n, \ldots, q_m^n$, and $q_1, \ldots, q_m$ are independent. Multibasic series are sometimes called polybasic series.

Since a multibasic series in bases $q_1, \ldots, q_m$ may contain products and quotients of q-shifted factorials $(a;q)_n$ with q replaced by $q_1^{k_1} \cdots q_m^{k_m}$ where $k_1, \ldots, k_m$ are arbitrary integers, the form of such a series could be quite complicated. Therefore, in working with multibasic series either the series are displayed explicitly or notations are employed which apply only to the series under consideration. For example, to shorten the displays of many of the formulas derived in §3.8 we employ the notation

$$\Phi\left[\begin{matrix} a_1, \ldots, a_r : c_{1,1}, \ldots, c_{1,r_1} : \ldots : c_{m,1}, \ldots, c_{m,r_m} \\ b_1, \ldots, b_s : d_{1,1}, \ldots, d_{1,s_1} : \ldots : d_{m,1}, \ldots, d_{m,s_m} \end{matrix}; q, q_1, \ldots, q_m; z\right] \tag{3.9.1}$$

to represent the $(m+1)$-basic hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} \times \prod_{j=1}^{m} \frac{(c_{j,1}, \ldots, c_{j,r_j}; q_j)_n}{(d_{j,1}, \ldots, d_{j,s_j}; q_j)_n} \left[(-1)^n q_j^{\binom{n}{2}}\right]^{s_j - r_j}. \tag{3.9.2}$$

The notation in (3.9.1) may be abbreviated by using the vector notations: $\mathbf{a} = (a_1, \ldots, a_r)$, $\mathbf{b} = (b_1, \ldots, b_s)$, $\mathbf{c}_j = (c_{j,1}, \ldots, c_{j,r_j})$, $\mathbf{d}_j = (d_{j,1}, \ldots, d_{j,s_j})$ and letting

$$\Phi\left[\begin{matrix} \mathbf{a} : \mathbf{c}_1 : \ldots : \mathbf{c}_m \\ \mathbf{b} : \mathbf{d}_1 : \ldots : \mathbf{d}_m \end{matrix}; q, q_1, \ldots, q_m; z\right] \tag{3.9.3}$$

denote the series in (3.9.2).

If in (2.2.2) we set

$$A_k = \frac{(a;q)_k}{(q^{-n};q)_k} z^k \prod_{j=1}^{m} \frac{(c_{j,1}, \ldots, c_{j,r_j}; q_j)_k}{(d_{j,1}, \ldots, d_{j,s_j}; q_j)_k} \left[(-1)^k q_j^{\binom{k}{2}}\right]^{s_j - r_j}, \tag{3.9.4}$$

then we obtain the expansion

$$\Phi\left[\begin{matrix} a, b, c : c_{1,1}, \ldots, c_{1,r_1} : \ldots : c_{m,1}, \ldots, c_{m,r_m} \\ aq/b, aq/c : d_{1,1}, \ldots, d_{1,s_1} : \ldots : d_{m,1}, \ldots, d_{m,s_m} \end{matrix}; q, q_1, \ldots, q_m; z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(aq/bc;q)_n (a;q)_{2n}}{(q, aq/b, aq/c; q)_n} \left(-\frac{bcz}{aq}\right)^n q^{-\binom{n}{2}}$$

$$\times \prod_{j=1}^{m} \frac{(c_{j,1},\ldots,c_{j,r_j};q_j)_n}{(d_{j,1},\ldots,d_{j,s_j};q_j)_n} \left[(-1)^n q_j^{\binom{n}{2}}\right]^{s_j-r_j}$$

$$\times \Phi\left[\begin{matrix} aq^{2n} & : c_{1,1}q_1^n,\ldots,c_{1,r_1}q_1^n : \ldots : \\ & : d_{1,1}q_1^n,\ldots,d_{1,s_1}q_1^n : \ldots : \end{matrix}\right.$$

$$\left.\begin{matrix} c_{m,1}q_m^n,\ldots,c_{m,r_m}q_m^n \\ d_{m,1}q_m^n,\ldots,d_{m,s_m}q_m^n \end{matrix} ;q,q_1,\ldots,q_m;\frac{bcz}{a}q^{-n-1}P_n\right] \tag{3.9.5}$$

where

$$P_n = \prod_{j=1}^{m} q_j^{n(s_j-r_j)}, \tag{3.9.6}$$

provided at least one c_{j,r_j} is a negative integer power of q_j so that the series terminate. Note that this formula is valid even if $a = q^{-k}$, $k = 0,1,\ldots$, in which case the upper limit of the sum on the right side can be replaced by $[k/2]$, where $[k/2]$ denotes the greatest integer less than or equal to $k/2$.

Similarly, use of the expansion formula (2.8.2) leads to the formula

$$\Phi\left[\begin{matrix} a,b,c,d : c_{1,1},\ldots,c_{1,r_1} : \ldots : c_{m,1},\ldots,c_{m,r_m} \\ aq/b,aq/c,aq/d : d_{1,1},\ldots,d_{1,s_1} : \ldots : d_{m,1},\ldots,d_{m,s_m} \end{matrix};q,q_1,\ldots,q_m;z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1-\lambda q^{2n})(\lambda,\lambda b/a,\lambda c/a,\lambda d/a;q)_n}{(1-\lambda)(q,aq/b,aq/c,aq/d;q)_n} \frac{(a;q)_{2n}}{(\lambda q;q)_{2n}} \left(\frac{az}{\lambda}\right)^n$$

$$\times \prod_{j=1}^{m} \frac{(c_{j,1},\ldots,c_{j,r_j};q_j)_n}{(d_{j,1},\ldots,d_{j,s_j};q)_n} \left[(-1)^n q_j^{\binom{n}{2}}\right]^{s_j-r_j}$$

$$\times \Phi\left[\begin{matrix} aq^{2n},\frac{a}{\lambda} : c_{1,1}q_1^n,\ldots,c_{1,r_1}q_1^n : \ldots : c_{m,1}q_m^n,\ldots,c_{m,r_m}q_m^n \\ \lambda q^{2n+1} : d_{1,1}q_1^n,\ldots,d_{1,s_1}q_1^n : \ldots : d_{m,1}q_m^n,\ldots,d_{m,s_m}q_m^n \end{matrix};q,q_1,\ldots,q_m;zP_n\right], \tag{3.9.7}$$

where $\lambda = qa^2/bcd$. In (3.9.7) the series on both sides need not terminate as long as they converge absolutely. Formulas (3.9.5) and (3.9.7) are multibasic extensions of formulas 4.3(1) and 4.3(6), respectively, in Bailey [1935].

3.10 Transformations of series with base q to series with base q^2

Following Nassrallah and Rahman [1981], we shall derive some quadratic transformation formulas for basic hypergeometric series by using the following special cases of (3.9.5) and (3.9.7):

$$\Phi\left[\begin{matrix} a^2,-aq^2,b^2,c^2 : a_1,\ldots,a_r,q^{-n} \\ -a,a^2q^2/b^2,a^2q^2/c^2 : b_1,\ldots,b_r,b_{r+1} \end{matrix};q^2,q;z\right]$$

$$= \sum_{j=0}^{n} \frac{(a^2q^2/b^2c^2, -aq^2; q^2)_j (a^2; q^2)_{2j} (a_1, \ldots, a_r, q^{-n}; q)_j}{(q^2, -a, a^2q^2/b^2, a^2q^2/c^2; q^2)_j (b_1, \ldots, b_r, b_{r+1}; q)_j} q^{-2\binom{j}{2}} \left(-\frac{b^2c^2z}{a^2q^2}\right)^j$$

$$\times \Phi \left[\begin{matrix} a^2q^{4j}, -aq^{2j+2} : a_1q^j, \ldots, a_rq^j, q^{j-n} \\ -aq^{2j} : b_1q^j, \ldots, b_rq^j, b_{r+1}q^j \end{matrix} ; q^2, q; \frac{b^2c^2z}{a^2q^{2j+2}} \right] \tag{3.10.1}$$

and

$$\Phi \left[\begin{matrix} a^2, -aq^2, b^2, c^2, d^2 : a_1, \ldots, a_r, q^{-n} \\ -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : b_1, \ldots, b_r, b_{r+1} \end{matrix} ; q^2, q; z \right]$$

$$= \sum_{j=0}^{n} \frac{1-\lambda q^{4j}}{1-\lambda} \frac{(\lambda, \lambda b^2/a^2, \lambda c^2/a^2, \lambda d^2/a^2, -aq^2; q^2)_j (a^2; q^2)_{2j}}{(q^2, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2, -a; q^2)_j (\lambda q^2; q^2)_{2j}}$$

$$\times \frac{(a_1, \ldots, a_r, q^{-n}; q)_j}{(b_1, \ldots, b_r, b_{r+1}; q)_j} \left(\frac{a^2z}{\lambda}\right)^j$$

$$\times \Phi \left[\begin{matrix} a^2q^{4j}, -aq^{2j+2}, a^2/\lambda : a_1q^j, \ldots, a_rq^j, q^{j-n} \\ -aq^{2j}, \lambda q^{4j+2} : b_1q^j, \ldots, b_rq^j, b_{r+1}q^j \end{matrix} ; q^2, q; z \right], \tag{3.10.2}$$

respectively, where $\lambda = \left(qa^2/bcd\right)^2$.

Let us first consider the $r = 1$ case of (3.10.1). If we set $a_1 = -aq/w$, $b_1 = w, b_2 = -aq^{n+1}$, $z = awq^{n+2}/b^2c^2$ and apply (1.2.40), the series on the right side of (3.10.1) reduces to a VWP-balanced ${}_6\phi_5$ series in base q, and hence can be summed by (2.4.2). This gives the transformation formula

$$\Phi \left[\begin{matrix} a^2, -aq^2, b^2, c^2 : -aq/w, q^{-n} \\ -a, a^2q^2/b^2, a^2q^2/c^2 : w, -aq^{n+1} \end{matrix} ; q^2, q; \frac{awq^{n+2}}{b^2c^2} \right]$$

$$= \frac{(w/a, -aq; q)_n}{(w, -q; q)_n} {}_5\phi_4 \left[\begin{matrix} a, aq, a^2q^2/w^2, a^2q^2/b^2c^2, q^{-2n} \\ a^2q^2/b^2, a^2q^2/c^2, aq^{1-n}/w, aq^{2-n}/w \end{matrix} ; q^2, q^2 \right]. \tag{3.10.3}$$

Note that the above ${}_5\phi_4$ series is balanced and that the Φ series on the left side of (3.10.3) can be written as

$${}_{10}W_9(-a; a, b, -b, c, -c, -aq/w, q^{-n}; q, awq^{n+2}/b^2c^2).$$

Formula (3.10.3) is a q-analogue of Bailey [1935, 4.5(1)]. By reversing the series on both sides of (3.10.3) and relabelling the parameters, this formula can be written, as in Jain and Verma [1980], in the form

$${}_{10}\phi_9 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, x, -x, y, -y, -q^{-n}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/x, -aq/x, aq/y, -aq/y, -aq^{n+1}, aq^{n+1} \end{matrix} ; q, -\frac{a^3q^{2n+3}}{bx^2y^2} \right]$$

$$= \frac{(a^2q^2, a^2q^2/x^2y^2; q^2)_n}{(a^2q^2/x^2, a^2q^2/y^2; q^2)_n} {}_5\phi_4 \left[\begin{matrix} q^{-2n}, x^2, y^2, -aq/b, -aq^2/b \\ x^2y^2q^{-2n}/a^2, a^2q^2/b^2, -aq, -aq^2 \end{matrix} ; q^2, q^2 \right]. \tag{3.10.4}$$

For a nonterminating extension of (3.10.4) see Jain and Verma [1982].

Since the ${}_5\phi_4$ series on the right side of (3.10.3) is balanced, it can be summed by (1.7.2) whenever it reduces to a ${}_3\phi_2$ series. Thus, we obtain the summation formulas:

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2 : -aq/w, q^{-n} \\ a, -a : w, -aq^{n+1} \end{matrix}; q^2, q; \frac{wq^{n-1}}{a}\right]$$
$$= \frac{(-aq, aq^2/w, w/aq; q)_n}{(-q, aq/w, w; q)_n}, \tag{3.10.5}$$

$$\Phi\left[\begin{matrix} a^2, -aq^2, b^2 : -aq^n/b^2, q^{-n} \\ -a, a^2q^2/b^2 : b^2q^{1-n}, -aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(q^n/b^2, -aq; q)_n (aq/b^2, aq^2/b^2; q^2)_n}{(aq^n/b^2, -q; q)_n (q/b^2, a^2q^2/b^2; q^2)_n}, \tag{3.10.6}$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2 : -aq^n/b^2, q^{-n} \\ a, -a, a^2q^2/b^2 : b^2q^{1-n}, -aq^{n+1} \end{matrix}; q^2, q; q\right]$$
$$= \frac{(-aq, a/b^2; q)_n (1/b^2; q^2)_n}{(-q, 1/b^2; q)_n (a^2q^2/b^2; q^2)_n} q^n \tag{3.10.7}$$

and

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2 : -aq^{n-1}/b^2, q^{-n} \\ a, -a, a^2q^2/b^2 : b^2q^{2-n}, -aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(-aq, a/qb^2; q)_n (aq/b^2, 1/b^2q^2; q^2)_n}{(-q, 1/qb^2; q)_n (a/qb^2, a^2q^2/b^2; q^2)_n} q^n. \tag{3.10.8}$$

These are q-analogues of formulas 4.5(1.1) - 4.5(1.4) in Bailey [1935]. Since the series on the left sides of (3.10.5) and (3.10.6) can also be written as VWP-balanced ${}_8\phi_7$ series in base q, which are transformable to balanced ${}_4\phi_3$ series by Watson's formula (2.5.1), formulas (3.10.5) and (3.10.6) are equivalent to the summation formulas

$${}_4\phi_3\left[\begin{matrix} a, qa^{\frac{1}{2}}, -w/qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, w, -a^{\frac{1}{2}}q^{1-n} \end{matrix}; q, q\right]$$
$$= \frac{(w/aq, -a^{\frac{1}{2}}, aq^2/w; q)_n}{(w, -a^{-\frac{1}{2}}, aq/w; q)_n} \tag{3.10.9}$$

and

$${}_4\phi_3\left[\begin{matrix} a, b, -bq^{1-n}, q^{-n} \\ aq/b, b^2q^{1-n}, -bq^{-n} \end{matrix}; q, q\right]$$
$$= \frac{(1+1/b)(1+a^{\frac{1}{2}}q^n/b)(a/b^2, qa^{\frac{1}{2}}/b, 1/b; q)_n}{(1+q^n/b)(1+a^{\frac{1}{2}}/b)(aq/b, a^{\frac{1}{2}}/b, 1/b^2; q)_n}, \tag{3.10.10}$$

respectively, which are closer to what one would expect q-analogues of formulas 4.5(1.1) and 4.5(1.2) in Bailey [1935] to look like.

It is also of interest to note that if we set $c^2 = aq$ in (3.10.3), rewrite the Φ series on the left side as an ${}_8\phi_7$ series in base q and transform it to a balanced ${}_4\phi_3$ series, we obtain

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, b, -w/b, q^{-n} \\ aq/b, -bq^{-n}, w \end{matrix}; q, q\right] \\
&= \frac{(w/a, -aq/b; q)_n}{(w, -q/b; q)_n}\, {}_4\phi_3\left[\begin{matrix} a, a^2q^2/w^2, aq/b^2, q^{-2n} \\ a^2q^2/b^2, aq^{1-n}/w, aq^{2-n}/w \end{matrix}; q^2, q^2\right], \qquad (3.10.11)
\end{aligned}
$$

which is a q-analogue of the $c = (1+a)/2$ case of Bailey [1935, 4.5(1)]. Using (2.10.4), the left side of (3.10.11) can be transformed to give

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, aq/b^2, -aq/w \\ aq/b, -aq/b, aq^{1-n}/w \end{matrix}; q, q\right] \\
&= {}_4\phi_3\left[\begin{matrix} q^{-2n}, a, aq/b^2, a^2q^2/w^2 \\ a^2q^2/b^2, aq^{1-n}/w, aq^{2-n}/w \end{matrix}; q^2, q^2\right]. \qquad (3.10.12)
\end{aligned}
$$

This formula was first proved by Singh [1959] and more recently by Askey and Wilson [1985]. The latter authors also wrote it in the form

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a^2, b^2, c, d \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -cd \end{matrix}; q, q\right] \\
&= {}_4\phi_3\left[\begin{matrix} a^2, b^2, c^2, d^2 \\ qa^2b^2, -cd, -qcd \end{matrix}; q^2, q^2\right], \qquad (3.10.13)
\end{aligned}
$$

provided that both series terminate.

Now that we have the summation formulas (3.10.5)–(3.10.8), we can use them to produce additional transformation formulas. Set $r = 3$ and $a_1 = -a_2 = qa^{\frac{1}{2}}, a_3 = -aq/w, b_1 = -b_2 = a^{\frac{1}{2}}$, $b_4 = -aq^{n+1}$ in (3.10.1). The Φ series on the right side can now be summed by (3.10.5) and this leads to the following q-analogue of Bailey [1935, 4.5(2)]

$$
\begin{aligned}
&\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2, c^2 : -aq/w, q^{-n} \\ a, -a, a^2q^2/b^2, a^2q^2/c^2 : w, -aq^{n+1} \end{matrix}; q^2, q; \frac{awq^{n+1}}{b^2c^2}\right] \\
&= \frac{(-aq, aq^2/w, w/aq; q)_n}{(-q, aq/w, w; q)_n} \\
&\quad \times {}_5\phi_4\left[\begin{matrix} aq, aq^2, a^2q^2/b^2c^2, a^2q^2/w^2, q^{-2n} \\ a^2q^2/b^2, a^2q^2/c^2, aq^{2-n}/w, aq^{3-n}/w \end{matrix}; q^2, q^2\right]. \qquad (3.10.14)
\end{aligned}
$$

Let us now turn to applications of (3.10.2). If we set $r = 1$, $a_1 = -\lambda q^{n+1}/a$, $b_1 = a^2q^{-n}/\lambda$, $b_2 = -aq^{n+1}$, $z = q$ in (3.10.2), where $\lambda = (qa^2/bcd)^2$, then the Φ series on the right side reduces to a balanced very-well-poised ${}_8\phi_7$ series in base q which can be summed by Jackson's formula (2.6.2). Thus, we obtain the transformation formula

$$
\Phi\left[\begin{matrix} a^2, -aq^2, b^2, c^2, d^2 : -\lambda q^{n+1}/a, q^{-n} \\ -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : a^2q^{-n}/\lambda, -aq^{n+1} \end{matrix}; q^2, q; q\right]
$$

$$= \frac{(-aq, \lambda q/a; q)_n (\lambda q^2/a^2; q^2)_n}{(-q, \lambda q/a^2; q)_n (\lambda q^2; q^2)_n}$$

$$\times {}_{10}\phi_9 \left[\begin{matrix} \lambda, q^2\lambda^{\frac{1}{2}}, -q^2\lambda^{\frac{1}{2}}, a, aq, \frac{\lambda b^2}{a^2}, \frac{\lambda c^2}{a^2}, \frac{\lambda d^2}{a^2}, \frac{\lambda^2 q^{2n+2}}{a^2}, q^{-2n} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, \frac{\lambda q^2}{a}, \frac{\lambda q}{a}, \frac{a^2q^2}{b^2}, \frac{a^2q^2}{c^2}, \frac{a^2q^2}{d^2}, \frac{a^2q^{-2n}}{\lambda}, \lambda q^{2n+2} \end{matrix} ; q^2, \frac{qa^2}{\lambda} \right]. \tag{3.10.15}$$

This is a q-analogue of Bailey [1935, 4.5(3)].

Similarly, setting $r = 3$ and choosing the parameters so that the Φ series on the right side of (3.10.2) can be summed by (3.10.7), we get the following q-analogue of Bailey [1935, 4.5(4)]

$$\Phi \left[\begin{matrix} a^2, aq^2, -aq^2, b^2, c^2, d^2 : -\lambda q^n/a, q^{-n} \\ a, -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : a^2q^{1-n}/\lambda, -aq^{n+1} \end{matrix} ; q^2, q; q \right]$$

$$= \frac{(\lambda/a, -aq; q)_n (\lambda/a^2; q^2)_n}{(\lambda/a^2, -q; q)_n (\lambda q^2; q^2)_n} q^n$$

$$\times {}_{10}\phi_9 \left[\begin{matrix} \lambda, q^2\lambda^{\frac{1}{2}}, -q^2\lambda^{\frac{1}{2}}, aq, aq^2, \frac{\lambda b^2}{a^2}, \frac{\lambda c^2}{a^2}, \frac{\lambda d^2}{a^2}, \frac{\lambda^2 q^{2n}}{a^2}, q^{-2n} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, \frac{\lambda q}{a}, \frac{\lambda}{a}, \frac{a^2q^2}{b^2}, \frac{a^2q^2}{c^2}, \frac{a^2q^2}{d^2}, \frac{a^2q^{2-2n}}{\lambda}, \lambda q^{2n+2} \end{matrix} ; q^2, \frac{qa^2}{\lambda} \right], \tag{3.10.16}$$

where $\lambda = (qa^2/bcd)^2$.

Exercises

3.1 Deduce from (3.10.13) that

$$ {}_3\phi_2 \left[\begin{matrix} a^2, b^2, z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}} \end{matrix} ; q, q \right] = {}_3\phi_2 \left[\begin{matrix} a^2, b^2, z^2 \\ a^2b^2q, 0 \end{matrix} ; q^2, q^2 \right], $$

provided that both series terminate. Show that this formula is a q-analogue of Gauss' quadratic transformation formula (3.1.7) when the series terminate.

3.2 Using the sum

$$ {}_2\phi_1 \left(q^{-n}, q^{1-n}; qb^2; q^2, q^2\right) = \frac{(b^2; q^2)_n}{(b^2; q)_n} q^{-\binom{n}{2}}, $$

prove that

$$\text{(i)} \quad {}_3\phi_2 \left[\begin{matrix} a, b, -b \\ b^2, az \end{matrix} ; q, -z \right] = \frac{(z; q)_\infty}{(az; q)_\infty} {}_2\phi_1 \left(a, aq; qb^2; q^2, z^2\right), \ |z| < 1.$$

Use this formula to prove that

(ii)
$$ {}_3\phi_2\left[\begin{matrix} a^2, & ab, & -ab \\ & a^2b^2, & -za^2 \end{matrix}; q, z\right] $$
$$ = \frac{(a^2z^2;q^2)_\infty}{(z,-a^2z;q)_\infty}\, {}_2\phi_2\left[\begin{matrix} a^2, & b^2 \\ a^2b^2q, & a^2z^2 \end{matrix}; q^2, a^2z^2q\right], \quad |z|<1. $$

Show that

(iii)
$$ \begin{aligned} &{}_2\phi_1(b,-b;b^2;q,z) \\ &= (-z;q)_\infty\ {}_2\phi_1(0,0;qb^2;q^2,z^2) \\ &= \frac{1}{(z;q)_\infty}\ {}_0\phi_1(-;qb^2;q^2,qb^2z^2). \end{aligned} $$

Formulas (i) and (ii), due to Jain [1981], are q-analogues of (3.1.5) and (3.1.6), respectively. T. Koornwinder in a letter (1990) suggested part (iii).

3.3 Show that
$$ {}_3\phi_2\left[\begin{matrix} a, q/a, z \\ c, -q \end{matrix}; q, q\right] $$
$$ = \frac{(-1,-qz/c;q)_\infty}{(-q/c,-z;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} c/a, ac/q, z^2 \\ c^2, 0 \end{matrix}; q^2, q^2\right], $$
when the series terminate. This is a q-analogue of
$$ {}_2F_1(a, 1-a; c; z) = (1-z)^{c-1}\ {}_2F_1\left((c-a)/2, (c+a-1)/2; c; 4z(1-z)\right), $$
when the series terminate.

3.4 Show that
$$ \sum_{k=0}^{n} \frac{(q^{-n}, b, -b; q)_k}{(q, b^2; q)_k}\, q^{nk-\binom{k}{2}} $$
vanishes if n is an odd integer. Evaluate the sum when n is even. Hence, or otherwise, show that
$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a,c;q)_{m+n}(b^2;q^2)_m}{(q;q)_m(q;q)_n(d;q)_{m+n}(b^2;q)_m}(-z)^m z^n $$
$$ = {}_4\phi_3\left[\begin{matrix} a, aq, c, cq \\ d, dq, qb^2 \end{matrix}; q^2, z^2\right], \quad |z|<1. $$

Deduce that
$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1-n}, a, aq \\ qb^2, d, dq \end{matrix}; q^2, q^2\right] $$
$$ = \frac{(d/a;q)_n}{(d;q)_n} a^n\ {}_4\phi_2\left[\begin{matrix} q^{-n}, a, b, -b \\ b^2, aq^{1-n}/d \end{matrix}; q, -\frac{q}{d}\right]. $$

(Jain [1981])

3.5 By using Sears' summation formula (2.10.12) show that

$$\sum_{r=0}^{\infty} \frac{(a, aq/e; q)_r}{(q, abq/e, acq/e; q)_r} A_r$$
$$= \frac{(aq/e, bq/e, cq/e, abcq/e; q)_\infty}{(q/e, abq/e, acq/e, bcq/e; q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k q^k}{(q, e, abcq/e; q)_k} \sum_{r=0}^{\infty} \frac{(aq^k; q)_r}{(q, abcq^{k+1}/e; q)_r} A_r$$
$$+ \frac{(a, b, c, abcq^2/e^2; q)_\infty}{(e/q, abq/e, acq/e, bcq/e; q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(aq/e, bq/e, cq/e; q)_k}{(q, q^2/e, abcq^2/e^2; q)_k} q^k \sum_{r=0}^{\infty} \frac{(aq^{k+1}/e; q)_r}{(q, abcq^{k+2}/e^2; q)_r} A_r,$$

where a, b, c, e are arbitrary parameters such that $e \neq q$, and $\{A_r\}$ is an arbitrary sequence such that the infinite series on both sides converge absolutely.

3.6 Prove that

$${}_3\phi_2 \left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix} ; q, q \right] + \frac{(q/e, a, b, c, dq/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, d; q)_\infty}$$
$$\times {}_3\phi_2 \left[\begin{matrix} aq/e, & bq/e, & cq/e \\ & q^2/e, & dq/e \end{matrix} ; q, q \right]$$
$$= \frac{(q/e, abq/e, acq/e, d/a; q)_\infty}{(d, aq/e, bq/e, cq/e; q)_\infty} {}_3\phi_2 \left[\begin{matrix} a, aq/e, abcq/de \\ abq/e, acq/e \end{matrix} ; q, \frac{d}{a} \right],$$

where $e \neq q$ and $|d/a| < 1$.

3.7 Prove that

$${}_3\phi_2 \left[\begin{matrix} a, aq/e, e/bc \\ abq/e, acq/e \end{matrix} ; q, -\frac{bcq}{e} \right]$$
$$= \frac{(aq, aq/e, bq/e, cq/e, -q; q)_\infty}{(q/e, abq/e, acq/e, bcq/e, -bcq/e; q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k \, (ab^2c^2q^{k+2}/e^2; q^2)_\infty}{(q, aq, e; q)_k (aq^{k+2}; q^2)_\infty} q^k,$$

provided $|bcq/e| < 1$.

3.8 Assuming that $|x| < 1$ and $a/b \neq q^j$, $j = 0, \pm 1, \pm 2, \ldots$, prove that

$${}_2\phi_1(a, b; c; q, x)$$
$$= \frac{(b, c/a, ax; q)_\infty}{(b/a, c, x; q)_\infty} {}_3\phi_2 \left[\begin{matrix} a, c/b, 0 \\ aq/b, ax \end{matrix} ; q, q \right]$$
$$+ \frac{(a, c/b, bx; q)_\infty}{(a/b, c, x; q)_\infty} {}_3\phi_2 \left[\begin{matrix} b, c/a, 0 \\ bq/a, bx \end{matrix} ; q, q \right].$$

Show that this is a q-analogue of the formula

$$
\begin{aligned}
&{}_2F_1(a,b;c;x)\\
&= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-x)^{-a}\ {}_2F_1(a,c-b;a-b+1;(1-x)^{-1})\\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-x)^{-b}\ {}_2F_1(b,c-a;b-a+1;(1-x)^{-1}).
\end{aligned}
$$

3.9 Show that

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} a, \lambda q, b \\ \lambda, q\lambda^2/b \end{matrix}; q, \frac{\lambda^2}{ab^2}\right]\\
&= \frac{1-\lambda+\lambda/b(1-\lambda/a)}{(1-\lambda)(1+\lambda/b)}\ \frac{(\lambda^2/b^2, q\lambda^2/ab;q)_\infty}{(q\lambda^2/b, \lambda^2/ab^2;q)_\infty}, \ |\lambda^2/ab^2|<1.
\end{aligned}
$$

3.10 Show that

$$
\begin{aligned}
&{}_8W_7\left(-\lambda; q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, a, b, -b; q, \lambda/ab^2\right)\\
&= \frac{(\lambda/a, \lambda q, -\lambda q, \lambda/b^2, \lambda q/ab, -\lambda q/ab;q)_\infty}{(\lambda, \lambda q/a, -\lambda q/a, \lambda q/b, -\lambda q/b, \lambda/ab^2;q)_\infty}, \ |\lambda/ab^2|<1.
\end{aligned}
$$

Show that this is a q-analogue of the formula

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & 1+\lambda/2, & b \\ & \lambda/2, & 1+\lambda-b \end{matrix}; 1\right]\\
&= \frac{\Gamma(\lambda/2)\Gamma\left(1+\frac{\lambda-a}{2}\right)\Gamma(1+\lambda-b)\Gamma(\lambda-a-2b)}{\Gamma(1+\lambda/2)\Gamma\left(\frac{\lambda-a}{2}\right)\Gamma(1+\lambda-a-b)\Gamma(\lambda-2b)}, \ \mathrm{Re}\,(\lambda-a-2b)>0.
\end{aligned}
$$

3.11 Derive Jackson's [1941] product formula

$$
\begin{aligned}
&{}_2\phi_1(a^2,b^2;qa^2b^2;q^2,z)\ {}_2\phi_1(a^2,b^2;qa^2b^2;q^2,qz)\\
&= {}_4\phi_3\left[\begin{matrix} a^2, b^2, ab, -ab \\ a^2b^2, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}} \end{matrix}; q, z\right], \quad |z|<1,\ |q|<1,
\end{aligned}
$$

and show that it has Clausen's [1828] formula

$$
\left[{}_2F_1\left(a,b;a+b+\frac{1}{2};z\right)\right]^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+\frac{1}{2} \end{matrix}; z\right], \quad |z|<1,
$$

as a limit case. Additional q-analogues of Clausen's formula are given in §8.8.

3.12 Prove that

$$
\begin{aligned}
&\Phi\left[\begin{matrix} q^{-2n}, -q^{2-n}, b^2, c^2 : d, -q^{1-n}/w \\ -q^{-n}, q^{2-2n}/b^2, q^{2-2n}/c^2 : -q^{1-n}/d, w \end{matrix}; q^2, q; \frac{wq^{2-n}}{b^2c^2d}\right]\\
&= \frac{(-1, w/d;q)_n}{(-d, w;q)_n} d^n\\
&\quad \times {}_5\phi_4\left[\begin{matrix} d^2, q^{2-2n}/b^2c^2, q^{2-2n}/w^2, q^{-n}, q^{1-n} \\ q^{2-2n}/b^2, q^{2-2n}/c^2, dq^{1-n}/w, dq^{2-n}/w \end{matrix}; q^2, q^2\right].
\end{aligned}
$$

3.13 If $\lambda = a^4q^2/b^2c^2d^2$, show that

$$\Phi\left[\begin{matrix} a^2, -q^2a, b^2, c^2, d^2 : -\lambda q^n/a, q^{-n} \\ -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : a^2q^{1-n}/\lambda, -aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$

$$= \frac{(-aq, \lambda/a; q)_n(\lambda/a^2, \lambda q^2/a; q^2)_n}{(-q, \lambda/a^2; q)_n(\lambda q^2, \lambda/a; q^2)_n}$$

$$\times {}_{10}W_9(\lambda; a, aq, \lambda b^2/a^2, \lambda c^2/a^2, \lambda d^2/a^2, \lambda^2q^{2n}/a^2, q^{-2n}; q^2, a^2q^3/\lambda)$$

and

$$\Phi\left[\begin{matrix} a^2, q^2a, -q^2a, b^2, c^2, d^2 : -\lambda q^{n-1}/a, q^{-n} \\ a, -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : a^2q^{2-n}/\lambda, -aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$

$$= \frac{(-aq, \lambda/aq; q)_n(\lambda/a^2q^2; q^2)_n(1-\lambda q^{2n-1}/a)}{(-q, \lambda/qa^2; q)_n(\lambda q^2; q^2)_n(1-\lambda q^{-1}/a)} q^n$$

$$\times {}_{10}W_9(\lambda; aq, aq^2, \lambda b^2/a^2, \lambda c^2/a^2, \lambda d^2/a^2, \lambda^2q^{2n-2}/a^2, q^{-2n}; q^2, a^2q^3/\lambda).$$

(Nassrallah and Rahman [1981])

3.14 Using (3.4.7) show that the q-Bessel function defined in Ex. 1.24 can be expressed as

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q, -x^2q^{\nu+1}/4; q)_\infty}\left(\frac{x}{2}\right)^\nu$$

$$\times \sum_{n=0}^{\infty} \frac{(-x^2q^\nu/4;q)_n(1+x^2q^{2n+\nu}/4)(-x^2/4;q)_n}{(q;q)_n(1+x^2q^\nu/4)(q^{\nu+1};q)_n}\left(\frac{-x^2q^{2\nu}}{4}\right)^n q^{2n^2}.$$

3.15 Prove that

$$J_\nu^{(2)}(x;q) = \frac{(-ix/2, -ixq^{\nu+1}/2; q)_\infty}{\Gamma_q(\nu+1)}\left(\frac{x}{2(1-q)}\right)^\nu$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1} \\ q^{2\nu+1}, -ixq^{\nu+1}/2 \end{matrix}; q, -ix/2\right]$$

$$= \frac{(ix/2;q^{1/2})_\infty}{\Gamma_q(\nu+1)}\left(\frac{x}{2(1-q)}\right)^\nu {}_2\phi_1\left[\begin{matrix} q^{\frac{\nu}{2}+\frac14}, -q^{\frac{\nu}{2}+\frac14} \\ q^{\nu+\frac12} \end{matrix}; q^{1/2}, \frac{ix}{2}\right]$$

for $|x| < 2$.
(Rahman [1987])

3.16 Show that

$${}_4\phi_3\left[\begin{matrix} a, -qa^{\frac12}, b, c \\ -a^{\frac12}, aq/b, aq/c \end{matrix}; q, \frac{axq}{bc}\right]$$

$$= \frac{(1-xa^{\frac12})(axq;q)_\infty}{(x;q)_\infty}\, {}_5\phi_4\left[\begin{matrix} a^{\frac12}, -qa^{\frac12}, (aq)^{\frac12}, -(aq)^{\frac12}, aq/bc \\ aq/b, aq/c, q/x, axq \end{matrix}; q, q\right]$$

$$+ \frac{(1-a^{\frac12})(aq, aq/bc, axq/b, axq/c; q)_\infty}{(aq/b, aq/c, axq/bc, 1/x; q)_\infty}$$

$$\times {}_5\phi_4\left[\begin{matrix} xa^{\frac12}, -xqa^{\frac12}, x(aq)^{\frac12}, -x(aq)^{\frac12}, axq/bc \\ axq/b, axq/c, qx, aqx^2 \end{matrix}; q, q\right].$$

3.17 If $\dfrac{(cz/ab;q^2)_\infty}{(z;q^2)_\infty}\ {}_2\phi_1(a,b;c;q,cz/abq)=\sum_{n=0}^\infty a_n z^n$,
show that

$$\begin{aligned}&{}_2\phi_1(c/a,c/b;cq;q^2,z)\ {}_2\phi_1(a,b;cq;q^2,cz/abq)\\&=\sum_{n=0}^\infty \frac{(c;q^2)_n}{(cq;q^2)_n}a_n z^n.\end{aligned}$$

(Singh [1959], Nassrallah [1982])

3.18 If $\dfrac{(cqz/ab;q^2)_\infty}{(z;q^2)_\infty}\ {}_2\phi_1(a,b;c;q,cz/ab)=\sum_{n=0}^\infty a_n z^n$, show that

$$\begin{aligned}&{}_2\phi_1(cq/a,cq/b;cq^2;q^2,z)\ {}_2\phi_1(a,b;c;q^2,cz/ab)\\&=\sum_{n=0}^\infty \frac{(cq;q^2)_n}{(cq^2;q^2)_n}a_n z^n.\end{aligned}$$

(Singh [1959], Nassrallah [1982])

3.19 If $\dfrac{(cqz/ab;q^2)_\infty}{(z;q^2)_\infty}\ {}_2\phi_1(a/q,b;c/q;q,cz/ab)=\sum_{n=0}^\infty a_n z^n$, show that

$$\begin{aligned}&{}_2\phi_1(cq/a,c/bq;c;q^2,z)\ {}_2\phi_1(a,b;c;q^2,cz/ab)\\&=\sum_{n=0}^\infty \frac{(c/q;q^2)_n}{(c;q^2)_n}a_n z^n.\end{aligned}$$

(Singh [1959], Nassrallah [1982])

3.20 Prove that

$$\sum_{k=0}^\infty \frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(b^{-1};q)_k}{(q;q)_k(abp;p)_k}b^k=0$$

when $\max(|p|,|q|,|b|)<1$, and extend this to the bibasic transformation formulas

$$\begin{aligned}&\sum_{k=0}^\infty \frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(c/b;q)_k}{(q;q)_k(abp;p)_k}b^k\\&=\frac{1-c}{1-b}\sum_{k=0}^\infty \frac{(ap;p)_k(c/b;q)_k}{(q;q)_k(abp;p)_k}(bq)^k\\&=\frac{1-c}{1-abp}\sum_{k=0}^\infty \frac{(ap;p)_k(cq/b;q)_k}{(q;q)_k(abp^2;p)_k}b^k\\&=\frac{(1-c)(ap;p)_\infty}{(1-b)(abp;p)_\infty}\sum_{k=0}^\infty \frac{(b;p)_k(cqp^k;q)_\infty}{(p;p)_k(bqp^k;q)_\infty}(ap)^k\end{aligned}$$

when $\max(|p|,|q|,|ap|,|b|)<1$.

(Gasper [1989a])

3.21 Derive the quadbasic transformation formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k}$$
$$\times \frac{(CP^{-n}/A,P^{-n}/BC;P)_k(Q^{-n},BQ^{-n}/A;Q)_k}{(Q^{-n}/C,BCQ^{-n}/A;Q)_k(P^{-n}/A,P^{-n}/B;P)_k} q^k$$
$$= \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n} \frac{(Q,AQ/B;Q)_n(AP/C,BCP;P)_n}{(AP,BP;P)_n(CQ,AQ/BC;Q)_n}$$
$$\times \sum_{k=0}^{n} \frac{(1-AP^kQ^k)(1-BP^kQ^{-k})}{(1-A)(1-B)} \frac{(A,B;P)_k(C,A/BC;Q)_k}{(Q,AQ/B;Q)_k(AP/C,BCP;P)_k}$$
$$\times \frac{(cp^{-n}/a,p^{-n}/bc;p)_k(q^{-n},bq^{-n}/a;q)_k}{(q^{-n}/c,bcq^{-n}/a;q)_k(p^{-n}/a,p^{-n}/b;p)_k} Q^k$$

for $n = 0, 1, \dots$. Use it to derive the mixed bibasic and hypergeometric transformation formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k}$$
$$\times \frac{(C-A-n)_k(-B-C-n)_k(\mu B-\mu A-n)_k(-n)_k}{(-\mu C-n)_k(\mu B+\mu C-\mu A-n)_k(-A-n)_k(-B-n)_k} q^k$$
$$= \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n} \frac{n!(\mu A+1-\mu B)_n(A+1-C)_n}{(A+1)_n(B+1)_n(\mu C+1)_n}$$
$$\times \frac{(B+C+1)_n}{(\mu A+1-\mu B-\mu C)_n}$$
$$\times \sum_{k=0}^{n} \frac{(A+k+k/\mu)(B+k-k\mu)}{AB} \frac{(A)_k(B)_k(\mu C)_k(\mu A-\mu B-\mu C)_k}{k!(\mu A+1-\mu B)_k(A+1-C)_k(B+C+1)_k}$$
$$\times \frac{(c/ap^n,1/bcp^n;p)_k(q^{-n},b/aq^n;q)_k}{(1/cq^n,bc/aq^n;q)_k(1/ap^n,1/bp^n;p)_k},$$

and the following transformation formula for a "*split-poised*" ${}_{10}\phi_9$ series

$${}_{10}\phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc, C/Aq^n, 1/BCq^n, B/Aq^n, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq, 1/Cq^n, BC/Aq^n, 1/Bq^n, 1/Aq^n \end{matrix} ; q, q \right]$$
$$= \frac{(aq,bq,cq,aq/bc,Aq/B,Aq/C,BCq;q)_n}{(Aq,Bq,Cq,Aq/BC,aq/b,aq/c,bcq;q)_n}$$
$$\times {}_{10}\phi_9 \left[\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, B, C, A/BC, c/aq^n, 1/bcq^n, b/aq^n, q^{-n} \\ \sqrt{A}, -\sqrt{A}, Aq/B, Aq/C, BCq, 1/cq^n, bc/aq^n, 1/bq^n, 1/aq^n \end{matrix} ; q, q \right].$$

(Gasper [1989a])

3.22 Using the observation that, for arbitrary (fixed) positive integers $m_1, \ldots, m_r$,

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r) z_1^{k_1} \cdots z_r^{k_r} = \sum_{M=0}^{\infty} \sum_{\substack{k_1m_1+\cdots+k_rm_r=M \\ k_1,\ldots,k_r\geq 0}} \Lambda(k_1,\ldots,k_r) z_1^{k_1} \cdots z_r^{k_r},$$

show that (3.7.14) implies the multivariable bibasic expansion formula

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r)\Omega_{k_1m_1+\cdots+k_rm_r}(x^{m_1}w^{m_1}z_1)^{k_1}\cdots(x^{m_r}w^{m_r}z_r)^{k_r}$$

$$= \sum_{n=0}^{\infty} \frac{(apq^n, bpq^{-n};p)_{n-1}}{(q, aq^n/b;q)_n}(-x)^n q^{n+\binom{n}{2}}$$

$$\times \sum_{j=0}^{\infty} \frac{(ap^nq^n, bp^nq^{-n};p)_j}{(q, aq^{2n+1}/b;q)_j}\Omega_{j+n}x^jq^j$$

$$\times \sum_{\substack{k_1m_1+\cdots+k_rm_r=M\leq n \\ k_1,\ldots,k_r\geq 0}} \frac{(1-ap^Mq^M)(1-bp^Mq^{-M})(q^{-n}, aq^n/b;q)_M}{(apq^n, bpq^{-n};p)_M}$$

$$\times \Lambda(k_1,\ldots,k_r)(w^{m_1}z_1)^{k_1}\cdots(w^{m_r}z_r)^{k_r},$$

which is equivalent to (3.7.14) and extends Srivastava [1984, (10)].
(Gasper [1989a])

3.23 Prove the following q-Lagrange inversion theorem:
If

$$G_n(x) = \sum_{j=n}^{\infty} b_{jn}x^j,$$

where b_{jn} is as defined in (3.6.20), and if

$$f(x) = \sum_{j=0}^{\infty} f_jx^j = \sum_{n=0}^{\infty} c_nG_n(x),$$

then

$$f_j = \sum_{n=0}^{j} b_{jn}c_n$$

and, vice versa,

$$c_n = \sum_{j=0}^{n} a_{nj}f_j,$$

where a_{nj} is as defined in (3.6.19).
(Gasper [1989a])

3.24 Derive (3.8.19)–(3.8.22).

3.25 Prove Gosper's formula

$$\sum_{n=0}^{\infty}\frac{(a^2b^2c^2q^{-1};q^2)_n(1-a^2b^2c^2q^{3n-1})(abcd,abcd^{-1};q^2)_n(a^2,b^2,c^2;q)_n}{(q;q)_n(1-a^2b^2c^2q^{-1})(abcd^{-1},abcd;q)_n(b^2c^2q,c^2a^2q,a^2b^2q;q^2)_n}q^n$$
$$=\frac{(a^2b^2c^2q,a^2q,b^2q,c^2q,dq/abc,bcdq/a,cdaq/b,dabq/c;q^2)_\infty}{(q,b^2c^2q,c^2a^2q,a^2b^2q,abcdq,adq/bc,bdq/ac,cdq/ab;q^2)_\infty}$$
$$-\frac{(a^2,b^2,c^2;q)_\infty(a^2b^2c^2q;q^2)_\infty}{(q;q)_\infty(b^2c^2q,c^2a^2q,a^2b^2q,abcdq,abcd^{-1}q;q^2)_\infty}$$
$$\times\sum_{n=0}^{\infty}(1-d^2q^{4n+2})\frac{(bcdq/a,cdaq/b,dabq/c;q^2)_n}{(adq/bc,dbq/ac,cdq/ab;q^2)_{n+1}}\left(-\frac{d}{abc}\right)^n q^{(n+1)^2}.$$

(Rahman [1993])

3.26 Show that

$$\sum_{k=0}^{\infty}\frac{(abcq;q)_k(1-abcq^{3k+1})(d,q/d;q)_k(abq,bcq,caq;q^2)_k}{(q^2;q^2)_k(1-abcq)(abcq^3/d,abcdq^2;q^2)_k(cq,aq,bq;q)_k}q^k$$
$$={}_8\phi_7\left[\begin{matrix}abcq,q^2\sqrt{abcq},-q^2\sqrt{abcq},d,q/d,abq,bcq,caq\\ \sqrt{abcq},-\sqrt{abcq},abcq^3/d,abcdq^2,cq^2,aq^2,bq^2\end{matrix};q^2,q^2\right]$$
$$+\frac{(abcq^3,abq,bcq,caq,d,q/d;q^2)_\infty}{(q^2,aq^2,bq^2,cq^2,abcq^3/d,abcdq^2;q^2)_\infty}\frac{q}{(1-aq)(1-bq)(1-cq)}$$
$$\times{}_4\phi_3\left[\begin{matrix}q^2,abcq^2,dq,q^2/d\\ aq^3,bq^3,cq^3\end{matrix};q^2,q^2\right].$$

3.27 Prove

$$\sum_{n=0}^{\infty}\frac{(bcdq^{-2};q^3)_n(1-bcdq^{4n-2})(b,c,d;q)_n}{(q;q)_n(1-bcdq^{-2})(cdq,bdq,bcq;q^3)_n}q^{n^2}$$
$$\times{}_4\phi_3\left[\begin{matrix}q^{-n},q^{1-n},q^{2-n},bcdq^{3n}\\ bq^2,cq^2,dq^2\end{matrix};q^3,q^3\right]$$
$$=\frac{(bcdq,bq,cq,dq;q^3)_\infty}{(q,cdq,bdq,bcq;q^3)_\infty}.$$

(Rahman [1993])

3.28 Show that

$$\sum_{n=0}^{\infty}\frac{(bcdq^{-1};q^3)_n(1-bcdq^{4n-1})(b,c,d;q)_n}{(q;q)_n(1-bcdq^{-1})(cdq^2,bdq^2,bcq^2;q^3)_n}q^{n^2+n}$$
$$\times{}_4\phi_3\left[\begin{matrix}q^{-n-1},q^{-n},q^{1-n},bcdq^{3n}\\ bq,cq,dq\end{matrix};q^3,q^3\right]$$
$$=\frac{(bcdq^2,bq^2,cq^2,dq^2;q^3)_\infty}{(q^2,cdq^2,bdq^2,bcq^2;q^3)_\infty}.$$

(Rahman [1993])

3.29 Derive the summation formulas:

(i)

$$\sum_{k=0}^{\infty} \frac{(1-aq^{5k})(a,b;q^2)_k(ab^2/q^3;q^3)_k(q^2/b;q)_k(aq^3/b;q^6)_k}{(1-a)(q^3,aq^3/b;q^3)_k(q^5/b^2;q^2)_k(ab,abq^2;q^4)_k} \left(-\frac{q^2}{b}\right)^k q^{\binom{k+1}{2}}$$
$$= \frac{(aq^2,qa^3/b;q^2)_\infty(ab^2,q^9/b^3;q^6)_\infty}{(ab,q^5/b^2;q^2)_\infty(q^3,aq^6/b;q^6)_\infty},$$

(ii)

$$\sum_{k=0}^{\infty} \frac{(1-acq^{5k})(a,q^4/a;q^2)_k(q^5/ac;q^3)_k(ac/q^2;q)_k}{(1-ac)(cq^3,a^2c/q;q^3)_k(a^2c^2/q^3;q^2)_k}$$
$$\times \frac{(a^2c^2/q;q^6)_k}{(q^4,q^6;q^4)_k}\left(-\frac{ac}{q^2}\right)^k q^{\binom{k+1}{2}}$$
$$= \frac{(acq^2,ac/q;q^2)_\infty(q^6,a^3c^2/q^3,ac^2q,a^2c^2/q,aq^4,q^8/a;q^6)_\infty}{(q^4,a^2c^2/q^3;q^2)_\infty(cq^3,cq^6,a^2cq^2,a^2c/q,acq,acq^4;q^6)_\infty},$$

(iii)

$$\sum_{k=0}^{\infty} \frac{(1-acq^{5k})(a,q^2/a;q^2)_k(q/ac;q^3)_k(ac;q)_k(a^2c^2q;q^6)_k}{(1-ac)(cq^3,a^2cq;q^3)_k(a^2c^2q;q^2)_k(q^2,q^4;q^4)_k}(-ac)^k q^{\binom{k+1}{2}}$$
$$= \frac{(acq^2,acq;q^2)_\infty(q^6,a^3c^2q^3,ac^2q^5,a^2c^2q,aq^2,q^4/a;q^6)_\infty}{(q^2,a^2c^2q;q^2)_\infty(cq^3,cq^6,a^2cq^4,a^2cq,acq^2,acq^5;q^6)_\infty}.$$

(Rahman [1989b])

3.30 Derive the quartic summation formula

$$\sum_{k=0}^{\infty} \frac{1-aq^{5k}}{1-a} \frac{(a,b;q)_k(q/b,q^2/b,q^3/b;q^3)_k(a^2b^2/q^2;q^4)_k}{(q^4,aq^4/b;q^4)_k(abq,ab,ab/q;q^2)_k(q^3/ab^2;q)_k} q^k$$
$$+ \frac{ab^3}{q^2} \frac{(aq,bq,1/b;q)_\infty(a^2b^2q^2;q^4)_\infty}{(ab,q^3/ab^2;q)_\infty(ab/q;q^2)_\infty(q^4,ab^3/q^2,aq^4/b;q^4)_\infty}$$
$$\times {}_1\phi_1\left[\begin{matrix} a^2b^2/q^2 \\ a^2b^2q^2 \end{matrix};q^4,ab^3q^2\right]$$
$$= \frac{(aq,ab^2/q^2;q)_\infty}{(ab;q)_\infty(ab/q;q^2)_\infty(aq^4/b,ab^3/q^2;q^4)_\infty}.$$

(Gasper [1989a])

3.31 Derive the cubic transformation formulas

(i)

$$\sum_{k=0}^{n} \frac{1-acq^{4k}}{1-ac} \frac{(a,b;q)_k(cq/b;q)_{2k}(a^2bcq^{3n},q^{-3n};q^3)_k}{(cq^3,acq^3/b;q^3)_k(ab;q)_{2k}(q^{1-3n}/ab,acq^{3n+1};q)_k} q^k$$
$$= \frac{(1-acq^2)(1-ab/q^2)(1-abq^{3n})(1-acq^{3n})}{(1-acq^{3n+2})(1-abq^{3n-2})(1-ab)(1-ac)}$$

$$\times \sum_{k=0}^{n} \frac{1-acq^{6k+2}}{1-acq^2} \frac{(aq^2, bq^2, cq^2/b, cq^3/b, a^2bcq^{3n}, q^{-3n}; q^3)_k}{(cq^3, acq^3/b, abq^3, abq^2, q^{5-3n}/ab, acq^{3n+5}; q^3)_k} q^{3k},$$

(ii)

$$\sum_{k=0}^{n} \frac{(1-aq^{4k})(1-bq^{-2k})}{(1-a)(1-b)} \frac{(a, 1/ab; q)_k (abq; q)_{2k} (c, a/bc; q^3)_k}{(q^3, a^2bq^3; q^3)_k (q/b; q)_{2k} (aq/c, bcq; q)_k} q^{3k}$$
$$= \frac{(aq, bq; q)_n (cq^3, aq^3/bc; q^3)_n}{(aq/c, bcq; q)_n (q^3, aq^3/b; q^3)_n}$$
$$\times {}_{10}W_9(a/b; q/ab^2, c, a/bc, aq^{n+1}, aq^{n+2}, aq^{n+3}, q^{-3n}; q^3, q^3),$$

where $n = 0, 1, \ldots$.
(Gasper and Rahman [1990])

3.32 Derive the cubic summation formula

$$\sum_{k=0}^{\infty} \frac{1-a^2q^{4k}}{1-a^2} \frac{(b, q^2/b; q)_k (a^2/q; q)_{2k} (c^3, a^2q^2/c^3; q^3)_k}{(a^2q^3/b, a^2bq; q^3)_k (q^2; q)_{2k} (a^2q/c^3, c^3/q; q)_k} q^k$$
$$= \frac{(bq^2, q^4/b, bc^3/q, c^3q/b, c^3/a^2, c^3q^2/a^2, a^2q, a^2q^3; q^3)_\infty}{(q^2, q^4, c^3q, bc^3/a^2, a^2q^3/b, a^2bq, c^3q^2/a^2b; q^3)_\infty}$$
$$- \frac{(b, bq, bq^2, q^2/b, q^3/b, q^4/b, a^2/q, a^2q, a^2q^3, c^3/a^2; q^3)_\infty}{(q^2, q^4, c^3/q, c^3q, a^2/c^3, a^2q/c^3, c^3q^3/a^2, c^3q^3/a^2b, a^3q^3/b; q^3)_\infty}$$
$$\times \frac{(c^6q/a^2; q^3)_\infty}{(a^2bq, bc^3/a^2; q^3)_\infty} {}_2\phi_1 \left[\begin{matrix} bc^3/a^2, c^3q^2/a^2b \\ c^6q/a^2 \end{matrix} ; q^3, q^3 \right]$$

and show that it has the $q \to 1^-$ limit case

$${}_7F_6 \left[\begin{matrix} a-1/2, a, b, 2-b, c, (2a+2-3c)/3, a/2+1 \\ 3/2, (2a-b+3)/3, (2a+b+1)/3, 3c-1, 2a+1-3c, a/2 \end{matrix} ; 1 \right]$$
$$= \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(c-\frac{1}{3}\right)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(\frac{2a-b+3}{3}\right)}{\Gamma\left(\frac{b+2}{3}\right)\Gamma\left(\frac{4-b}{3}\right)\Gamma\left(\frac{b+3c-1}{3}\right)\Gamma\left(\frac{3c-b+1}{3}\right)\Gamma\left(\frac{2a+1}{3}\right)}$$
$$\times \frac{\Gamma\left(\frac{2a+b+1}{3}\right)\Gamma\left(\frac{2a-3c+3}{3}\right)\Gamma\left(\frac{2a-3c+1}{3}\right)}{\Gamma\left(\frac{2a+3}{3}\right)\Gamma\left(\frac{2a-3c-b+3}{3}\right)\Gamma\left(\frac{2a-3c+b+1}{3}\right)}.$$

(Gasper and Rahman [1990])

3.33 Derive the quartic transformation formula

$$\sum_{k=0}^{\infty} \frac{1-a^2b^2q^{5k-2}}{1-a^2b^2/q^2} \frac{(a, b; q)_k (ab/q, ab, abq; q^3)_k (a^2b^2/q^2; q^4)_k}{(ab^2q^2, a^2bq^2; q^4)_k (abq, ab, ab/q; q^2)_k (q; q)_k} q^k$$
$$= \frac{(aq, b; q)_\infty (a^2b^2q^2; q^4)_\infty}{(q; q)_\infty (abq; q^2)_\infty (b, ab^2q^2, a^2bq^2; q^4)_\infty} {}_1\phi_1 \left[\begin{matrix} a \\ aq^4 \end{matrix} ; q^4, bq^4 \right].$$

(Gasper and Rahman [1990])

3.34 Show that

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, c^2, a, qa \\ q^2a^2, cq^{-n}, cq^{1-n} \end{matrix} ; q^2, q^2 \right] = \frac{(-q, qa/c; q)_n}{(-aq, q/c; q)_n}.$$

3.35 For $k = 1, 2, \ldots$, show that

$$\sum_{n=0}^{\infty} \frac{(a;q^k)_n (b;q)_{kn}}{(q^k;q^k)_n (c;q)_{kn}} t^n = \frac{(b;q)_\infty (at;q^k)_\infty}{(c;q)_\infty (t;q^k)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_n (t;q^k)_n}{(q;q)_n (at;q^k)_n} b^n.$$

(Andrews [1966c])

3.36 For the q-exponential function defined in (1.3.33) prove Suslov's addition formula

$$\mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(x; \alpha)\mathcal{E}_q(y; \alpha),$$

where

$$\mathcal{E}_q(x, y; \alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} (\alpha e^{-i\phi})^n \left(-q^{\frac{1-n}{2}} e^{i\theta+i\phi}, -q^{\frac{1-n}{2}} e^{i\phi-i\theta}; q\right)_n$$

with $x = \cos\theta$, $y = \cos\phi$, $0 \le \theta \le \pi$ and $0 \le \phi \le \pi$.

(Suslov [1997, 2003])

3.37 Derive the quadratic transformation formula

$$\mathcal{E}(x; \alpha) = \frac{(-\alpha; q^{1/2})_\infty}{(q\alpha^2; q^2)_\infty} \, {}_2\phi_1 \left[\begin{matrix} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2} \end{matrix} ; q^{1/2}, -\alpha \right].$$

(Ismail and Stanton [2002])

Notes

§3.4 Bressoud [1987] contains some transformation formulas for terminating ${}_{r+1}\phi_r(a_1, a_2, \ldots, a_{r+1}; b_1, \ldots, b_r; q; z)$ series that are *almost poised* in the sense that $b_k a_{k+1} = a_1 q^{\delta_k}$ with $\delta_k = 0, 1$ or 2 for $1 \le k \le r$. Transformations for *level basic series*, that is ${}_{r+1}\phi_r$ series in which $a_1 b_k = q a_{k+1}$ for $1 \le k \le r$, are considered in Gasper [1985].

§3.5 For a comprehensive list of q-analogues of the quadratic transformation formulas in §2.11 of Erdélyi [1953], see Rahman and Verma [1993].

§3.6 Agarwal and Verma [1967a,b] derived transformation formulas for certain sums of bibasic series by applying the theorem of residues to contour integrals of the form (4.9.2) considered in Chapter 4. Inversion formulas are also considered in Carlitz [1973] and W. Chu [1994b, 1995] and, in connection with the Bailey lattice, in A.K. Agarwal, Andrews and Bressoud [1987].

§3.7 Jackson [1928] applied his q-analogue of the Euler's transformation formula (the $p = q$ case of (3.7.11)) to the derivation of transformation formulas and theta function series. Jackson [1942, 1944] and Jain [1980a] also derived q-analogues of some of the double hypergeometric function expansions in Burchnall and Chaundy [1940, 1941].

§3.8 Gosper [1988a] stated a strange q-series transformation formula containing bases q^2, q^3, and q^6. Krattenthaler [1989b] independently derived the terminating case of (3.8.18) and terminating special cases of some of the other summation formulas in this section. For further results on cubic and quintic summation and transformation formulas, see Rahman [1989d, 1992b, 1997].

§3.9 For multibasic series containing bibasic shifted factorials of the form $(a;p,q)_{r,s} = \prod_{j=0}^{r-1} \prod_{k=0}^{s-1}(1-ap^jq^k)$ and connections with Schur functions and permutation statistics, see Désarménien and Foata [1985–1988].

§3.10 Jain and Verma [1986] contains nonterminating versions of some of Nassrallah and Rahman's [1981] transformation formulas.

Ex. 3.11 q-Differential equations for certain products of basic hypergeometric series are considered in Jackson [1911].

Exercises 3.17–3.19 These exercises are q-analogues of the Cayley [1858] and Orr [1899] theorems (also see Bailey [1935, Chapter X], Burchnall and Chaundy [1949], Edwards [1923], Watson [1924], and Whipple [1927, 1929]). Other q-analogues are given in N. Agarwal [1959], and Jain and Verma [1987].

Ex. 3.20 The formula obtained by writing the last series as a multiple of the series with argument bp is equivalent to the bibasic identity (21.9) in Fine [1988], and it is a special case of the Fundamental Lemma in Andrews [1966a, p. 65]. Applications of the Fundamental Lemma to mock theta functions and partitions are contained in Andrews [1966a,b]. Agarwal [1969a] extended Andrews' Fundamental Lemma and pointed out some expansion formulas that follow from his extension.

Ex. 3.23 For additional material on Lagrange inversion, see Andrews [1975b, 1979a], Bressoud [1983b], Cigler [1980], Fürlinger and Hofbauer [1985], Garsia [1981], Garsia and Remmel [1986], Gessel [1980], Gessel and Stanton [1983, 1986], Hofbauer [1982, 1984], Krattenthaler [1984, 1988, 1989a], Paule [1985b], and Stanton [1988].

4

BASIC CONTOUR INTEGRALS

4.1 Introduction

Our first objective in this chapter is to give q-analogues of Barnes' [1908] contour integral representation for the hypergeometric function

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds, \quad (4.1.1) $$

where $|\arg(-z)| < \pi$, Barnes' [1908] first lemma

$$ \begin{aligned} &\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) \, ds \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \end{aligned} \quad (4.1.2) $$

and Barnes' [1910] second lemma

$$ \begin{aligned} &\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(1-d-s)\Gamma(-s)}{\Gamma(e+s)} \, ds \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1-d+a)\Gamma(1-d+b)\Gamma(1-d+c)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}, \end{aligned} \quad (4.1.3) $$

where $e = a+b+c-d+1$.

In (4.1.1) the contour of integration is the imaginary axis directed upward with indentations, if necessary, to ensure that the poles of $\Gamma(-s)$, i.e. $s = 0, 1, 2, \ldots$, lie to the right of the contour and the poles of $\Gamma(a+s)\Gamma(b+s)$, i.e. $s = -a-n, -b-n$, with $n = 0, 1, 2, \ldots$, lie to the left of the contour (as shown in Fig. 4.1 at the end of this section). The assumption that there exists such a contour excludes the possibility that a or b is zero or a negative integer. Similarly, in (4.1.2), (4.1.3) and the other contour integrals in this book it is assumed that the parameters are such that the contour of integration can be drawn separating the increasing and decreasing sequences of poles.

Barnes' first and second lemmas are integral analogues of Gauss' ${}_2F_1$ summation formula (1.2.11) and Saalschütz's formula (1.7.1), respectively. In Askey and Roy [1986] it was pointed out that Barnes' first lemma is an extension of the beta integral (1.11.8). To see this, replace b by $b - i\omega$, d by $d + i\omega$ and then set $s = \omega x$ in (4.1.2). Then let $\omega \to \infty$ and use Stirling's formula to obtain the beta integral in the form

$$ \int_{-\infty}^{\infty} x_+^{a+c-1} (1-x)_+^{b+d-1} dx = B(a+c, b+d), \quad (4.1.4) $$

where $\mathrm{Re}(a+c) > 0$, $\mathrm{Re}(b+d) > 0$ and

$$x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{4.1.5}$$

It is for this reason that Askey and Roy call (4.1.2) Barnes' beta integral. Following Watson [1910], we shall give a q-analogue of (4.1.1) in §4.2, that is, a Barnes-type integral representation for ${}_2\phi_1(a, b; c; q, z)$. It will be used in §4.3 to derive an analytic continuation formula for the ${}_2\phi_1$ series. We shall give q-analogues of (4.1.2) and (4.1.3) in §4.4. The rest of the chapter will be devoted to generalizations of these integral representations, other types of basic contour integrals, and to the use of these integrals to derive general transformation formulas for basic hypergeometric series.

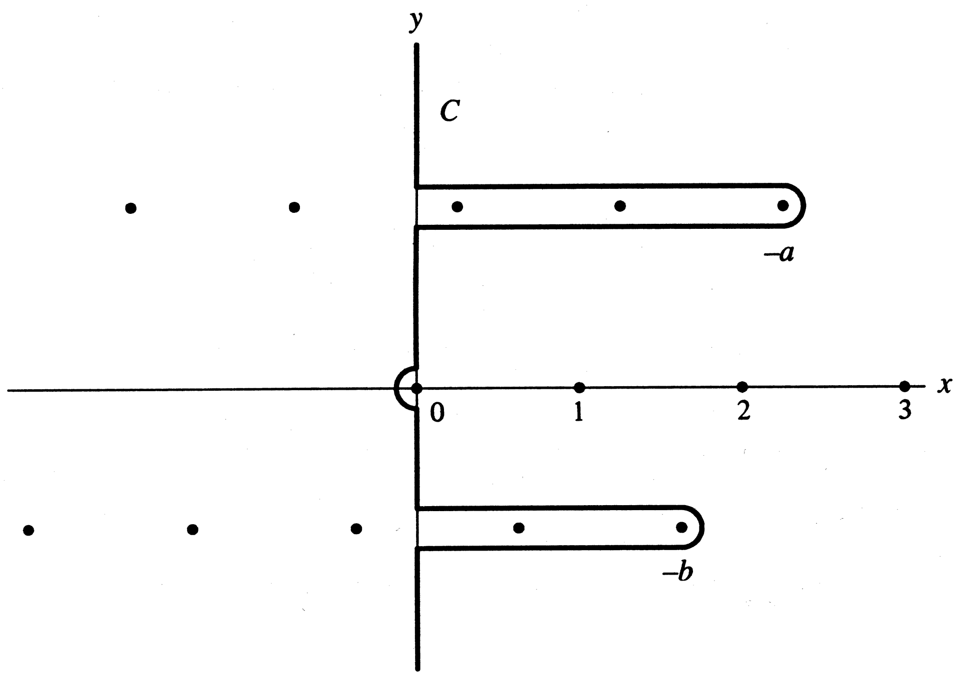


Fig. 4.1

4.2 Watson's contour integral representation for ${}_2\phi_1(a, b; c; q, z)$ series

For the sake of simplicity we shall assume in this and the following five sections that $0 < q < 1$ and write

$$q = e^{-\omega}, \quad \omega > 0. \tag{4.2.1}$$

This is not a severe restriction for most applications because the results derived for $0 < q < 1$ can usually be extended to complex q in the unit disc by using analytic continuation. The restriction $0 < q < 1$ has the advantage of simplifying the proofs by enabling one to use contours parallel to the imaginary axis. Extensions to complex q in the unit disc will be considered in §4.8.

For $0 < q < 1$ Watson [1910] showed that Barnes' contour integral in (4.1.1) has a q-analogue of the form

$$\begin{aligned} &{}_2\phi_1(a, b; c; q, z) \\ &= \frac{(a, b; q)_\infty}{(q, c; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{\left(q^{1+s}, cq^s; q\right)_\infty}{(aq^s, bq^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, ds, \end{aligned} \tag{4.2.2}$$

where $|z| < 1, |\arg(-z)| < \pi$. The contour of integration (denote it by C) runs from $-i\infty$ to $i\infty$ (in Watson's paper the contour is taken in the opposite direction) so that the poles of $\left(q^{1+s}; q\right)_\infty / \sin \pi s$ lie to the right of the contour and the poles of $1/(aq^s, bq^s; q)_\infty$, i.e. $s = \omega^{-1} \log a - n + 2\pi i m \omega^{-1}, s = \omega^{-1} \log b - n + 2\pi i m \omega^{-1}$ with $n = 0, 1, 2, \ldots,$ and $m = 0, \pm 1, \pm 2, \ldots,$ when a and b are not zero, lie to the left of the contour and are at least some $\epsilon > 0$ distance away from the contour.

To prove (4.2.2) first observe that by the triangle inequality,

$$|1 - |a|e^{-\omega\, \mathrm{Re}(s)}| \le |1 - aq^s| \le 1 + |a|e^{-\omega\, \mathrm{Re}(s)}$$

and so

$$\begin{aligned} &\left| \frac{\left(q^{1+s}, cq^s; q\right)_\infty}{(aq^s, bq^s; q)_\infty} \right| \\ &\le \prod_{n=0}^{\infty} \frac{\left(1 + e^{-(n+1+\mathrm{Re}(s))\omega}\right)\left(1 + |c|e^{-(n+\mathrm{Re}(s))\omega}\right)}{\left(1 - |a|e^{-(n+\mathrm{Re}(s))\omega}\right)\left(1 - |b|e^{-(n+\mathrm{Re}(s))\omega}\right)}, \end{aligned} \tag{4.2.3}$$

which is bounded on C. Hence the integral in (4.2.2) converges if $\mathrm{Re}[s \log(-z) - \log(\sin \pi s)] < 0$ on C for large $|s|$, i.e. if $|\arg(-z)| < \pi$.

Now consider the integral in (4.2.2) with C replaced by a contour C_R consisting of a large clockwise-oriented semicircle of radius R with center at the origin that lies to the right of C, is terminated by C and is bounded away from the poles (as shown in Fig. 4.2).

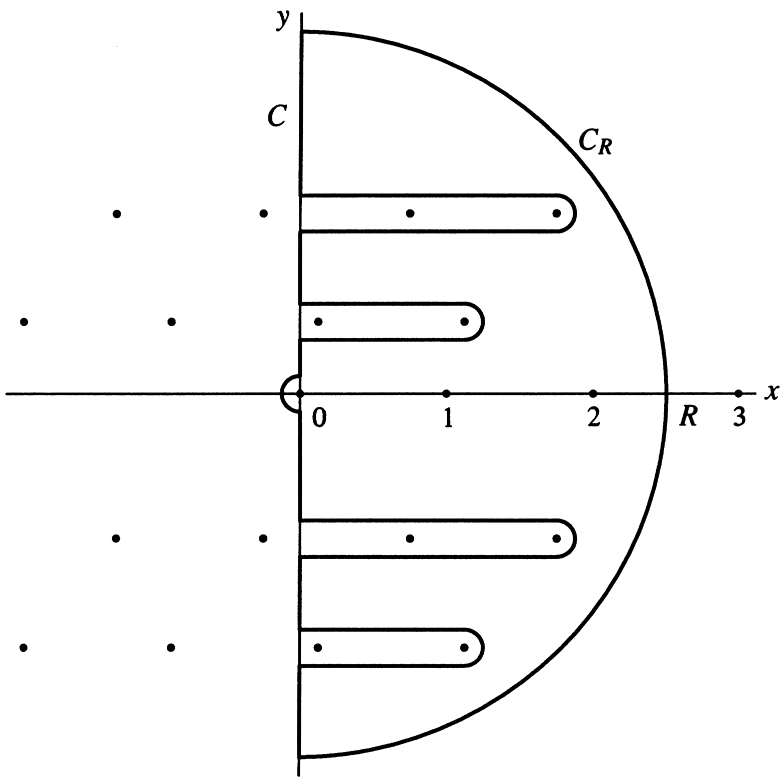


Fig. 4.2

Setting $s = Re^{i\theta}$, we have for $|z| < 1$ that

$$\begin{aligned}
&\text{Re}\left[\log \frac{(-z)^s}{\sin \pi s}\right] \\
&= R\left[\cos\theta\ \log|z| - \sin\theta\ \arg(-z) - \pi|\sin\theta|\right] + O(1) \\
&\leq -R\left[\sin\theta\ \arg(-z) + \pi|\sin\theta|\right] + O(1).
\end{aligned}$$

Hence, when $|z| < 1$ and $|\arg(-z)| < \pi - \delta$, $0 < \delta < \pi$, we have

$$\frac{(-z)^s}{\sin \pi s} = O\left[\exp(-\delta R|\sin\theta|)\right] \tag{4.2.4}$$

on C_R as $R \to \infty$, and it follows that the integral in (4.2.2) with C replaced by C_R tends to zero as $R \to \infty$, under the above restrictions. Therefore, by applying Cauchy's theorem to the closed contour consisting of C_R and that part of C terminated above and below by C_R and letting $R \to \infty$, we obtain that $-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ldots ds$ equals the sum of the residues of the integrand at $n = 0, 1, 2, \ldots$. Since

$$\lim_{s\to n}(s-n)\frac{(q^{1+s}, cq^s; q)_\infty}{(aq^s, bq^s; q)_\infty}\frac{\pi(-z)^s}{\sin \pi s} = \frac{(q^{1+n}, cq^n; q)_\infty}{(aq^n, bq^n, q)_\infty}z^n,$$

this completes the proof of Watson's formula (4.2.2).

It should be noted that the contour of integration in (4.2.2) can be replaced by other suitably indented contours parallel to the imaginary axis. To see that (4.2.2) is a q-analogue of (4.1.1) it suffices to use (1.10.1) and the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{4.2.5}$$

to rewrite (4.2.2) in the form

$$\begin{aligned}&{}_2\phi_1\left(q^a, q^b; q^c; q, z\right)\\ &= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma_q(a+s)\Gamma_q(b+s)\Gamma(-s)\Gamma(1+s)}{\Gamma_q(c+s)\Gamma_q(1+s)} (-z)^s \, ds.\end{aligned} \tag{4.2.6}$$

4.3 Analytic continuation of ${}_2\phi_1(a, b; c; q, z)$

Since the integral in (4.2.2) defines an analytic function of z which is single-valued when $|\arg(-z)| < \pi$, the right side of (4.2.2) gives the analytic continuation of the function represented by the series ${}_2\phi_1(a, b; c; q, z)$ when $|z| < 1$. As in the ordinary hypergeometric case, we shall denote this analytic continuation of ${}_2\phi_1(a, b; c; q, z)$ to the domain $|\arg(-z)| < \pi$ again by ${}_2\phi_1(a, b; c; q, z)$.

Barnes [1908] used (4.1.1) to show that if $|\arg(-z)| < \pi$ and $a, b, c, a-b$ are not integers, then the analytic continuation for $|z| > 1$ of the series which defines ${}_2F_1(a, b; c; z)$ for $|z| < 1$ is given by the equation

$$\begin{aligned}{}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\, {}_2F_1(a, 1+a-c;\ 1+a-b; z^{-1})\\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\, {}_2F_1(b, 1+b-c;\ 1+b-a; z^{-1}),\end{aligned} \tag{4.3.1}$$

where, as elsewhere in this section, the symbol "=" is used in the sense "is the analytic continuation of". To illustrate the extension of Barnes' method to ${}_2\phi_1$ series we shall now give Watson's [1910] derivation of the following q-analogue of (4.3.1):

$$\begin{aligned}&{}_2\phi_1(a, b; c; q, z)\\ &= \frac{(b, c/a; q)_\infty (az, q/az; q)_\infty}{(c, b/a; q)_\infty (z, q/z; q)_\infty}\, {}_2\phi_1(a, aq/c; aq/b; q, cq/abz)\\ &\quad + \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty} \frac{(bz, q/bz; q)_\infty}{(z, q/z; q)_\infty}\, {}_2\phi_1(b, bq/c; bq/a; q, cq/abz),\end{aligned} \tag{4.3.2}$$

provided that $|\arg(-z)| < \pi$, c and a/b are not integer powers of q, and $a, b, z \neq 0$.

First consider the integral

$$I_1 = \frac{1}{2\pi i} \int \frac{\left(q^{1+s}, cq^s; q\right)_\infty}{\left(aq^s, bq^s; q\right)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, ds \tag{4.3.3}$$

along three line-segments A_1, A_2, B, whose equations are:

$$A_1: \ \mathrm{Im}(s) = m_1, \quad A_2: \ \mathrm{Im}(s) = -m_2, \quad B: \ \mathrm{Re}(s) = -M, \tag{4.3.4}$$

where m_1, m_2, M are large positive constants chosen so that A_1, A_2, B are at least a distance $\epsilon > 0$ away from each pole and zero of

$$g(s) = \frac{\left(q^{1+s}, cq^s; q\right)_\infty}{\left(aq^s, bq^s; q\right)_\infty} \tag{4.3.5}$$

and it is assumed that A_1, A_2, B are terminated by each other and by the contour of the integral in (4.2.2), i.e. $\mathrm{Re}(s) = 0$ with suitable indentations.

From an asymptotic formula for $(a;q)_\infty$ with $q = e^{-\omega}, \omega > 0$, due to Littlewood [1907, §12], it follows that if $\mathrm{Re}(s) \to -\infty$ with $|s - s_0| > \epsilon$ for some fixed $\epsilon > 0$ and any zero s_0 of $(q^s;q)_\infty$, then

$$\mathrm{Re}[\log(q^s;q)_\infty] = \frac{\omega}{2}(\mathrm{Re}(s))^2 + \frac{\omega}{2}\,\mathrm{Re}(s) + O(1). \tag{4.3.6}$$

This implies that

$$g(s) = O\left(\left|\frac{ab}{cq}\right|^{\mathrm{Re}(s)}\right), \tag{4.3.7}$$

when $\mathrm{Re}(s) \to -\infty$ with s bounded away from the zeros and poles of $g(s)$. By using this asymptotic expansion and the method of §4.2 it can be shown that if $|z| > |cq/ab|$, then the value of the integral I_1 in (4.3.3) taken along the contours A_1, A_2, B tends to zero as $m_1, m_2, M \to \infty$.

Hence, by Cauchy's theorem, the value of I_1, taken along the contour C of §4.2, equals the sum of the residues of the integrand at its poles to the left of C when $|z| > |cq/ab|$. Set $\alpha = -\omega^{-1}\log a,\ \beta = -\omega^{-1}\log b$ so that $a = q^\alpha, b = q^\beta$. Since the residue of the integrand at $-\alpha - n + 2\pi i m\omega^{-1}$ is

$$\frac{\left(a^{-1}q^{1-n}, ca^{-1}q^{-n}, q^{n+1}; q\right)_\infty}{\left(q, q, ba^{-1}q^{-n}; q\right)_\infty}\pi\omega^{-1}(-z)^{-\alpha-n}q^{n(n+1)/2}$$
$$\times \exp\left\{2m\pi i\omega^{-1}\log(-z)\right\}\csc\left(2m\pi^2 i\omega^{-1} - \alpha\pi\right),$$

we have

$$\begin{aligned} I_1 = &\sum_{m=-\infty}^{\infty} \csc\left(2m\pi^2 i\omega^{-1} - \alpha\pi\right)\exp\left\{2m\pi i\omega^{-1}\log(-z)\right\} \\ &\times \frac{\pi\omega^{-1}(c/a, q/a; q)_\infty}{(b/a, q; q)_\infty}(-z)^{-\alpha}\ {}_2\phi_1(a, aq/c; aq/b; q, cq/abz) \\ &+ \text{idem }(a;b). \end{aligned} \tag{4.3.8}$$

Thus it remains to evaluate the above sums over m when $|\arg(-z)| < \pi$. Letting $c = b$ in (4.3.8) and using (4.2.2), we find that the analytic continuation of ${}_2\phi_1(a, b; b; q, z)$ is

$$\sum_{m=-\infty}^{\infty} \csc\left(\alpha\pi - 2m\pi^2 i\omega^{-1}\right)\exp\left\{2m\pi i\omega^{-1}\log(-z)\right\}$$
$$\times \frac{\pi\omega^{-1}(a, q/a; q)_\infty}{(q, q; q)_\infty}(-z)^{-\alpha}\ {}_2\phi_1(a, aq/b; aq/b; q, q/az).$$

Since, by the q-binomial theorem,

$$ {}_2\phi_1(a, b; b; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty} $$

and the products converge for all values of z, it follows that

$$ \sum_{m=-\infty}^{\infty} \csc\left(\alpha\pi - 2m\pi^2 i\omega^{-1}\right) \exp\left\{2m\pi i\omega^{-1}\log(-z)\right\} (-z)^{-\alpha} = \frac{\omega(q, q, az, q/az; q)_\infty}{\pi(a, q/a, z, q/z; q)_\infty}. \tag{4.3.9} $$

Using (4.3.9) in (4.3.8) we finally obtain (4.3.2).

4.4 q-analogues of Barnes' first and second lemmas

Assume, as before, that $0 < q < 1$, and consider the integral

$$ I_2 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1-c+s}, q^{1-d+s}; q\right)_\infty}{\left(q^{a+s}, q^{b+s}; q\right)_\infty} \frac{\pi q^s\, ds}{\sin\pi(c-s)\sin\pi(d-s)}, \tag{4.4.1} $$

where, as usual, the contour of integration runs from $-i\infty$ to $i\infty$ so that the increasing sequences of poles of the integrand (i.e. $c+n, d+n$ with $n = 0, 1, 2, \ldots$) lie to the right and the decreasing sequences of poles (i.e. the zeros of $\left(q^{a+s}, q^{b+s}; q\right)_\infty$) lie to the left of the contour. By using Cauchy's theorem as in §4.2 to evaluate this integral as the sum of the residues at the poles $c+n, d+n$ with $n = 0, 1, 2, \ldots$, we find that

$$ I_2 = \frac{\pi q^c}{\sin\pi(c-d)} \frac{\left(q, q^{1+c-d}; q\right)_\infty}{\left(q^{a+c}, q^{b+c}; q\right)_\infty} {}_2\phi_1\left(q^{a+c}, q^{b+c}; q^{1+c-d}; q, q\right) + \text{idem}\ (c; d). \tag{4.4.2} $$

Applying the formula (2.10.13) to (4.4.2), we get

$$ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1-c+s}, q^{1-d+s}; q\right)_\infty}{\left(q^{a+s}, q^{b+s}; q\right)_\infty} \frac{\pi q^s\, ds}{\sin\pi(c-s)\sin\pi(d-s)} = \frac{q^c}{\sin\pi(c-d)} \frac{\left(q, q^{1+c-d}, q^{d-c}, q^{a+b+c+d}; q\right)_\infty}{\left(q^{a+c}, q^{a+d}, q^{b+c}, q^{b+d}; q\right)_\infty}, \tag{4.4.3} $$

which is Watson's [1910] q-analogue of Barnes' first lemma (4.1.2), as can be seen by rewriting it in terms of q-gamma functions.

A q-analogue of Barnes' second lemma (4.1.3) can be derived by proceeding as in Agarwal [1953b]. Set $c = n$ and $d = c - a - b$ in (4.4.3) to obtain

$$ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1-n+s}, q^{1-c+a+b+s}; q\right)_\infty}{\left(q^{a+s}, q^{b+s}; q\right)_\infty} \frac{\pi q^s\, ds}{\sin\pi s \sin\pi(c-a-b-s)} = \csc\pi(c-a-b) \frac{\left(q^{1+a+b-c}, q^{c-a-b}, q^c, q; q\right)_\infty}{\left(q^a, q^b, q^{c-a}, q^{c-b}; q\right)_\infty} \times (-1)^n \frac{\left(q^a, q^b; q\right)_n}{(q^c; q)_n} q^{n(c-a-b)-\binom{n}{2}} \tag{4.4.4} $$

for $n=0,1,2,\ldots$. Next, replace c by d in (4.4.4), multiply both sides by $(-1)^n q^{n(e-c)+\binom{n}{2}} (q^c;q)_n / (q,q^e;q)_n$, sum over n and change the order of integration and summation (which is easily justified if $|q^{e-c+s}|<1$) to obtain

$$\begin{aligned}
&\csc\pi(d-a-b)\frac{\left(q^{1+a+b-d},q^{d-a-b},q^d,q;q\right)_\infty}{(q^a,q^b,q^{d-a},q^{d-b};q)_\infty} \\
&\quad\times {}_3\phi_2\left(q^a,q^b,q^c;q^d,q^e;q,q^{d+e-a-b-c}\right) \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s},q^{1-d+a+b+s};q\right)_\infty}{(q^{a+s},q^{b+s};q)_\infty}\frac{\pi q^s}{\sin\pi s\sin\pi(d-a-b-s)} \\
&\quad\times {}_2\phi_1\left(q^{-s},q^c;q^e;q,q^{e-c+s}\right)ds \\
&= \frac{(q^{e-c};q)_\infty}{(q^e;q)_\infty}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s},q^{e+s},q^{1-d+a+b+s};q\right)_\infty}{(q^{a+s},q^{b+s},q^{e-c+s};q)_\infty} \\
&\quad\times\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(d-a-b+s)},
\end{aligned}\tag{4.4.5}$$

by the q-Gauss formula (1.5.1). Now take $c=d$. Then the series on the left of (4.4.5) can be summed by the q-Gauss formula to give, after an obvious change in parameters,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s},q^{d+s},q^{e+s};q\right)_\infty}{(q^{a+s},q^{b+s},q^{c+s};q)_\infty}\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(d+s)} \\
&= \csc\pi d\;\frac{\left(q,q^d,q^{1-d},q^{e-a},q^{e-b},q^{e-c};q\right)_\infty}{\left(q^a,q^b,q^c,q^{1+a-d},q^{1+b-d},q^{1+c-d};q\right)_\infty},
\end{aligned}\tag{4.4.6}$$

where $d+e=1+a+b+c$, which is Agarwal's q-analogue of Barnes' second lemma. This integral converges if q is so small that

$$\mathrm{Re}\left[s\log q-\log(\sin\pi s\sin\pi(d+s))\right]<0 \tag{4.4.7}$$

on the contour for large $|s|$.

4.5 Analytic continuation of ${}_{r+1}\phi_r$ series

By employing Cauchy's theorem as in §4.2, we find that if $|z|<1$ and $|\arg(-z)|<\pi$, then

$$\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,z\right] \\
&= \frac{(a_1,a_2,\ldots,a_{r+1};q)_\infty}{(q,b_1,\ldots,b_r;q)_\infty} \\
&\quad\times\left(\frac{-1}{2\pi i}\right)\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s},b_1q^s,\ldots,b_rq^s;q\right)_\infty}{(a_1q^s,a_2q^s,\ldots,a_{r+1}q^s;q)_\infty}\frac{\pi(-z)^s\,ds}{\sin\pi s},
\end{aligned}\tag{4.5.1}$$

where, as before, only the poles of the integrand at $0,1,2,\ldots$, lie to the right of the contour. As in the $r=1$ case, the right side of (4.5.1) gives the analytic continuation of the ${}_{r+1}\phi_r$ series on the left side to the domain $|\arg(-z)|<\pi$.

Also, as in §4.3, it can be shown that if $|z| > |b_1 \cdots b_r q/a_1 \cdots a_{r+1}|$, then the integral $I_3 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \ldots ds$ is equal to the sum of the residues of the integrand at those poles of the integrand which lie on the left of the contour. Set $\alpha_1 = -\omega^{-1}\log a_1$. Since the residue of the integrand at $-\alpha_1 - n + 2\pi i m\omega^{-1}$ is

$$\frac{\left(q^{n+1}, a_1^{-1}q^{1-n}, b_1 a_1^{-1}q^{-n}, \ldots, b_r a_1^{-1}q^{-n}; q\right)_\infty}{\left(q, q, a_2 a_1^{-1}q^{-n}, \ldots, a_{r+1}a_1^{-1}q^{-n}; q\right)_\infty}\pi\omega^{-1}(-z)^{-\alpha_1-n}q^{n(n+1)/2}$$
$$\times \csc(2m\pi^2 i\omega^{-1} - \alpha_1\pi)\exp\left\{2m\pi i\omega^{-1}\log(-z)\right\},$$

by proceeding as in the proof of (4.3.2) and using (4.3.9) and (4.5.1) we obtain the expansion

$$\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ \\ b_1, \ldots, b_r\end{matrix}; q, z\right] \\
&= \frac{(a_2, \ldots, a_{r+1}, b_1/a_1, \ldots, b_r/a_1, a_1 z, q/a_1 z; q)_\infty}{(b_1, \ldots, b_r, a_2/a_1, \ldots, a_{r+1}/a_1, z, q/z; q)_\infty} \\
&\quad\times {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_1q/b_1, \ldots, a_1q/b_r \\ a_1q/a_2, \ldots, a_1q/a_{r+1}\end{matrix}; q, \frac{qb_1\cdots b_r}{za_1\cdots a_{r+1}}\right] \\
&\quad + \text{idem}\ (a_1; a_2, \ldots, a_{r+1}),
\end{aligned} \tag{4.5.2}$$

where the equality holds in the "is the analytic continuation of" sense. The symbol "idem $(a_1; a_2, \ldots, a_{r+1})$" after an expression stands for the sum of the r expressions obtained from the preceding expression by interchanging a_1 with each a_k, $k = 2, 3, \ldots, r+1$.

4.6 Contour integrals representing well-poised series

Let us replace a, b, c, d and e in (4.4.6) by $a+n, b+n, c+n, d+n$ and $e+2n$, respectively, where

$$e = 1 + a + b + c - d, \tag{4.6.1}$$

and transform the integration variable s to $s-n$, where n is a non-negative integer. Then we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{n-i\infty}^{n+i\infty}\frac{\left(q^{1+s-n}, q^{d+s}, q^{e+s+n}; q\right)_\infty}{\left(q^{a+s}, q^{b+s}, q^{c+s}; q\right)_\infty}\frac{\pi q^{s-n}\,ds}{\sin\pi s\sin\pi(d+s)} \\
&= \csc\pi d\ \frac{\left(q, q^{d+n}, q^{1-d-n}, q^{e-a+n}, q^{e-b+n}, q^{e-c+n}; q\right)_\infty}{\left(q^{a+n}, q^{b+n}, q^{c+n}, q^{e-a-b}, q^{e-a-c}, q^{e-b-c}; q\right)_\infty}.
\end{aligned} \tag{4.6.2}$$

The limits of integration $n \pm i\infty$ can be replaced by $\pm i\infty$ because we always indent the contour of integration to separate the increasing and decreasing sequences of poles. Thus, it follows from (4.6.2) that

$$\begin{aligned}
&\csc(\pi d)\ \frac{\left(q, q^d, q^{1-d}, e^{e-a}, q^{e-b}, q^{e-c}; q\right)_\infty}{\left(q^a, q^b, q^c, q^{e-b-c}, q^{e-c-a}, q^{e-a-b}; q\right)_\infty}\frac{\left(q^a, q^b, q^c; q\right)_n}{\left(q^{e-a}, q^{e-b}, q^{e-c}; q\right)_n}q^{(1-d)n} \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s}, q^{d+s}, q^{e+s}; q\right)_\infty}{\left(q^{a+s}, q^{b+s}, q^{c+s}; q\right)_\infty}\frac{\pi q^{(n+1)s}(q^{-s}; q)_n\,ds}{\sin\pi s\sin\pi(d+s)(q^{e+s}; q)_n},
\end{aligned} \tag{4.6.3}$$

and hence, by termwise integration, we obtain Agarwal's [1953b] formula

$$\begin{aligned}
&\csc(\pi d)\,\frac{(q,q^d,q^{1-d},q^{e-a},q^{e-b},q^{e-c};q)_\infty}{(q^a,q^b,q^c,q^{e-b-c},q^{e-c-a},q^{e-a-b};q)_\infty}\\
&\quad\times\,{}_{r+4}\phi_{r+3}\left[\begin{matrix}q^A,q^a,q^b,q^c,a_1,\ldots,a_r\\ q^{e-a},q^{e-b},q^{e-c},b_1,\ldots,b_r\end{matrix};q,zq^{-d}\right]\\
&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(q^{1+s},q^{d+s},q^{e+s};q)_\infty}{(q^{a+s},q^{b+s},q^{c+s};q)_\infty}\,\frac{\pi q^s}{\sin\pi s\sin\pi(d+s)}\\
&\quad\times\,{}_{r+2}\phi_{r+1}\left[\begin{matrix}q^A,a_1,\ldots,a_r,q^{-s}\\ b_1,\ldots,b_r,q^{e+s}\end{matrix};q,zq^{s-1}\right]ds, \qquad (4.6.4)
\end{aligned}$$

where $e=1+a+b+c-d$, $|z|<|q^d|$ and it is assumed that (4.4.7) holds.

Therefore, if we can sum the series on the right of (4.6.4), then we can find a simpler contour integral representing the series on the left. In particular, if we let

$$\begin{aligned}
&r=4, e=1+A, a_1=q^{1+A/2}=-a_2, a_3=q^d, a_4=q^e,\\
&b_1=q^{A/2}=-b_2, b_3=q^{1-d+A}, b_4=q^{1-e+A}, z=q^{2-d-e+A},
\end{aligned}$$

then we get a VWP-balanced ${}_6\phi_5$ series which can be summed by (2.7.1). This yields Agarwal's [1953b] contour integral representation for a VWP-balanced ${}_8\phi_7$ series:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}q^A,q^{1+A/2},-q^{1+A/2},q^a,q^b,q^c,q^d,q^e\\ q^{A/2},-q^{A/2},q^{1+A-a},q^{1+A-b},q^{1+A-c},q^{1+A-d},q^{1+A-e}\end{matrix};q,q^B\right]\\
&=\sin\pi(a+b+c-A)\\
&\quad\times\frac{(q^{1+A},q^a,q^b,q^c,q^{1+A-a-b},q^{1+A-b-c},q^{1+A-c-a},q^{1+A-d-e};q)_\infty}{(q,q^{a+b+c-A},q^{1+A-a-b-c},q^{1+A-a},q^{1+A-b},q^{1+A-c},q^{1+A-d},q^{1+A-e};q)_\infty}\\
&\quad\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(q^{1+s},q^{1+A-d+s},q^{1+A-e+s},q^{a+b+c-A+s};q)_\infty}{(q^{a+s},q^{b+s},q^{c+s},q^{1+A-d-e+s};q)_\infty}\\
&\quad\times\frac{\pi q^s\,ds}{\sin\pi s\sin\pi(a+b+c-A+s)}, \qquad (4.6.5)
\end{aligned}$$

where $B=2+2A-a-b-c-d-e$, provided $\mathrm{Re}(B)>0$ and

$$\mathrm{Re}\,[s\log q-\log(\sin\pi s\sin\pi(a+b+c-A+s))]<0.$$

If we evaluate the integral in (4.6.5) by considering the residues at the poles of $1/[\sin\pi s\sin\pi(a+b+c-A+s)]$ lying to the right of the contour, then we obtain the transformation (2.10.10) of a VWP-balanced ${}_8\phi_7$ series in terms of the sum of two balanced ${}_4\phi_3$ series. In addition, if we replace A, d, e and a by $\lambda, \lambda+d-A, \lambda+e-A$ and $\lambda+a-A$, respectively, and take $\lambda+a+d+e=1+2A$ in (4.6.5), then the integral in (4.6.5) remains unchanged. This gives Bailey's transformation formula (2.10.1) between two VWP-balanced ${}_8\phi_7$ series.

4.7 A contour integral analogue of Bailey's summation formula

By replacing A, a, b, c, d, e in (4.6.5) by a, d, e, f, b, c, respectively, we obtain the formula

$$
\begin{aligned}
{}_8W_7\left(q^a; q^b, q^c, q^d, q^e, q^f; q, q\right) &= \sin\pi(d+e+f-a) \\
&\times \frac{\left(q^{1+a}, q^d, q^e, q^f, q^{1+a-d-e}, q^{1+a-d-f}, q^{1+a-e-f}; q\right)_\infty}{(q, q^{1+a-b}, q^{1+a-c}, q^{1+a-d}, q^{1+a-e}, q^{1+a-f}, q^{1+a-d-e-f}; q)_\infty} \\
&\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1+s}, q^{1+a-b+s}, q^{1+a-c+s}; q\right)_\infty}{(q^{d+s}, q^{e+s}, q^{f+s}; q)_\infty} \frac{\pi q^s\, ds}{\sin\pi s \sin\pi(d+e+f-a+s)},
\end{aligned}
\tag{4.7.1}
$$

provided the series is VWP-balanced, i.e.,

$$
1+2a = b+c+d+e+f. \tag{4.7.2}
$$

Since $1+2(2b-a) = b+(b+c-a)+(b+d-a)+(b+e-a)+(b+f-a)$ by (4.7.2), it follows that (4.7.2) remains unchanged if we replace a, c, d, e, f by $2b-a, b+c-a, b+d-a, b+e-a, b+f-a$, respectively, and keep b unaltered. Then (4.7.1) gives

$$
\begin{aligned}
{}_8W_7&\left(q^{2b-a}; q^b, q^{b+c-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}; q, q\right) = \sin\pi c \\
&\times \frac{\left(q^{1+2b-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}, q^{1+a-d-e}, q^{1+a-d-f}, q^{1+a-e-f}; q\right)_\infty}{(q, q^{1+b-a}, q^{1+b-c}, q^{1+b-d}, q^{1+b-e}, q^{1+b-f}, q^c; q)_\infty} \\
&\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1+b-a+s}, q^{1+b-c+s}, q^{1+s}; q\right)_\infty}{(q^{b+d-a+s}, q^{b+e-a+s}, q^{b+f-a+s}; q)_\infty} \frac{\pi q^s\, ds}{\sin\pi s \sin\pi(c-s)} \\
&= \sin\pi c\, \frac{\left(q^{1+2b-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}; q\right)_\infty}{(q, q^{1+b-a}, q^{1+b-c}, q^c; q)_\infty} \\
&\times \frac{\left(q^{1+a-d-e}, q^{1+a-d-f}, q^{1+a-e-f}; q\right)_\infty}{(q^{1+b-d}, q^{1+b-e}, q^{1+b-f}; q)_\infty} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(q^{1+s}; q)_\infty}{(q^{d+s}; q)_\infty} \\
&\times \frac{\left(q^{1+a-c+s}, q^{1+a-b+s}; q\right)_\infty}{(q^{e+s}, q^{f+s}; q)_\infty} \frac{\pi q^s\, ds}{\sin\pi(a-b+s)\sin\pi(c+b-a-s)},
\end{aligned}
\tag{4.7.3}
$$

where the second integral in (4.7.3) follows from the first by a change of the integration variable $s \to a-b+s$. Combining (4.7.1) and (4.7.3) and simplifying, we obtain

$$
\begin{aligned}
&\frac{\left(q^{1+a-b}, q^{1+a-c}, q^{1+a-d}, q^{1+a-e}, q^{1+a-f}; q\right)_\infty}{(q^{1+a}, q^c, q^d, q^e, q^f; q)_\infty} \\
&\quad \times {}_8W_7\left(q^a; q^b, q^c, q^d, q^e, q^f; q, q\right) \\
&\quad - q^{b-a}\frac{\left(q^{1+b-a}, q^{1+b-c}, q^{1+b-d}, q^{1+b-e}, q^{1+b-f}; q\right)_\infty}{(q^{1+2b-a}, q^{b+c-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}; q)_\infty}
\end{aligned}
$$

$$
\begin{aligned}
&\times {}_8W_7\left(q^{2b-a}; q^b, q^{b+c-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}; q, q\right) \\
&= \frac{\left(q^{1+a-d-e}, q^{1+a-e-f}, q^{1+a-d-f}; q\right)_\infty}{(q, q^c, q^{b+c-a}; q)_\infty} \\
&\quad\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1+s}, q^{1+a-b+s}, q^{1+a-c+s}; q\right)_\infty}{(q^{d+s}, q^{e+s}, q^{f+s}; q)_\infty} \frac{\pi q^s \sin\pi(a-b)\, ds}{\sin\pi s \sin\pi(a-b+s)},
\end{aligned}
\tag{4.7.4}
$$

when (4.7.2) holds.

Evaluating the above integral via (4.4.6), we obtain Bailey's summation formula (2.11.7).

Agarwal's [1953b] formula

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1+s}, q^{\frac{1}{2}a+s}, -q^{\frac{1}{2}a+s}, q^{1+a-b+s}, q^{1+a-c+s}; q\right)_\infty}{\left(q^{a+s}, q^{1+\frac{1}{2}a+s}, -q^{1+\frac{1}{2}a+s}, q^{b+s}, q^{c+s}; q\right)_\infty} \\
&\qquad\times \frac{\left(q^{1+a-d+s}, q^{1+a-e+s}, q^{1+a-f+s}; q\right)_\infty}{(q^{d+s}, q^{e+s}, q^{f+s}; q)_\infty} \frac{\pi q^s\, ds}{\sin\pi s \sin\pi(a-b+s)} \\
&\quad= \csc\pi(a-b) \frac{\left(q, q^{1+a-b}, q^{b-a}, q^{1+a-d-e}, q^{1+a-e-f}; q\right)_\infty}{(q^b, q^c, q^d, q^e, q^f; q)_\infty} \\
&\qquad\times \frac{\left(q^{1+a-d-f}, q^{1+a-c-d}, q^{1+a-c-e}, q^{1+a-c-f}; q\right)_\infty}{(q^{b+c-a}, q^{b+d-a}, q^{b+e-a}, q^{b+f-a}; q)_\infty},
\end{aligned}
\tag{4.7.5}
$$

where $1+2a = b+c+d+e+f$, follows directly from (2.11.7) by considering the residues of the integrand of the above integral at the poles to the right of the contour, i.e. at $s = n, b-a+n$ with $n = 0, 1, 2, \ldots$. Thus (4.7.5) gives an integral analogue of Bailey's summation formula (2.11.7). The integral in (4.7.5) converges if q is so small that

$$
\operatorname{Re}\left[s\log q - \log(\sin\pi s \sin\pi(a-b+s))\right] < 0 \tag{4.7.6}
$$

on the contour for large $|s|$.

4.8 Extensions to complex q inside the unit disc

The previous basic contour integrals can be extended to complex q inside the unit disc by using suitable contours. For $0 < |q| < 1$, let

$$
\log q = -\left(\omega_1 + i\omega_2\right), \tag{4.8.1}
$$

where $\omega_1 = -\log|q| > 0$ and $\omega_2 = -\operatorname{Arg} q$.

Thus $q = e^{-(\omega_1+i\omega_2)}$. Then a modification of the proof in §4.2 (see Watson [1910]) shows that if $0 < |q| < 1$ and $|z| < 1$, then formula (4.2.2) extends to

$$
\begin{aligned}
&{}_2\phi_1(a, b; c; q, z) \\
&\quad= \frac{(a, b; q)_\infty}{(q, c; q)_\infty}\left(\frac{-1}{2\pi i}\right)\int_C \frac{\left(q^{1+s}, cq^s; q\right)_\infty}{(aq^s, bq^s; q)_\infty} \frac{\pi(-z)^s}{\sin\pi s}\, ds,
\end{aligned}
\tag{4.8.2}
$$

where C is an upward directed contour parallel to the line $\mathrm{Re}(s(\omega_1 + i\omega_2)) = 0$ with indentations, to ensure that the increasing sequence of poles $0, 1, 2, \ldots$, of the integrand lie to the right, and the decreasing sequences of poles lie to the left of C.

Since the above integral converges if $\mathrm{Re}[s \log(-z) - \log(\sin \pi s)] < 0$ on C for large $|s|$, i.e., if

$$|\arg(-z) - \omega_2 \omega_1^{-1} \log |z|| < \pi, \tag{4.8.3}$$

it is required that z satisfies (4.8.3) in order for (4.8.2) to hold. This restriction means that the z-plane has a cut in the form of the spiral whose equation in polar coordinates is $r = e^{\omega_1 \theta/\omega_2}$.

Analogously, when $0 < |q| < 1$, the contours in the q-analogues of Barnes' first and second lemmas given in §4.4 and the contours in the other integrals in §§4.4–4.7 must be replaced by upward directed contours parallel to the line $\mathrm{Re}(s(\omega_1 + i\omega_2)) = 0$ with indentations to separate the increasing and decreasing sequences of poles.

4.9 Other types of basic contour integrals

Let $q = e^{-\omega}$ with $\omega > 0$ and suppose that

$$P(z) = \frac{(a_1 z, \ldots, a_A z,\ b_1/z, \ldots, b_B/z; q)_\infty}{(c_1 z, \ldots, c_C z,\ d_1/z, \ldots, d_D/z; q)_\infty} \tag{4.9.1}$$

has only simple poles. During the 1950's Slater [1952c,d, 1955] considered contour integrals of the form

$$I_m \equiv I_m(A, B; C, D) = \frac{\omega}{2\pi i} \int_{-i\pi/\omega}^{i\pi/\omega} P(q^s) q^{ms} ds \tag{4.9.2}$$

with $m = 0$ or 1. However, here we shall let m be an arbitrary integer. It is assumed that none of the poles of $P(q^s)$ lie on the lines $\mathrm{Im} s = \pm\pi/\omega$ and that the contour of integration runs from $-i\pi/\omega$ to $i\pi/\omega$ and separates the increasing sequences of poles in $|\mathrm{Im}\ s| < \pi/\omega$ from those that are decreasing.

By setting $i\theta = -s\omega$ the integral I_m can also be written in the "exponential" form

$$I_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} P\left(e^{i\theta}\right) e^{im\theta} d\theta \tag{4.9.3}$$

with suitable indentations, if necessary, in the contour of integration. Similarly, setting $z = q^s$ we obtain that

$$I_m = \frac{1}{2\pi i} \int_K P(z) z^{m-1} dz, \tag{4.9.4}$$

where the contour K is a deformation of the (positively oriented) unit circle so that the poles of $1/\left(c_1 z, \ldots, c_C z; q\right)_\infty$ lie outside the contour and the origin and poles of $1/\left(d_1/z, \ldots, d_D/z; q\right)_\infty$ lie inside the contour. Special cases of (4.9.3) and (4.9.4) have been considered by Askey and Roy [1986].

Although each of the above three types of integrals can be used to derive transformation formulas for basic hypergeometric series, we shall prefer to use mainly the contour integrals of the type in (4.9.4) since they are easier to work with, especially when the assumption in Slater [1952c,d, 1966] that $0 < q < 1$ is replaced by only assuming that $|q| < 1$, which is the case we wish to consider in the remainder of this chapter.

4.10 General basic contour integral formulas

Our main objective in this section is to see what formulas can be derived by applying Cauchy's theorem to the integrals I_m in (4.9.4).

Let $|q| < 1$ and let δ be a positive number such that $\delta \neq |d_j q^n|$ for $j = 1, 2, \ldots, D$, and $\delta \neq |c_j^{-1} q^{-n}|$ for $j = 1, 2, \ldots, C$ when $n = 0, 1, 2, \ldots$. Also let C_N be the circle $|z| = \delta |q|^N$, where N is a positive integer. Then C_N does not pass through any of the poles of $P(z)$ and we have that

$$\begin{aligned} |P\left(\delta q^N\right)\left(\delta q^N\right)^{m-1}| &= \left|\frac{(a_1\delta, \ldots, a_A\delta, b_1/\delta, \ldots, b_B/\delta; q)_\infty}{(c_1\delta, \ldots, c_C\delta, d_1/\delta, \ldots, d_D/\delta; q)_\infty}\right| \\ &\times \left|\frac{(c_1\delta, \ldots, c_C\delta, q\delta/b_1, \ldots, q\delta/b_B; q)_N}{(a_1\delta, \ldots, a_A\delta, q\delta/d_1, \ldots, q\delta/d_D; q)_N}\left(\frac{b_1\cdots b_B q^{m-1}}{d_1\cdots d_D}\right)^N\right| \\ &\times \delta^{m-1}\left|\delta^N q^{\binom{N+1}{2}}\right|^{D-B} = O\left(\left|\frac{b_1\cdots b_B q^{m-1}}{d_1\cdots d_D}\right|^N \left|\delta^N q^{\binom{N+1}{2}}\right|^{D-B}\right). \end{aligned} \tag{4.10.1}$$

Since C_N is of length $O(|q|^N)$ it follows from (4.10.1) that if $D > B$ or if $D = B$ and

$$\left|\frac{b_1\cdots b_B q^m}{d_1\cdots d_D}\right| < 1, \tag{4.10.2}$$

then

$$\lim_{N\to\infty}\int_{C_N} P(z) z^{m-1} dz = 0. \tag{4.10.3}$$

Hence, by applying Cauchy's residue theorem to the region between K and C_N for sufficiently large N and letting $N \to \infty$, we find that if $D > B$ or if $D = B$ and (4.10.2) holds, then I_m equals the sum of the residues of $P(z)z^{m-1}$ at the poles of $1/\left(d_1/z, \ldots, d_D/z; q\right)_\infty$. Therefore, since

$$\underset{z=dq^n}{\text{Residue}}\left(\frac{1}{(d/z;q)_\infty}\right) = \frac{(-1)^n d q^{2n+\binom{n}{2}}}{(q;q)_n (q;q)_\infty}, \qquad n = 0, 1, 2, \ldots, \tag{4.10.4}$$

it follows that

$$\begin{aligned} I_m = &\frac{(a_1 d_1, \ldots, a_A d_1, b_1/d_1, \ldots, b_B/d_1; q)_\infty}{(q, c_1 d_1, \ldots, c_C d_1, d_2/d_1, \ldots, d_D/d_1; q)_\infty} d_1^m \\ &\times \sum_{n=0}^\infty \frac{(c_1 d_1, \ldots, c_C d_1, q d_1/b_1, \ldots, q d_1/b_B; q)_n}{(q, a_1 d_1, \ldots, a_A d_1, q d_1/d_2, \ldots, q d_1/d_D; q)_n} \\ &\times \left(-d_1 q^{(n+1)/2}\right)^{n(D-B)} \left(\frac{b_1\cdots b_B q^m}{d_1\cdots d_D}\right)^n \\ &+ \text{idem } (d_1; d_2, \ldots, d_D) \end{aligned} \tag{4.10.5}$$

if $D > B$, or if $D = B$ and (4.10.2) holds.

In addition, by considering the residues of $P(z)z^{m-1}$ outside of K or by just using the inversion $z \to z^{-1}$ and renaming the parameters, we obtain

$$\begin{aligned} I_m = &\frac{(b_1c_1, \ldots, b_Bc_1, a_1/c_1, \ldots, a_A/c_1; q)_\infty}{(q, d_1c_1, \ldots, d_Dc_1, c_2/c_1, \ldots, c_C/c_1; q)_\infty} c_1^{-m} \\ &\times \sum_{n=0}^{\infty} \frac{(d_1c_1, \ldots, d_Dc_1, qc_1/a_1, \ldots, qc_1/a_A; q)_n}{(q, b_1c_1, \ldots, b_Bc_1, qc_1/c_2, \ldots, qc_1/c_C; q)_n} \\ &\times \left(-c_1q^{(n+1)/2}\right)^{n(C-A)} \left(\frac{a_1 \cdots a_A q^{-m}}{c_1 \cdots c_C}\right)^n \\ &+ \text{idem } (c_1; c_2, \ldots, c_C) \end{aligned} \tag{4.10.6}$$

if $C > A$, or if $C = A$ and

$$\left|\frac{a_1 \cdots a_A q^{-m}}{c_1 \cdots c_C}\right| < 1. \tag{4.10.7}$$

In the special case when $C = A$ we can use the ${}_r\phi_s$ notation to write (4.10.5) in the form

$$\begin{aligned} I_m(A, B; A, D) = &\frac{(a_1d_1, \ldots, a_Ad_1, b_1/d_1, \ldots, b_B/d_1; q)_\infty}{(q, c_1d_1, \ldots, c_Ad_1, d_2/d_1, \ldots, d_D/d_1; q)_\infty} d_1^m \\ &\times {}_{A+B}\phi_{A+D-1} \left[\begin{matrix} c_1d_1, \ldots, c_Ad_1, qd_1/b_1, \ldots, qd_1/b_B \\ a_1d_1, \ldots, a_Ad_1, qd_1/d_2, \ldots, qd_1/d_D \end{matrix}; q, t(qd_1)^{D-B}\right] \\ &+ \text{idem } (d_1; d_2, \ldots, d_D) \end{aligned} \tag{4.10.8}$$

where $t = b_1b_2 \cdots b_Bq^m/d_1 \cdots d_D$, if $D > B$, or if $D = B$ and (4.10.2) holds.

Similarly, from the $D = B$ case of (4.10.6) we have

$$\begin{aligned} I_m(A, B; C, B) = &\frac{(b_1c_1, \ldots, b_Bc_1, a_1/c_1, \ldots, a_A/c_1; q)_\infty}{(q, d_1c_1, \ldots, d_Bc_1, c_2/c_1, \ldots, c_C/c_1; q)_\infty} c_1^{-m} \\ &\times {}_{A+B}\phi_{B+C-1} \left[\begin{matrix} d_1c_1, \ldots, d_Bc_1, qc_1/a_1, \ldots, qc_1/a_A \\ b_1c_1, \ldots, b_Bc_1, qc_1/c_2, \ldots, qc_1/c_C \end{matrix}; q, u(qc_1)^{C-A}\right] \\ &+ \text{idem } (c_1; c_2, \ldots, c_C) \end{aligned} \tag{4.10.9}$$

if $C > A$, or if $C = A$ and (4.10.7) holds, where $u = a_1 \cdots a_Aq^{-m}/c_1 \cdots c_C$.

Evaluations of I_m which follow from these formulas will be considered in §4.11.

From (4.10.8) and (4.10.9) it follows that if $C = A$ and $D = B$, then we have the transformation formula

$$\begin{aligned} &\frac{(a_1d_1, \ldots, a_Ad_1, b_1/d_1, \ldots, b_B/d_1; q)_\infty}{(c_1d_1, \ldots, c_Ad_1, d_2/d_1, \ldots, d_B/d_1; q)_\infty} d_1^m \\ &\quad\times {}_{A+B}\phi_{A+B-1} \left[\begin{matrix} c_1d_1, \ldots, c_Ad_1, qd_1/b_1, \ldots, qd_1/b_B \\ a_1d_1, \ldots a_Ad_1, qd_1/d_2, \ldots, qd_1/d_B \end{matrix}; q, \frac{b_1 \cdots b_Bq^m}{d_1 \cdots d_B}\right] \\ &\quad+ \text{idem } (d_1; d_2, \ldots, d_B) \\ &= \frac{(b_1c_1, \ldots, b_Bc_1, a_1/c_1, \ldots, a_A/c_1; q)_\infty}{(d_1c_1, \ldots, d_Bc_1, c_2/c_1, \ldots, c_A/c_1; q)_\infty} c_1^{-m} \end{aligned}$$

$$\times_{A+B}\,\phi_{A+B-1}\left[\begin{matrix}d_1c_1,\ldots,d_Bc_1,qc_1/a_1,\ldots,qc_1/a_A\\ b_1c_1,\ldots,b_Bc_1,qc_1/c_2,\ldots,qc_1/c_A\end{matrix};q,\frac{a_1\cdots a_Aq^{-m}}{c_1\cdots c_A}\right]$$
$$+\text{ idem }(c_1;c_2,\ldots,c_A) \tag{4.10.10}$$

provided that $|b_1\cdots b_Bq^m|<|d_1\cdots d_B|, |a_1\cdots a_Aq^{-m}|<|c_1\cdots c_A|$ and $m=0,\pm1,\pm2,\ldots$.

In some applications it is useful to have a variable z in the argument of the series which is independent of the parameters in the series. This can be accomplished by replacing A by $A+1, B$ by $B+1$ and setting $b_{B+1}=z$ and $a_{A+1}=q/z$ in (4.10.10). More generally, doing this to the $m=0$ case of (4.10.5) and of (4.10.6) gives the rather general transformation formula

$$\begin{aligned}&\frac{(a_1d_1,\ldots,a_Ad_1,b_1/d_1,\ldots,b_B/d_1,z/d_1,qd_1/z;q)_\infty}{(c_1d_1,\ldots,c_Cd_1,d_2/d_1,\ldots,d_D/d_1;q)_\infty}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(c_1d_1,\ldots,c_Cd_1,qd_1/b_1,\ldots,qd_1/b_B;q)_n}{(q,a_1d_1,\ldots,a_Ad_1,qd_1/d_2,\ldots,qd_1/d_D;q)_n}\\ &\quad\times\left(-d_1q^{(n+1)/2}\right)^{n(D-B-1)}\left(\frac{b_1\cdots b_Bz}{d_1\cdots d_D}\right)^n\\ &\quad+\text{ idem }(d_1;d_2,\ldots,d_D)\\ &=\frac{(b_1c_1,\ldots,b_Bc_1,a_1/c_1,\ldots,a_A/c_1,c_1z,q/c_1z;q)_\infty}{(d_1c_1,\ldots,d_Dc_1,c_2/c_1,\ldots,c_C/c_1;q)_\infty}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(d_1c_1,\ldots,d_Dc_1,qc_1/a_1,\ldots,qc_1/a_A;q)_n}{(q,b_1c_1,\ldots,b_Bc_1,qc_1/c_2,\ldots,qc_1/c_C;q)_n}\\ &\quad\times\left(-c_1q^{(n+1)/2}\right)^{n(C-A-1)}\left(\frac{a_1\cdots a_Aq}{c_1\cdots c_Cz}\right)^n\\ &\quad+\text{ idem }(c_1;c_2,\ldots,c_C),\end{aligned} \tag{4.10.11}$$

where, for convergence,

(i) $D>B+1,$ or $D=B+1$ and $\left|\dfrac{b_1\cdots b_Bz}{d_1\cdots d_D}\right|<1$

and

(ii) $C>A+1,$ or $C=A+1$ and $\left|\dfrac{a_1\cdots a_Aq}{c_1\cdots c_Cz}\right|<1.$

This is formula (5.2.20) in Slater [1966]. Observe that by replacing z in (4.10.11) by zq^m and using the identity

$$(qd/zq^m,zq^m/d;q)_\infty=(-1)^m(d/z)^mq^{-\binom{m}{2}}(qd/z,zd;q)_\infty \tag{4.10.12}$$

we obtain from (4.10.11) the formula that would have been derived by using (4.10.5) and (4.10.6) with m an arbitrary integer.

4.11 Some additional extensions of the beta integral

Askey and Roy [1986] used Ramanujan's summation formula (2.10.17) to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(ce^{i\theta}/\beta, qe^{i\theta}/c\alpha, c\alpha e^{-i\theta}, q\beta e^{-i\theta}/c;q\right)_\infty}{\left(ae^{i\theta}, be^{i\theta}, \alpha e^{-i\theta}, \beta e^{-i\theta};q\right)_\infty}\,d\theta$$
$$= \frac{(ab\alpha\beta, c, q/c, c\alpha/\beta, q\beta/c\alpha;q)_\infty}{(a\alpha, a\beta, b\alpha, b\beta, q;q)_\infty}, \tag{4.11.1}$$

where $\max(|q|,|a|,|b|,|\alpha|,|\beta|) < 1$ and $c\alpha\beta \neq 0$; and they extended it to the contour integral form

$$\frac{1}{2\pi i}\int_K \frac{(cz/\beta, qz/c\alpha, c\alpha/z, q\beta/cz;q)_\infty}{(az, bz, \alpha/z, \beta/z;q)_\infty}\frac{dz}{z}$$
$$= \frac{(ab\alpha\beta, c, q/c, c\alpha/\beta, q\beta/c\alpha;q)_\infty}{(a\alpha, a\beta, b\alpha, b\beta, q;q)_\infty} \tag{4.11.2}$$

where $a\alpha, a\beta, b\alpha, b\beta \neq q^{-n}, n = 0,1,2,\ldots, c\alpha\beta \neq 0$, and K is a deformation of the unit circle as described in §4.9. These formulas can also be derived from the $A = B = D = 2, m = 0$ case of (4.10.8) by setting $a_1 = c/\beta, a_2 = q/c\alpha, b_1 = c\alpha, b_2 = q\beta/c, c_1 = a, c_2 = b, d_1 = \alpha$ and $d_2 = \beta$ and then using the summation formula (2.10.13) for the sum of the two ${}_2\phi_1$ series resulting on the right side. In Askey and Roy [1986] it is also shown how Barnes' beta integral (4.1.2) can be obtained as a limit case of (4.11.1).

Analogously, application of the summation formula (2.10.11) to the $A = 3, B = D = 2, m = 0$ case of (4.10.8) gives

$$\frac{1}{2\pi i}\int_K \frac{(\delta z, qz/\gamma, \gamma z/\alpha\beta, \gamma/z, q\alpha\beta/\gamma z;q)_\infty}{(az, bz, cz, \alpha/z, \beta/z;q)_\infty}\frac{dz}{z}$$
$$= \frac{(\gamma/\alpha, q\alpha/\gamma, \gamma/\beta, q\beta/\gamma, \delta/a, \delta/b, \delta/c;q)_\infty}{(a\alpha, a\beta, b\alpha, b\beta, c\alpha, c\beta, q;q)_\infty}, \tag{4.11.3}$$

where $\delta = abc\alpha\beta, abc\alpha\beta\gamma \neq 0$, and

$$a\alpha, a\beta, b\alpha, b\beta, c\alpha, c\beta \neq q^{-n}, \qquad n = 0,1,2,\ldots$$

Note that (4.11.2) follows from the $c \to 0$ case of (4.11.3).

In addition, application of Bailey's summation formula (2.11.7) gives the more general formula

$$\frac{1}{2\pi i}\int_K \frac{\left(za^{\frac12}, -za^{\frac12}, qaz/b, qaz/c, qaz/d, qaz/f;q\right)_\infty}{\left(qza^{\frac12}, -qza^{\frac12}, bz, cz, dz, fz;q\right)_\infty}$$
$$\times \frac{(qz/\gamma, \gamma z/\alpha\beta, \gamma/z, q\alpha\beta/\gamma z;q)_\infty}{(a\alpha z, a\beta z, \alpha/z, \beta/z;q)_\infty}\frac{dz}{z}$$
$$= \frac{(\gamma/\alpha, q\alpha/\gamma, \gamma/\beta, q\beta/\gamma, aq/cd, aq/bd, aq/bc, aq/bf, aq/cf, aq/df;q)_\infty}{(a\alpha\beta, b\alpha, c\alpha, d\alpha, f\alpha, b\beta, c\beta, d\beta, f\beta, q;q)_\infty} \tag{4.11.4}$$

where $aq = bcdf\alpha\beta, bcdf\alpha\beta\gamma \neq 0$,

$$a\alpha\beta, b\alpha, c\alpha, d\alpha, f\alpha, b\beta, c\beta, d\beta, f\beta \neq q^{-n}, \qquad n = 0, 1, 2, \ldots,$$

and K is as described in §4.9; see Gasper [1989c].

4.12 Sears' transformations of well-poised series

Sears [1951d, (7.2)] used series manipulations of well-poised series to derive the transformation formula

$$\begin{aligned}
&\frac{\left(qa_1/a_{M+2}, \ldots, qa_1/a_{2M},\ q/a_{M+2}, \ldots, q/a_{2M}, a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, q/a_1^{\frac{1}{2}}, -q/a_1^{\frac{1}{2}}; q\right)_\infty}{(a_1, \ldots, a_{M+1}, a_2/a_1, \ldots, a_{M+1}/a_1; q)_\infty} \\
&\quad \times {}_{2M}\phi_{2M-1}\left[\begin{matrix} a_1, a_2, \ldots, a_{2M} \\ qa_1/a_2, \ldots, qa_1/a_{2M} \end{matrix}; q, -x\right] \\
&= a_2 \frac{(qa_2/a_{M+2}, \ldots, qa_2/a_{2M}, qa_1/a_2a_{M+2}, \ldots, qa_1/a_2a_{2M}; q)_\infty}{(a_1/a_2, a_2, a_3/a_2, \ldots, a_{M+1}/a_2, a_2^2/a_1, a_2a_3/a_1, \ldots, a_2a_{M+1}/a_1; q)_\infty} \\
&\quad \times \left(a_1^{\frac{1}{2}}/a_2, -a_1^{\frac{1}{2}}/a_2, qa_2/a_1^{\frac{1}{2}}, -qa_2/a_1^{\frac{1}{2}}; q\right)_\infty \\
&\quad \times {}_{2M}\phi_{2M-1}\left[\begin{matrix} a_2^2/a_1, a_2, a_2a_3/a_1, \ldots, a_2a_{2M}/a_1 \\ qa_2/a_1, qa_2/a_3, \ldots, qa_2/a_{2M} \end{matrix}; q, -x\right] \\
&\quad + \text{idem } (a_2; a_3, \ldots, a_{M+1}),
\end{aligned} \tag{4.12.1}$$

where $x = (qa_1)^M/a_1a_2\cdots a_{2M}$. Slater [1952c] observed that this formula could also be derived from (4.10.10) by taking $A = B = M+1, m = 1$, choosing the parameters such that $P(z)$ in (4.9.1) becomes

$$\begin{aligned}
&\frac{\left(qa_1z/a_{M+2}, \ldots, qa_1z/a_{2M}, qza_1^{\frac{1}{2}}, -qza_1^{\frac{1}{2}}; q\right)_\infty}{(a_1z, \ldots, a_{M+1}z; q)_\infty} \\
&\quad \times \frac{\left(q/za_{M+2}, \ldots, q/za_{2M}, 1/za_1^{\frac{1}{2}}, -1/za_1^{\frac{1}{2}}; q\right)_\infty}{(1/z, a_2/za_1, \ldots, a_{M+1}/za_1; q)_\infty},
\end{aligned} \tag{4.12.2}$$

and then using the fact that

$$\begin{aligned}
&(a, -a, q/a, -q/a; q)_\infty - a^2(qa, -qa, 1/a, -1/a; q)_\infty \\
&= 2(a, -a, q/a, -q/a; q)_\infty
\end{aligned} \tag{4.12.3}$$

to combine the terms with the same ${}_{2M}\phi_{2M-1}$ series.

Similarly, taking $A = B = M+2$ and $m = 1$ in (4.10.10) and choosing the parameters such that $P(z)$ in (4.9.1) becomes

$$\begin{aligned}
&\frac{\left(qa_1z/a_{M+3}, \ldots, qa_1z/a_{2M}, qza_1^{\frac{1}{2}}, -qza_1^{\frac{1}{2}}, z(qa_1)^{\frac{1}{2}}, -z(qa_1)^{\frac{1}{2}}, q\right)_\infty}{(a_1z, \ldots, a_{M+2}z; q)_\infty} \\
&\quad \times \frac{\left(q/za_{M+3}, \ldots, q/za_{2M}, 1/za_1^{\frac{1}{2}}, -1/za_1^{\frac{1}{2}}, q^{\frac{1}{2}}/za_1^{\frac{1}{2}}, -q^{\frac{1}{2}}/za_1^{\frac{1}{2}}; q\right)_\infty}{(1/z, a_2/za_1, \ldots, a_{M+2}/za_1; q)_\infty},
\end{aligned} \tag{4.12.4}$$

we obtain

$$\frac{(qa_1/a_{M+3},\ldots,qa_1/a_{2M},q/a_{M+3},\ldots,q/a_{2M},q/a_1;q)_\infty}{(a_2,\ldots,a_{M+2},a_2/a_1,\ldots,a_{M+2}/a_1;q)_\infty}$$
$$\times\ {}_{2M}\phi_{2M-1}\left[\begin{matrix}a_1,a_2,\ldots,a_{2M}\\ qa_1/a_2,\ldots,qa_1/a_{2M}\end{matrix};q,x\right]$$
$$=a_2\frac{(qa_2/a_{M+3},\ldots,qa_2/a_{2M},qa_1/a_2a_{M+3},\ldots,qa_1/a_2a_{2M};q)_\infty}{(a_1/a_2,a_3/a_2,\ldots,a_{M+2}/a_2,a_2a_3/a_1,\ldots,a_2a_{M+2}/a_1;q)_\infty}$$
$$\times\frac{(a_1/a_2^2,qa_2^2/a_1;q)_\infty}{(a_2,a_2^2/a_1;q)_\infty}\ {}_{2M}\phi_{2M-1}\left[\begin{matrix}a_2^2/a_1,a_2,a_2a_3/a_1,\ldots,a_2a_{2M}/a_1\\ qa_2/a_1,qa_2/a_3,\ldots,qa_2/a_{2M}\end{matrix};q,x\right]$$
$$+\ \text{idem}\ (a_2;a_3,\ldots,a_{M+2}), \tag{4.12.5}$$

where $x=(qa_1)^M/a_1\cdots a_{2M}$, which is formula (7.3) in Sears [1951d].

Finally, if we take $A=B=M+2$ and $m=1$ in (4.10.10) and choose the parameters such that $P(z)$ in (4.9.1) becomes

$$\frac{\left(qa_1z/a_{M+3},\ldots,qa_1z/a_{2M+1},qza_1^{\frac12},-qza_1^{\frac12},\pm q^{\frac12}za_1^{\frac12};q\right)_\infty}{(a_1z,\ldots,a_{M+2}z;q)_\infty}$$
$$\times\frac{\left(q/za_{M+3},\ldots,q/za_{2M+1},1/za_1^{\frac12},-1/za_1^{\frac12},\pm q^{\frac12}/za_1^{\frac12};q\right)_\infty}{(1/z,a_2/za_1,\ldots,a_{M+2}/za_1;q)_\infty}, \tag{4.12.6}$$

we obtain

$$\frac{(qa_1/a_{M+3},\ldots,qa_1/a_{2M+1},q/a_{M+3},\ldots,q/a_{2M+1};q)_\infty}{(a_1,\ldots,a_{M+2},a_2/a_1,\ldots,a_{M+2}/a_1;q)_\infty}$$
$$\times\left(a_1^{\frac12},-a_1^{\frac12},q/a_1^{\frac12},-q/a_1^{\frac12},\pm(a_1q)^{\frac12},\pm(q/a_1)^{\frac12};q\right)_\infty$$
$$\times\ {}_{2M+1}\phi_{2M}\left[\begin{matrix}a_1,a_2,\ldots,a_{2M+1}\\ qa_1/a_2,\ldots,qa_1/a_{2M+1}\end{matrix};q,\mp y\right]$$
$$=a_2\frac{(qa_2/a_{M+3},\ldots,qa_2/a_{2M+1},qa_1/a_2a_{M+3},\ldots,qa_1/a_2a_{2M+1};q)_\infty}{(a_1/a_2,a_3/a_2,\ldots,a_{M+2}/a_2,a_2,a_2^2/a_1,a_2a_3/a_1,\ldots,a_2a_{M+2}/a_1;q)_\infty}$$
$$\times\left(a_1^{\frac12}/a_2,-a_1^{\frac12}/a_2,qa_2/a_1^{\frac12},-qa_2/a_1^{\frac12},\pm a_2(q/a_1)^{\frac12},\pm(qa_1)^{\frac12}/a_2;q\right)_\infty$$
$$\times\ {}_{2M+1}\phi_{2M}\left[\begin{matrix}a_2^2/a_1,a_2,a_2a_3/a_1,\ldots,a_2a_{2M+1}/a_1\\ \\ qa_2/a_1,qa_2/a_3,\ldots,qa_2/a_{2M+1}\end{matrix};q,\mp y\right]$$
$$+\ \text{idem}\ (a_2;a_3,\ldots,a_{M+2}), \tag{4.12.7}$$

where $y=(qa_1)^{M+\frac12}/a_1\cdots a_{2M+1}$, which are formulas (7.4) and (7.5) in Sears [1951d].

Exercises

4.1 Let Re $c > 0$, Re $d > 0$, and $\mathrm{Re}(x+y) > 1$. Show that Cauchy's [1825] beta integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{ds}{(1+cs)^x(1-ds)^y} = \frac{\Gamma(x+y-1)(1+d/c)^{1-y}(1+c/d)^{1-x}}{(c+d)\Gamma(x)\Gamma(y)}$$

has a q-analogue of the form

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(-csq^x, dsq^y;q)_\infty}{(-cs, ds;q)_\infty}\, ds \\ &= \frac{\Gamma_q(x+y-1)}{\Gamma_q(x)\Gamma_q(y)} \frac{(-cq^x/d, -dq^y/c;q)_\infty}{(c+d)(-cq/d, -dq/c;q)_\infty},\end{aligned}$$

where $0 < q < 1$.
(Wilson [1985])

4.2 Prove that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\left(q^{1+s}, -q^{a+s}, q^{a-b+1+s}, -q^{a-b+1+s}, q^{c+d+e-a+s};q\right)_\infty}{(q^{c+s}, q^{d+s}, q^{e+s}, -q^{a+1+s}, -q^{a-2b+s};q)_\infty} \\ &\quad\times \frac{\pi q^s\, ds}{\sin \pi s \sin \pi(c+d+e-a+s)} \\ &= \csc \pi(c+d+e-a) \frac{\left(q, q^{c+d+e-a}, q^{1+a-c-d-e}, q^{1+a-b};q\right)_\infty}{(q^{1+a}, -q^{1+a}, -q^{a-2b}, q^c;q)_\infty} \\ &\quad\times \frac{\left(-q^{1+a-b}, -q^a, q^{1+a-c}, q^{1+a-d}, q^{1+a-e};q\right)_\infty}{(q^d, q^e, q^{1+a-c-d}, q^{1+a-c-e}, q^{1+a-d-e};q)_\infty} \\ &\quad\times {}_{10}W_9\left(q^a; iq^{1+a/2}, -iq^{1+a/2}, q^b, -q^b, q^c, q^d, q^e;q, -q^{1+2a-2b-c-d-e}\right),\end{aligned}$$

where $1+2a-2b > c+d+e$.

4.3 Show that

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix} ac, bc, ad \\ abcg, acdh \end{matrix}; q, gh\right] \\ &= \frac{(q, ac, bc, ag, bg, ch;q)_\infty}{(f, q/f, cf/g, qg/cf, abcg, acdh;q)_\infty} \\ &\quad\times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(fe^{i\theta}/g, qe^{i\theta}/cf, dhe^{i\theta}, qge^{-i\theta}/f, cfe^{-i\theta};q\right)_\infty}{(ae^{i\theta}, be^{i\theta}, he^{i\theta}, ce^{-i\theta}, ge^{-i\theta};q)_\infty}\, d\theta.\end{aligned}$$

4.4 Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(fe^{i\theta},ke^{i\theta}/d,qde^{-i\theta}/k,cke^{-i\theta},qe^{i\theta}/ck,abcdghe^{i\theta}/f;q\right)_\infty}{(ae^{i\theta},be^{i\theta},ce^{-i\theta},de^{-i\theta},ge^{i\theta},he^{i\theta};q)_\infty}\,d\theta$$
$$=\frac{(k,q/k,ck/d,qd/ck,cf,df,acdg,bcdg,cdgh,abcdh/f;q)_\infty}{(q,ac,ad,bc,bd,cg,dg,ch,dh,cdfg;q)_\infty}$$
$$\times\,{}_8W_7(cdfg/q;cg,dg,f/a,f/b,f/h;q,abcdh/f).$$

4.5 Prove that

$${}_4\phi_3\left[\begin{matrix}q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta}\\ ab,ac,ad\end{matrix};q,q\right]$$
$$=\frac{(q;q)_\infty}{(q^{\frac12},q^{\frac12},ab,ac,bc;q)_\infty}\left|\frac{\left(ae^{i\theta},be^{i\theta},ce^{i\theta};q\right)_\infty}{\left(q^{\frac12}e^{2i\theta};q\right)_\infty}\right|^2$$
$$\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(q^{\frac12}e^{i\theta+i\phi}/\sigma,q^{\frac12}\sigma e^{-i\theta-i\phi},\sigma q^{\frac12}e^{i\theta-i\phi},q^{\frac12}e^{i\phi-i\theta}/\sigma,abce^{i\phi}/\sigma;q\right)_\infty}{(ae^{i\phi}/\sigma,be^{i\phi}/\sigma,ce^{i\phi}/\sigma,\sigma e^{i\theta-i\phi},\sigma e^{-i\theta-i\phi};q)_\infty}$$
$$\times\frac{(d\sigma e^{-i\phi},bc;q)_n}{\left(abce^{i\phi}/\sigma,ad;q\right)_n}\left(\frac{a}{\sigma}e^{i\phi}\right)^n d\phi$$

and, more generally,

$$=\frac{(q,az,a/z,bz,b/z,cz,c/z;q)_\infty}{(\kappa,q/\kappa,\kappa z^2,q/\kappa z^2,ab,ac,bc;q)_\infty}$$
$$\times\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(\kappa ze^{i\phi}/\sigma,q\sigma e^{-i\phi}/\kappa z,\kappa\sigma ze^{-i\phi},qe^{i\phi}/\kappa\sigma z,abce^{i\phi}/\sigma;q\right)_\infty}{(ae^{i\phi}/\sigma,be^{i\phi}/\sigma,ce^{i\phi}/\sigma,\sigma ze^{-i\phi},\sigma e^{-i\phi}/z;q)_\infty}$$
$$\times\frac{\left(d\sigma e^{-i\phi},bc;q\right)_n}{(abce^{i\phi}/\sigma,ad;q)_n}\left(\frac{a}{\sigma}e^{i\phi}\right)^n\,d\phi$$

for $n=0,1,2,\ldots$, where $z=e^{i\theta}$ and κ is an arbitrary parameter.

4.6 Prove that

$$a^{-1}\frac{(aq/e,aq/f,aq/g,aq/h,q/ae,q/af,q/ag,q/ah;q)_\infty}{(qa^2,ab,ac,ad,b/a,c/a,d/a;q)_\infty}$$
$$\times\,{}_{10}W_9\left(a^2;ab,ac,ad,ae,af,ag,ah;q,q^3/abcdefgh\right)$$
$$+\,\text{idem}\,(a;b,c,d)=0,$$

where $|q^3|<|abcdefgh|$.

4.7 Prove that

$$a_1^{-1}\frac{(a_1q/b_1, a_1q/b_2, \ldots, a_1q/b_r, q/a_1b_1, \ldots, q/a_1b_r; q)_\infty}{(qa_1^2, a_1a_2, \ldots, a_1a_r, a_2/a_1, \ldots, a_r/a_1; q)_\infty}$$
$$\times\ {}_{2r+2}W_{2r+1}\left(a_1^2; a_1a_2, \ldots, a_1a_r, a_1b_1, \ldots, a_1b_r; q, q^{r-1}/a_1\cdots a_rb_1\cdots b_r\right)$$
$$+ \text{idem }(a_1; a_2, \ldots, a_r) = 0,$$

where $r = 1, 2, \ldots,$ and $|q^{r-1}| < |a_1 \cdots a_rb_1 \cdots b_r|$.

4.8 Show that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(-csq^{n+1}, bds, \alpha s; q)_\infty}{(-cs, ds, b\alpha sq^{n-1}; q)_\infty}\,ds$$
$$= \frac{(-\alpha/c, -bd/c; q)_\infty}{(c+d)(-dq/c, -b\alpha/cq; q)_\infty}\frac{(b, \alpha/d; q)_n}{(q, -cq/d; q)_n},$$

where $\mathrm{Re}(c, d, b\alpha) > 0$ and $n = 0, 1, \ldots$. Show that the q-Cauchy beta integral in Ex. 4.1 follows from this formula by letting $n \to \infty$ and then setting $b = q^y, \alpha = -cq^x$.

4.9 Extend the integral in Ex. 4.8 to

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(-acs, ac^2s/\alpha, ac^2s/\beta, ac^2s/\gamma, ac^2s/\delta, ac^2s/\lambda; q)_\infty}{(-cs, \alpha s, \beta s, \gamma s, \delta s, \lambda s; q)_\infty}$$
$$\times\left(1 - \frac{a^2c^2s^2}{q}\right)\,ds$$
$$= \frac{(a/q, -ac/\alpha, -ac/\beta, -ac/\gamma, -ac/\delta, -ac/\lambda; q)_\infty}{c(q, -\alpha/c, -\beta/c, -\gamma/c, -\delta/c, -\lambda/c; q)_\infty}$$
$$\times\frac{(q^2/a, ac^2/\alpha\beta, ac^2/\alpha\gamma, ac^2/\beta\gamma; q)_n}{(-cq/\alpha, -cq/\beta, -cq/\gamma, -a^2c^3/\alpha\beta\gamma q; q)_n},$$

where $\mathrm{Re(c}, \alpha, \beta, \gamma, \delta, \lambda) > 0$, $a^3c^5 = -\alpha\beta\gamma\delta\lambda q^2$, $ac = -\lambda q^{n+1}$, and $n = 1, 2 \ldots$.

4.10 Show that

$$\frac{1}{2\pi i}\int_K \frac{(q^2z/a\alpha\gamma, q^2z/b\alpha\gamma, q^2z/c\alpha\gamma, \gamma/z; q)_\infty}{(az, bz, cz, \alpha/z; q)_\infty}\frac{1 - qz^2/\alpha\gamma}{z}\,dz$$
$$= \frac{(a\gamma, b\gamma, c\gamma, \alpha q/\gamma; q)_\infty}{(a\alpha, b\alpha, c\alpha, q; q)_\infty},$$

where $q^2 = abc\alpha q^2, |\gamma/\alpha| < 1$, and the contour K is as defined in §4.9.

4.11 Show that

$$\frac{1}{2\pi i}\int_K \frac{(bqz, qz/\gamma, \gamma/z; q)_\infty}{(az, bz, \alpha/z; q)_\infty} \times (qz/\gamma q^m; q)_m (a_1 z; q)_{m_1} \cdots (a_r z; q)_{m_r} \frac{dz}{z}$$
$$= \frac{(\gamma/\alpha, \alpha q/\gamma, bq/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty} \times (q/b\gamma q^m; q)_m (a_1/b; q)_{m_1} \cdots (a_r/b; q)_{m_r} (b\alpha)^{m+m_1+\cdots+m_r},$$

provided $|\gamma/\alpha| < 1$, where $m, m_1, \ldots, m_r$ are nonnegative integers, $q = a\gamma q^{m+m_1+\cdots+m_r}$ and K is as defined in §4.9.

4.12 Show that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(-acs, dqs; q)_\infty}{(-cs, ds; q)_\infty} (a_1 s; q)_{m_1} \cdots (a_r s; q)_{m_r}\, ds$$
$$= \frac{(-ac/d; q)_\infty}{(c+d)(-cq/d; q)_\infty} (a_1/d; q)_{m_1} \cdots (a_r/d; q)_{m_r}$$

provided $|aq^{-(m_1+\cdots+m_r)}| < 1$, Re $(c, d, a_1, \cdots, a_r) > 0$, and $m_1, \ldots, m_r$ are nonnegative integers.

4.13 Show that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(-acs, -ac^2 s/f, ac^2 s/\alpha, ac^2 s/\beta; q)_\infty}{(-cs, -fs, \alpha s, \beta s; q)_\infty} \times \frac{(ac^2 s/\gamma, ac^2 s/\delta; q)_\infty}{(\gamma s, \delta s; q)_\infty} (1 - ac^2 s^2/q)\, ds$$
$$= \frac{(a/q, ac/f, -ac/\alpha, -ac/\beta, -ac/\gamma, -ac/\delta; q)_\infty}{c(q, f/c, -\alpha/c, -\beta/c, -\gamma/c, -\delta/c; q)_\infty} \times \frac{(q^2/a, ac^2/\alpha\beta, ac^2/\alpha\gamma, ac^2/\beta\gamma; q)_n}{(-cq/\alpha, -cq/\beta, -cq/\gamma, -a^2c^3/\alpha\beta\gamma q; q)_n}$$
$$+ \frac{(ac^2/f^2 q, ac/f, -ac^2/f\alpha, -ac^2/f\beta, -ac^2/f\gamma, -ac^2/f\delta; q)_\infty}{f(q, c/f, -\alpha/f, -\beta/f, -\gamma/f, -\delta/f; q)_\infty} \times \frac{(f^2q^2/ac^2, ac^2/\alpha\beta, ac^2/\alpha\gamma, ac^2/\beta\gamma; q)_n}{(-fq/\alpha, -fq/\beta, -fq/\gamma, -a^2c^4/f\alpha\beta\gamma q; q)_n}$$

provided $ac = fq^{n+1}, a^3c^5 = f\alpha\beta\gamma\delta q^2$, Re $(c, f, \alpha, \beta, \gamma, \delta) > 0$, the integrand has only simple poles, and $n = 0, 1, \ldots$.

(For the formulas in Exercises 4.8 – 4.13, and related formulas, see Gasper [1989c].)

Notes

§4.4 Kalnins and Miller [1988, 1989] exploited symmetry (recurrence relation) techniques similar to those used by Nikiforov and Suslov [1986], Nikiforov, Suslov and Uvarov [1991], and Nikiforov and Uvarov [1988] to give another proof of (4.4.3) and of (4.11.1).

§4.6 Contour integrals of the types considered in this section were used by Agarwal [1953c] to give simple proofs of the two-term and three-term transformation formulas for ${}_8\phi_7$ series.

§4.12 Sears [1951b] also derived the hypergeometric limit cases of the transformation formulas in this section. Applications of (4.12.1) to some formulas in partition theory are given in M. Jackson [1949].

Exercises 4.1–4.5 For additional q-beta integrals, see Askey [1988a,b, 1989e], Gasper [1989c], and Rahman and Suslov [1994a,b, 1996a, 1998].

5

BILATERAL BASIC HYPERGEOMETRIC SERIES

5.1 Notations and definitions

The general bilateral basic hypergeometric series in base q with r numerator and s denominator parameters is defined by

$$\begin{aligned} {}_r\psi_s(z) &\equiv {}_r\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] \\ &= \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n. \end{aligned} \tag{5.1.1}$$

In (5.1.1) it is assumed that q, z and the parameters are such that each term of the series is well-defined (i.e., the denominator factors are never zero, $q \neq 0$ if $s < r$, and $z \neq 0$ if negative powers of z occur). Note that a bilateral basic hypergeometric series is a series $\sum_{n=-\infty}^{\infty} v_n$ such that $v_0 = 1$ and v_{n+1}/v_n is a rational function of q^n. By applying (1.2.28) to the terms with negative n, we obtain that

$$\begin{aligned} {}_r\psi_s(z) &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{(q/b_1, q/b_2, \ldots, q/b_s; q)_n}{(q/a_1, q/a_2, \ldots, q/a_r; q)_n} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r z}\right)^n. \end{aligned} \tag{5.1.2}$$

Let $R = |b_1 \cdots b_s / a_1 \cdots a_r|$. If $s < r$ and $|q| < 1$, then the first series on the right side of (5.1.2) diverges for $z \neq 0$; if $s < r$ and $|q| > 1$, then the first series converges for $|z| < R$ and the second series converges for all $z \neq 0$. When $r < s$ and $|q| < 1$ the first series converges for all z, but the second series converges only when $|z| > R$. If $r < s$ and $|q| > 1$, the second series diverges for all $z \neq 0$. If $r = s$, which is the most important case, and $|q| < 1$, the first series converges when $|z| < 1$ and the second when $|z| > R$; on the other hand, if $|q| > 1$ the second series converges when $|z| > 1$ and the first when $|z| < R$.

We shall assume throughout this chapter that $|q| < 1$, so that the region of convergence of the bilateral series

$$\begin{aligned} {}_r\psi_r(z) &= \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{(q/b_1, \ldots, q/b_r; q)_n}{(q/a_1, \ldots, q/a_r; q)_n} \left(\frac{b_1 \cdots b_r}{a_1 \cdots a_r z}\right)^n \end{aligned} \tag{5.1.3}$$

is the annulus

$$\left|\frac{b_1\cdots b_r}{a_1\cdots a_r}\right| < |z| < 1. \tag{5.1.4}$$

When $b_j = q$ for some j, the second series on the right sides of (5.1.2) and (5.1.3) vanish and the first series become basic hypergeometric series. If we replace the index of summation n in (5.1.1) by $k+n$, where k is an integer, then it follows that

$$\begin{aligned}
&{}_r\psi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right]\\
&= \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k} z^k \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r}\\
&\quad\times {}_r\psi_s\left[\begin{matrix}a_1q^k,\ldots,a_rq^k\\ b_1q^k,\ldots,b_sq^k\end{matrix};q,zq^{k(s-r)}\right].
\end{aligned} \tag{5.1.5}$$

When r and s are small we shall frequently use the single-line notation

$${}_r\psi_s(z) \equiv {}_r\psi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z).$$

An ${}_r\psi_r$ series will be called *well-poised* if $a_1b_1 = a_2b_2 = \cdots = a_rb_r$, and *very-well-poised* if it is well-poised and $a_1 = -a_2 = qb_1 = -qb_2$. Corresponding to the definition of a VWP-balanced ${}_{r+1}\phi_r$ series in §2.1, we call a very-well-poised ${}_r\psi_r$ series *very-well-poised-balanced* (*VWP-balanced*) if

$$(a_3a_4\cdots a_r)qz = (\pm a_1 q^{-\frac{1}{2}})^{r-2} \tag{5.1.6}$$

with either the plus or minus sign. The very-well-poised bilateral basic hypergeometric series in (5.3.1), (5.5.2), (5.5.3), and in §5.6 are VWP-balanced. Note that if in (5.1.6) we replace r by $r+1$, a_1 by $qa_1^{\frac{1}{2}}$, and then a_3 by a_1, then (5.1.6) reduces to the condition (2.1.12) for a ${}_{r+1}W_r$ series to be VWP-balanced. A well-poised ${}_r\psi_r$ series will be called *well-poised-balanced* (*WP-balanced*) if

$$(a_1a_2\cdots a_r)z = -(\pm(a_1b_1)^{\frac{1}{2}})^r \tag{5.1.7}$$

with either the plus or minus sign. The well-poised bilateral basic hypergeometric series in (5.3.3), (5.3.4), (5.5.1), (5.5.4), and in (5.5.5) are WP-balanced.

5.2 Ramanujan's sum for ${}_1\psi_1(a;b;q,z)$

The bilateral summation formula

$${}_1\psi_1(a;b;q,z) = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \qquad |b/a| < |z| < 1, \tag{5.2.1}$$

which is an extension of the q-binomial formula (1.3.2), was first given by Ramanujan (see Hardy [1940]). In Chapter 2 we saw that this formula follows as a special case of Sears' ${}_3\phi_2$ summation formula (2.10.12). Andrews [1969, 1970a], Hahn [1949b], M. Jackson [1950b], Ismail [1977] and Andrews and Askey [1978] published different proofs of (5.2.1). The proof given here is due to Andrews and Askey [1978].

The first step is to regard ${}_1\psi_1(a;b;q,z)$ as a function of b, say, $f(b)$. Then

$$\begin{aligned} f(b) &= {}_1\psi_1(a;b;q,z) \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b;q)_n}{(q/a;q)_n} (b/az)^n \end{aligned} \tag{5.2.2}$$

so that, by (5.1.4), the two series are convergent when $|b/a| < |z| < 1$. As a function of b, $f(b)$ is clearly analytic for $|b| < \min(1, |az|)$ when $|z| < 1$. Since

$$\begin{aligned} &{}_1\psi_1(a;b;q,z) - a\, {}_1\psi_1(a;b;q,qz) \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{(a;q)_n}{(b;q)_n} - \frac{aq^n(a;q)_n}{(b;q)_n} \right\} z^n \\ &= \sum_{n=-\infty}^{\infty} \frac{(a;q)_{n+1}}{(b;q)_n} z^n \\ &= z^{-1}(1-b/q) \sum_{n=-\infty}^{\infty} \frac{(a;q)_{n+1}}{(b/q;q)_{n+1}} z^{n+1} \\ &= z^{-1}(1-b/q)\, {}_1\psi_1(a;b/q;q,z), \end{aligned}$$

we get

$$f(bq) - z^{-1}(1-b)f(b) = a\, {}_1\psi_1(a;bq;q,qz). \tag{5.2.3}$$

However,

$$\begin{aligned} &a\, {}_1\psi_1(a;bq;q,qz) \\ &= a \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(bq;q)_n} (qz)^n \\ &= -ab^{-1} \sum_{n=-\infty}^{\infty} \frac{(a;q)_n(1-bq^n-1)}{(bq;q)_n} z^n \\ &= -ab^{-1}(1-b)f(b) + ab^{-1}f(bq). \end{aligned} \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4) gives the functional equation

$$(1-ab^{-1})f(bq) = (1-b)(z^{-1}-ab^{-1})f(b),$$

that is,

$$f(b) = \frac{1-b/a}{(1-b)(1-b/az)} f(bq). \tag{5.2.5}$$

Iterating (5.2.5) $n-1$ times we get

$$f(b) = \frac{(b/a;q)_n}{(b,b/az;q)_n} f(bq^n). \tag{5.2.6}$$

Since $f(b)$ is analytic for $|b| < \min(1, |az|)$, by letting $n \to \infty$ we obtain

$$f(b) = \frac{(b/a;q)_\infty}{(b,b/az;q)_\infty} f(0). \tag{5.2.7}$$

However, since

$$f(q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

by (1.3.2), on setting $b = q$ in (5.2.7) we find that

$$f(0) = \frac{(q, q/az; q)_\infty}{(q/a;q)_\infty}\, f(q) = \frac{(q, q/az, az; q)_\infty}{(q/a, z; q)_\infty}.$$

Substituting this in (5.2.7) we obtain formula (5.2.1).

Jacobi's triple-product identity (1.6.1) is a limit case of Ramanujan's sum. First replace a and z in (5.2.1) by a^{-1} and az, respectively, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{(a^{-1};q)_n}{(b;q)_n}\,(az)^n = \frac{(q, ab, z, q/z; q)_\infty}{(b, aq, az, b/z; q)_\infty}, \tag{5.2.8}$$

when $|b| < |z| < |a^{-1}|$. Now set $b = 0$, replace q by q^2, z by zq, and then take $a \to 0$ to get (1.6.1).

5.3 Bailey's sum of a very-well-poised ${}_6\psi_6$ series

Bailey [1936] proved that

$$\begin{aligned} &{}_6\psi_6 \left[\begin{matrix} qa^{\frac12}, -qa^{\frac12}, b, c, d, e \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e \end{matrix} ; q, \frac{qa^2}{bcde} \right] \\ &= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}, \end{aligned} \tag{5.3.1}$$

provided $|qa^2/bcde| < 1$. Since this VWP-balanced ${}_6\psi_6$ reduces to a VWP-balanced ${}_6\phi_5$ series when one of the parameters b, c, d, e equals a, (5.3.1) can be regarded as an extension of the ${}_6\phi_5$ summation formula (2.7.1).

There are several known proofs of (5.3.1). Bailey's proof depends crucially on the identity

$$\begin{aligned} &\frac{(aq/d, aq/e, aq/f, q/ad, q/ae, q/af; q)_\infty}{a(qa^2, ab, ac, b/a, c/a; q)_\infty} \\ &\quad \times {}_8W_7(a^2; ab, ac, ad, ae, af; q, q^2/abcdef) + \text{ idem } (a; b, c) = 0, \end{aligned} \tag{5.3.2}$$

when $|q^2/abcdef| < 1$, which is easily proved by using the q-integral representation (2.10.19) of an ${}_8\phi_7$ series (see Exercise 2.15). If we set $c = q/a$, the first and third series in (5.3.2) combine to give, via (5.1.3),

$$\begin{aligned} &\frac{(aq/d, aq/e, aq/f, q/ad, q/ae, q/af; q)_\infty}{a(qa^2, q/a^2, q, ab, b/a; q)_\infty} \\ &\quad \times {}_6\psi_6 \left[\begin{matrix} qa, -qa, ab, ad, ae, af \\ a, -a, aq/b, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{q}{bdef} \right], \end{aligned}$$

while, by (2.7.1), the second series reduces to

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} b^2, bq, -bq, bd, be, bf \\ b, -b, bq/d, bq/e, bq/f \end{matrix}; q, \frac{q}{bdef}\right] \\
&= \frac{(qb^2, q/de, q/df, q/ef; q)_\infty}{(bq/d, bq/e, bq/f, q/bdef; q)_\infty}.
\end{aligned}
$$

This gives (5.3.1) after we replace a^2, ab, ad, ae, af by a, b, c, d, e, respectively, and use the same square root of a everywhere.

Slater and Lakin [1956] gave a proof of (5.3.1) via a Barnes type integral and a second proof via a q-difference operator. Andrews [1974a] gave a simpler proof and Askey [1984c] showed that it can be obtained from a simple difference equation. The simplest proof was given by Askey and Ismail [1979] who only used the ${}_6\phi_5$ sum (2.7.1) and an argument based on the properties of analytic functions.

Setting $e = a^{\frac{1}{2}}$ in (5.3.1), we obtain

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}},\ b,\ c,\ d \\ -a^{\frac{1}{2}},\ aq/b,\ aq/c,\ aq/d \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] \\
&= \frac{(aq, aq/bc, aq/bd, aq/cd, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c, qa^{\frac{1}{2}}/d, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, qa^{\frac{3}{2}}/bcd; q)_\infty} \qquad (5.3.3)
\end{aligned}
$$

provided $|qa^{\frac{3}{2}}/bcd| < 1$. This is an extension of the q-Dixon formula (2.7.2).

If we set $d = a^{\frac{1}{2}}$, $e = -a^{\frac{1}{2}}$ in (5.3.1) and simplify, we get the sum of a WP-balanced ${}_2\psi_2$ series

$$
\begin{aligned}
&{}_2\psi_2(b, c; aq/b, aq/c; q, -aq/bc) \\
&= \frac{(aq/bc; q)_\infty (aq^2/b^2, aq^2/c^2, q^2, aq, q/a; q^2)_\infty}{(aq/b, aq/c, q/b, q/c, -aq/bc; q)_\infty}, \quad \left|\frac{aq}{bc}\right| < 1. \qquad (5.3.4)
\end{aligned}
$$

5.4 A general transformation formula for an ${}_r\psi_r$ series

In this section we shall derive a transformation formula for an ${}_r\psi_r$ series from those for ${}_{r+1}\phi_r$ series in Chapter 4. First observe that (5.1.3) gives

$$
\begin{aligned}
&{}_r\psi_r(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r; q, z) \\
&= {}_{r+1}\phi_r\left[\begin{matrix} q,\ a_1, \ldots,\ a_r \\ b_1, \ldots, b_r \end{matrix}; q, z\right] + z^{-1}\prod_{k=1}^{r}\frac{b_k - q}{a_k - q} \\
&\quad \times {}_{r+1}\phi_r\left[\begin{matrix} q,\ q^2/b_1, \ldots, q^2/b_r \\ q^2/a_1, \ldots, q^2/a_r \end{matrix}; q, \frac{b_1\cdots b_r}{a_1\cdots a_r z}\right]. \qquad (5.4.1)
\end{aligned}
$$

In (4.10.11) let us now make the following specialization of the parameters

$$\begin{aligned}
C = A+1, \qquad D &= B+1, \qquad A = B = r, \\
c_1 d_1 = c_2 d_2 &= \cdots = c_{A+1} d_{A+1} = q, \\
qd_1/b_1 = \alpha_1, qd_1/b_2 &= \alpha_2, \ldots, qd_1/b_A = \alpha_A, \\
a_1 d_1 = \beta_1, a_2 d_1 &= \beta_2, \ldots, a_A d_1 = \beta_A, \\
\frac{b_1 \cdots b_A z}{d_1 \cdots d_{A+1}} &= x.
\end{aligned} \tag{5.4.2}$$

Then, combining the pairs of the resulting ${}_{r+1}\phi_r$ series in (4.10.11) via (5.4.1), simplifying the coefficients and relabelling the parameters, we obtain Slater's [1952b, (4)] transformation formula

$$\begin{aligned}
&\frac{(b_1, b_2, \ldots, b_r, q/a_1, q/a_2, \ldots, q/a_r, dz, q/dz; q)_\infty}{(c_1, c_2, \ldots, c_r, q/c_1, q/c_2, \ldots, q/c_r; q)_\infty} \, {}_r\psi_r \left[\begin{matrix} a_1, \ a_2, \ldots, a_r \\ b_1, \ b_2, \ldots, b_r \end{matrix} ; q, z \right] \\
&= \frac{q}{c_1} \frac{(c_1/a_1, c_1/a_2, \ldots, c_1/a_r, qb_1/c_1, qb_2/c_1, \ldots, qb_r/c_1, dc_1 z/q, q^2/dc_1 z; q)_\infty}{(c_1, q/c_1, c_1/c_2, \ldots, c_1/c_r, qc_2/c_1, \ldots, qc_r/c_1; q)_\infty} \\
&\quad \times {}_r\psi_r \left[\begin{matrix} qa_1/c_1, qa_2/c_1, \ldots, qa_r/c_1 \\ qb_1/c_1, qb_2/c_1, \ldots, qb_r/c_1 \end{matrix} ; q, z \right] \\
&\quad + \text{ idem } (c_1; c_2, \ldots, c_r),
\end{aligned} \tag{5.4.3}$$

where $d = a_1 a_2 \cdots a_r / c_1 c_2 \cdots c_r$, $\left| \dfrac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1$.

Note that the c's are absent in the ${}_r\psi_r$ series on the left side of (5.4.3). This gives us the freedom to choose the c's in any convenient way. For example, if we set $c_j = qa_j$, where j is an integer between 1 and r, then the j^{th} series on the right becomes an ${}_r\phi_{r-1}$ series. So if we set $c_j = qa_j$, $j = 1, 2, \ldots, r$, in (5.4.3), then we get an expansion of an ${}_r\psi_r$ series in terms of r ${}_r\phi_{r-1}$ series:

$$\begin{aligned}
&\frac{(b_1, b_2, \ldots, b_r, q/a_1, q/a_2, \ldots, q/a_r, z, q/z; q)_\infty}{(qa_1, qa_2, \ldots, qa_r, 1/a_1, 1/a_2, \ldots, 1/a_r; q)_\infty} \, {}_r\psi_r \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z \right] \\
&= \frac{a_1^{r-1} (q, qa_1/a_2, \ldots, qa_1/a_r, b_1/a_1, b_2/a_1, \ldots, b_r/a_1, a_1 z, q/a_1 z; q)_\infty}{(qa_1, 1/a_1, a_1/a_2, \ldots, a_1/a_r, qa_2/a_1, \ldots, qa_r/a_1; q)_\infty} \\
&\quad \times {}_r\phi_{r-1} \left[\begin{matrix} qa_1/b_1, qa_1/b_2, \ldots, qa_1/b_r \\ qa_1/a_2, \ldots, qa_1/a_r \end{matrix} ; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r z} \right] \\
&\quad + \text{ idem } (a_1; a_2, \ldots, a_r),
\end{aligned} \tag{5.4.4}$$

provided $\left| \dfrac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1$.

On the other hand, if we set $c_j = b_j$, $j = 1, 2, \ldots, r$ in (5.4.3), then we obtain the expansion formula

$$\frac{(q/a_1, q/a_2, \ldots, q/a_r, dz, q/dz; q)_\infty}{(q/b_1, q/b_2, \ldots, q/b_r; q)_\infty} {}_r\psi_r \left[\begin{matrix} a_1, & a_2, \ldots, a_r \\ b_1, & b_2, \ldots, b_r \end{matrix}; q, z\right]$$
$$= \frac{q}{b_1} \frac{(q, b_1/a_1, b_1/a_2, \ldots, b_1/a_r, db_1z/q, q^2/db_1z; q)_\infty}{(b_1, q/b_1, b_1/b_2, \ldots, b_1/b_r; q)_\infty} \tag{5.4.5}$$
$$\times {}_r\phi_{r-1} \left[\begin{matrix} qa_1/b_1, \ldots, qa_r/b_1 \\ qb_2/b_1, \ldots, qb_r/b_1 \end{matrix}; q, z\right] + \text{idem } (b_1; b_2, \ldots, b_r),$$

with $d = a_1a_2 \cdots a_r/b_1b_2 \cdots b_r$.

5.5 A general transformation formula for a very-well-poised ${}_{2r}\psi_{2r}$ series

Using (4.12.1) and (5.4.1) as in §5.4, we obtain Slater's [1952b] expansion of a WP-balanced ${}_{2r}\psi_{2r}$ series in terms of r other WP-balanced ${}_{2r}\psi_{2r}$ series:

$$\frac{(q/b_1, \ldots, q/b_{2r}, aq/b_1, \ldots, aq/b_{2r}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, qa^{-\frac{1}{2}}, -qa^{-\frac{1}{2}}; q)_\infty}{(a, a_1, \ldots, a_r, aq/a_1, \ldots, aq/a_r, q/a, q/a_1, \ldots, q/a_r, a_1/a, \ldots, a_r/a; q)_\infty}$$
$$\times {}_{2r}\psi_{2r} \left[\begin{matrix} b_1, & b_2, \ldots, b_{2r} \\ aq/b_1, & aq/b_2, \ldots, aq/b_{2r} \end{matrix}; q, -\frac{a^r q^r}{b_1 \cdots b_{2r}}\right]$$
$$= \frac{a_1(a_1q/b_1, \ldots, a_1q/b_{2r}, aq/a_1b_1, \ldots, aq/a_1b_{2r}; q)_\infty}{(a_1, q/a_1, a/a_1, qa_1/a, a_2/a_1, \ldots, a_r/a_1, qa_1/a_2, \ldots, qa_1/a_r; q)_\infty}$$
$$\times \frac{(a^{\frac{1}{2}}/a_1, -a^{\frac{1}{2}}/a_1, qa_1a^{-\frac{1}{2}}, -qa_1a^{-\frac{1}{2}}; q)_\infty}{(a_1^2/a, qa/a_1^2, a_1a_2/a, \ldots, a_1a_r/a, qa/a_1a_2, \ldots, qa/a_1a_r; q)_\infty}$$
$$\times {}_{2r}\psi_{2r} \left[\begin{matrix} a_1b_1/a, & a_1b_2/a, \ldots, a_1b_{2r}/a \\ a_1q/b_1, & a_1q/b_2, \ldots, a_1q/b_{2r} \end{matrix}; q, -\frac{a^r q^r}{b_1 \cdots b_{2r}}\right]$$
$$+ \text{idem}(a_1; a_2, \ldots, a_r). \tag{5.5.1}$$

For the very-well-poised case when $a_1 = b_1 = qa^{\frac{1}{2}}$, $a_2 = b_2 = -qa^{\frac{1}{2}}$, the first two terms on the right side vanish and we get

$$\frac{(q/b_3, \ldots, q/b_{2r}, aq/b_3, \ldots, aq/b_{2r}; q)_\infty}{(aq, q/a, a_3, \ldots, a_r, q/a_3, \ldots, q/a_r; q)_\infty}$$
$$\times (a_3/a, \ldots, a_r/a, aq/a_3, \ldots, aq/a_r; q)_\infty^{-1}$$
$$\times {}_{2r}\psi_{2r} \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b_3, \ldots, b_{2r} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b_3, \ldots, aq/b_{2r} \end{matrix}; q, \frac{a^{r-1}q^{r-2}}{b_3 \cdots b_{2r}}\right]$$
$$= \frac{(a_3q/b_3, \ldots, a_3q/b_{2r}, aq/a_3b_3, \ldots, aq/a_3b_{2r}; q)_\infty}{(a_3, q/a_3, a_3/a, aq/a_3, qa_3^2/a, aq/a_3^2; q)_\infty}$$
$$\times (a_4/a_3, \ldots, a_r/a_3, qa_3/a_4, \ldots, qa_3/a_r; q)_\infty^{-1}$$
$$\times (a_3a_4/a, \ldots, a_3a_r/a, aq/a_3a_4, \ldots, aq/a_3a_r; q)_\infty^{-1}$$
$$\times {}_{2r}\psi_{2r} \left[\begin{matrix} qa_3a^{-\frac{1}{2}}, -qa_3a^{-\frac{1}{2}}, a_3b_3/a, \ldots, a_3b_{2r}/a \\ a_3a^{-\frac{1}{2}}, -a_3a^{-\frac{1}{2}}, qa_3/b_3, \ldots, qa_3/b_{2r} \end{matrix}; q, \frac{a^{r-1}q^{r-2}}{b_3 \cdots b_{2r}}\right]$$

$$+ \text{ idem } (a_3; a_4, \ldots, a_r). \tag{5.5.2}$$

In particular, for $r=3$ we have the following transformation formula for VWP-balanced ${}_6\psi_6$ series

$$\begin{aligned}
&\frac{(q/b_3,\ldots,q/b_6,aq/b_3,\ldots,aq/b_6;q)_\infty}{(aq,q/a,a_3,q/a_3,a_3/a,aq/a_3;q)_\infty} \\
&\quad\times {}_6\psi_6\left[\begin{matrix} qa^{\frac12},\ -qa^{\frac12},\ b_3,\ b_4,\ b_5,\ b_6 \\ a^{\frac12},\ -a^{\frac12}, aq/b_3,\ aq/b_4,\ aq/b_5,\ aq/b_6\end{matrix};q,\frac{a^2q}{b_3b_4b_5b_6}\right] \\
&= \frac{(a_3q/b_3,\ldots,a_3q/b_6,aq/a_3b_3,\ldots,aq/a_3b_6;q)_\infty}{(a_3,q/a_3,a_3/a,aq/a_3,qa_3^2/a,aq/a_3^2;q)_\infty} \\
&\quad\times {}_6\psi_6\left[\begin{matrix} qa_3a^{-\frac12},-qa_3a^{-\frac12},a_3b_3/a,a_3b_4/a,a_3b_5/a,a_3b_6/a \\ a_3a^{-\frac12},-a_3a^{-\frac12},qa_3/b_3,qa_3/b_4,qa_3/b_5,qa_3/b_6\end{matrix};q,\frac{a^2q}{b_3b_4b_5b_6}\right].
\end{aligned} \tag{5.5.3}$$

If we now set $a_3 = b_6$, then the ${}_6\psi_6$ series on the right side becomes a ${}_6\phi_5$ with sum

$$\frac{(qb_6^2/a,aq/b_3b_4,aq/b_3b_5,aq/b_4b_5;q)_\infty}{(qb_6/b_3,qb_6/b_4,qb_6/b_5,qa^2/b_3b_4b_5b_6;q)_\infty}.$$

This provides another derivation of the ${}_6\psi_6$ sum (5.3.1); see M. Jackson [1950a]. As in Slater [1952b], Sears' formulas (4.12.5) and (4.12.7) can be used to obtain the transformation formulas

$$\begin{aligned}
&\frac{(q/b_1,\ldots,q/b_{2r},aq/b_1,\ldots,aq/b_{2r};q)_\infty}{(a_1,\ldots,a_{r+1},q/a_1,\ldots,q/a_{r+1},a_1/a,\ldots,a_{r+1}/a,aq/a_1,\ldots,aq/a_{r+1};q)_\infty} \\
&\quad\times {}_{2r}\psi_{2r}\left[\begin{matrix} b_1,\ldots,b_{2r} \\ aq/b_1,\ldots,aq/b_{2r}\end{matrix};q,\frac{a^rq^r}{b_1b_2\cdots b_{2r}}\right] \\
&= \frac{(a_1q/b_1,\ldots,a_1q/b_{2r},aq/a_1b_1,\ldots,aq/a_1b_{2r};q)_\infty}{(a_2/a_1,\ldots,a_{r+1}/a_1,qa_1/a_2,\ldots,qa_1/a_{r+1},aq/a_1a_2,\ldots,aq/a_1a_{r+1};q)_\infty} \\
&\quad\times (a_1,q/a_1,aq/a_1,a_1/a,a_1a_2/a,\ldots,a_1a_{r+1}/a;q)_\infty^{-1} \\
&\quad\times {}_{2r}\psi_{2r}\left[\begin{matrix} a_1b_1/a,a_1b_2/a,\ldots,a_1b_{2r}/a \\ a_1q/b_1,a_1q/b_2,\ldots,a_1q/b_{2r}\end{matrix};q,\frac{q^ra^r}{b_1b_2\cdots b_{2r}}\right] \\
&\quad+ \text{ idem } (a_1;a_2,\ldots,a_{r+1}),
\end{aligned} \tag{5.5.4}$$

and

$$\begin{aligned}
&\frac{(q/b_1,\ldots,q/b_{2r-1},aq/b_1,\ldots,aq/b_{2r-1};q)_\infty}{(a,a_1,\ldots,a_r,q/a,q/a_1,\ldots,q/a_r;q)_\infty} \\
&\quad\times \frac{(a^{\frac12},-a^{\frac12},q/a^{\frac12},-q/a^{\frac12},\pm(aq)^{\frac12},\pm(q/a)^{\frac12};q)_\infty}{(a_1/a,\ldots,a_r/a,aq/a_1,\ldots,aq/a_r;q)_\infty} \\
&\quad\times {}_{2r-1}\psi_{2r-1}\left[\begin{matrix} b_1,\ldots,b_{2r-1} \\ aq/b_1,\ldots,aq/b_{2r-1}\end{matrix};q,\frac{\mp q^{r-\frac12}a^{r-\frac12}}{b_1b_2\cdots b_{2r-1}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a_1(a_1q/b_1,\ldots,a_1q/b_{2r-1},aq/a_1b_1,\ldots,aq/a_1b_{2r-1};q)_\infty}{(aq/a_1^2,aq/a_1a_2,\ldots,aq/a_1a_r,a_2/a_1,\ldots,a_r/a_1;q)_\infty} \\
&\quad\times \frac{(a^{\frac12}/a_1,-a^{\frac12}/a_1,a_1q/a^{\frac12},-a_1q/a^{\frac12};q)_\infty}{(a/a_1,q/a_1,a_1^2/a,a_1a_2/a,\ldots,a_1a_r/a;q)_\infty} \\
&\quad\times \frac{(\pm(aq)^{\frac12}/a_1,\pm a_1(q/a)^{\frac12};q)_\infty}{(qa_1/a,a_1,qa_1/a_2,\ldots,qa_1/a_r;q)_\infty} \\
&\quad\times {}_{2r-1}\psi_{2r-1}\left[\begin{matrix}a_1b_1/a,\ldots,a_1b_{2r-1}/a\\ qa_1/b_1,\ldots,qa_1/b_{2r-1}\end{matrix};q,\frac{\mp q^{r-\frac12}a^{r-\frac12}}{b_1b_2\cdots b_{2r-1}}\right] \\
&\quad+\text{idem }(a_1;a_2,\ldots,a_r).
\end{aligned}
\tag{5.5.5}
$$

5.6 Transformation formulas for very-well-poised $_8\psi_8$ and $_{10}\psi_{10}$ series

In this section we consider two special cases of (5.5.2) that may be regarded as extensions of the transformation formulas for very-well-poised $_8\phi_7$ and $_{10}\phi_9$ series derived in Chapter 2. First, set $r=4$ in (5.5.2) and replace b_3, b_4, b_5, b_6, b_7, b_8 by b, c, d, e, f, g, respectively, choose $a_3=f$, $a_4=g$ and simplify to get

$$
\begin{aligned}
&\frac{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e;q)_\infty}{(f,g,f/a,g/a,aq,q/a;q)_\infty} \\
&\quad\times {}_8\psi_8\left[\begin{matrix}qa^{\frac12},-qa^{\frac12},b,c,d,e,f,g\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f,aq/g\end{matrix};q,\frac{a^3q^2}{bcdefg}\right] \\
&= \frac{(q,aq/bf,aq/cf,aq/df,aq/ef,fq/b,fq/c,fq/d,fq/e;q)_\infty}{(f,q/f,aq/f,f/a,g/f,fg/a,qf^2/a;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix}f^2/a,qfa^{-\frac12},-qfa^{-\frac12},fb/a,fc/a,fd/a,fe/a,fg/a\\ fa^{-\frac12},-fa^{-\frac12},fq/b,fq/c,fq/d,fq/e,fq/g\end{matrix};q,\frac{a^3q^2}{bcdefg}\right] \\
&\quad+\text{idem }(f;g),
\end{aligned}
\tag{5.6.1}
$$

where $\left|\frac{a^3q^2}{bcdefg}\right|<1$.

Replacing a, b, c, d, e, f, g by a^2, ba, ca, da, ea, fa, ga, respectively, we may rewrite (5.6.1) as

$$
\begin{aligned}
&\frac{(aq/b,aq/c,aq/d,aq/e,q/ab,q/ac,q/ad,q/ae;q)_\infty}{(fa,ga,f/a,g/a,qa^2,q/a^2;q)_\infty} \\
&\quad\times {}_8\psi_8\left[\begin{matrix}qa,-qa,ba,ca,da,ea,fa,ga\\ a,-a,aq/b,aq/c,aq/d,aq/e,aq/f,aq/g\end{matrix};q,\frac{q^2}{bcdefg}\right] \\
&= \frac{(q,q/bf,q/cf,q/df,q/ef,qf/b,qf/c,qf/d,qf/e;q)_\infty}{(fa,q/fa,aq/f,f/a,g/f,fg,qf^2;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix}f^2,qf,-qf,fb,fc,fd,fe,fg\\ f,-f,fq/b,fq/c,fq/d,fq/e,fq/g\end{matrix};q,\frac{q^2}{bcdefg}\right]
\end{aligned}
$$

$$+ \text{idem}\ (f; g), \tag{5.6.2}$$

provided $|q^2/bcdefg| < 1$. Note that no a's appear in the ${}_8\phi_7$ series on the right side of (5.6.2). This is essentially the same as eq. (2.2) in M. Jackson [1950a].

For the next special case of (5.5.2) we take $r = 5$ and replace $b_3, \ldots, b_{10}$ by b, c, d, e, f, g, h, k, respectively, choose $a_3 = g$, $a_4 = h$, $a_5 = k$, and finally, replace $a, b, \ldots, k$ by $a^2, ba, \ldots, ka$ and simplify. This gives

$$\frac{(aq/b, aq/c, aq/d, aq/e, aq/f, q/ab, q/ac, q/ad, q/ae, q/af; q)_\infty}{(ag, ah, ak, g/a, h/a, k/a, qa^2, q/a^2; q)_\infty}$$
$$\times {}_{10}\psi_{10}\left[\begin{matrix} qa, -qa, ba, ca, da, ea, fa, ga, ha, ka \\ a, -a, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, aq/k \end{matrix}; q, \frac{q^3}{bcdefghk}\right]$$
$$= \frac{(q, q/bg, q/cg, q/dg, q/eg, q/fg, qg/b, qg/c, qg/d, qg/e, qg/f; q)_\infty}{(gh, gk, k/g, h/g, ag, q/ag, g/a, aq/g, qg^2; q)_\infty}$$
$$\times {}_{10}\phi_9\left[\begin{matrix} g^2, qg, -qg, gb, gc, gd, ge, gf, gh, gk \\ g, -g, qg/b, qg/c, qg/d, qg/e, qg/f, qg/h, qg/k \end{matrix}; q, \frac{q^3}{bcdefghk}\right]$$
$$+ \text{idem}\ (g; h, k), \tag{5.6.3}$$

where $|q^3/bcdefghk| < 1$. Notice that all of the very-well-poised series in this section are VWP-balanced.

Exercises

5.1 Show that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_\infty}{(-q; q)_\infty}.$$

5.2 Letting $c \to \infty$ in (5.3.4) and setting $a = 1$, $b = -1$, show that

$$1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n} = \left[\frac{(q; q)_\infty}{(-q; q)_\infty}\right]^2.$$

5.3 In (5.3.1) set $b = a$, $c = d = e = -1$ and then let $a \to 1$ to show that

$$1 + 8\sum_{n=1}^{\infty} \frac{(-q)^n}{(1 + q^n)^2} = \left[\frac{(q; q)_\infty}{(-q; q)_\infty}\right]^4.$$

See section 8.11 for applications of Exercises 5.1-5.3 to Number Theory.

5.4 Set $b = c = d = e = -1$ and then let $a \to 1$ in (5.3.1) to obtain

$$1 + 16\sum_{n=1}^{\infty} \frac{q^{2n}(4 - q^n - q^{-n})}{(1 + q^n)^4}$$
$$= \left[\frac{(q; q)_\infty}{(-q; q)_\infty}\right]^8.$$

5.5 Show that

$$\sum_{n=-\infty}^{\infty} q^{4n^2} z^{2n}(1+zq^{4n+1}) = (q^2, -zq, -q/z; q^2)_\infty, \qquad z \neq 0.$$

5.6 Prove the quintuple product identity

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n}(1+zq^n)$$
$$= (q, -z, -q/z; q)_\infty (qz^2, q/z^2; q^2)_\infty, \qquad z \neq 0.$$

See the Notes for this exercise.

5.7 Show that

$$\sum_{n=-\infty}^{\infty} \frac{(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})^2} q^{5n+1}$$
$$= \frac{q(1-q^4)}{(1-q)^2(1-q^3)^2} {}_6\psi_6 \left[\begin{matrix} q^7, & -q^7, & q, & q, & q^3, & q^3 \\ q^2, & -q^2, & q^8, & q^8, & q^6, & q^6 \end{matrix}; q^5, q^5\right].$$

Deduce that

$$\sum_{n=0}^{\infty} \left\{ \frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2} \right\}$$
$$= q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \qquad |q| < 1.$$

See Andrews [1974a] for the above formulas.

5.8 Deduce (5.4.4) directly from (4.5.2).

5.9 Deduce (5.3.1) from (5.4.5) by using (2.7.1).

5.10 Show that

$$\frac{q}{e} \frac{(e/a, e/b, e/ab, qc/e, q^2/e, q^2f/e; q)_\infty}{(e, q/e, e/f, qf/e; q)_\infty} {}_2\psi_2 \left[\begin{matrix} e/c, e/q \\ e/a, e/b \end{matrix}; q, q\right] + \text{idem } (e; f)$$
$$= \frac{(q, q/a, q/b, c/a, c/b, c/ef, qef/c; q)_\infty}{(e, f, q/e, q/f, c/ab; q)_\infty}.$$

5.11 Show that

$${}_8\psi_8 \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, c, d, e, f, aq^{-n}, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/c, aq/d, aq/e, aq/f, q^{n+1}, aq^{n+1} \end{matrix}; q, \frac{a^2q^{2n+2}}{cdef}\right]$$
$$= \frac{(aq, q/a, aq/cd, aq/ef; q)_n}{(q/c, q/d, aq/e, aq/f; q)_n}$$
$$\times {}_4\psi_4 \left[\begin{matrix} e, f, aq^{n+1}/cd, q^{-n} \\ aq/c, aq/d, q^{n+1}, ef/aq^n \end{matrix}; q, q\right], \quad n = 0, 1, \ldots,$$

and deduce the limit cases

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{matrix} e, f \\ aq/c, aq/d \end{matrix}; q, \frac{aq}{ef}\right] \\
&= \frac{(q/c, q/d, aq/e, aq/f; q)_\infty}{(aq, q/a, aq/cd, aq/ef; q)_\infty} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(c, d, e, f; q)_n}{(1-a)(aq/c, aq/d, aq/e, aq/f; q)_n} \left(\frac{qa^3}{cdef}\right)^n q^{n^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} &= \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \frac{1-aq^{2n}}{1-a} (-1)^n a^{2n} q^{n(5n-1)/2} \\
&= \frac{(aq^5, a^2q^2, a^2q^3; q^5)_\infty}{(aq;q)_\infty} {}_3\phi_2\left[\begin{matrix} a/q, a, aq \\ a^2q^2, a^2q^3 \end{matrix}; q^5, aq^5\right],
\end{aligned}
$$

which reduces to the first and second Rogers–Ramanujan identities when $a = 1$ and $a = q$, respectively.
(Bailey [1950a] and Garrett, Ismail and Stanton [1999])

5.12 Using (5.6.2) and (2.11.7), show that

$$
\begin{aligned}
&{}_8\psi_8\left[\begin{matrix} qa, -qa, ab, ac, ad, ae, af, ag \\ a, -a, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g \end{matrix}; q, q\right] \\
&= \frac{(q, qa^2, q/a^2, ag, g/a, q/bc, q/bd, q/be, q/bf, q/cd, q/ce, q/cf; q)_\infty}{(bg, cg, dg, eg, fg, aq/b, aq/c, aq/d, aq/e, aq/f, q/ab, q/ac; q)_\infty} \\
&\quad \times \frac{(q/de, q/df, q/ef; q)_\infty}{(q/ad, q/ae, q/af; q)_\infty},
\end{aligned}
$$

provided $bcdefg = q$ and

$$
\frac{(bf, q/bf, cf, q/cf, df, q/df, ef, q/ef, ag, q/ag, g/a, aq/g; q)_\infty}{(bg, q/bg, cg, q/cg, dg, q/dg, eg, q/eg, af, q/af, f/a, aq/f; q)_\infty} = 1.
$$

Following Gosper [1988b], we may call this the bilateral Jackson formula.

5.13 Deduce from Ex. 5.12 the bilateral q-Saalschütz formula

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(a, b, c; q)_n}{(d, e, f; q)_n} q^n \\
&= \frac{(q, d/a, d/b, d/c, e/a, e/b, e/c, q/f; q)_\infty}{(d, e, aq/f, bq/f, cq/f, q/a, q/b, q/c; q)_\infty},
\end{aligned}
$$

provided $def = abcq^2$ and

$$
\begin{aligned}
&(e/a, aq/e, e/b, bq/e, e/c, cq/e, f, q/f; q)_\infty \\
&= (f/a, aq/f, f/b, bq/f, f/c, cq/f, e, q/e; q)_\infty.
\end{aligned}
$$

5.14 Show that

$$\int_{-c}^{d} \frac{(-qt/c, qt/d; q)_\infty}{(-at/c, bt/d; q)_\infty} d_q t$$
$$= \frac{d(1-q)}{1-b}\ {}_1\psi_1(q/a; bq; q, -ad/c)$$
$$= \frac{cd(1-q)(q, ab, -c/d, -d/c; q)_\infty}{(c+d)(a, b, -bc/d, -ad/c; q)_\infty}$$

when $|ab| < |ad/c| < 1$.
(Andrews and Askey [1981])

5.15 Show that

$$\int_{-\infty}^{\infty} \frac{(ct, -dt; q)_\infty}{(at, -bt; q)_\infty} d_q t$$
$$= \frac{2(1-q)(c/a, d/b, -c/b, -d/a, ab, q/ab; q)_\infty (q^2; q^2)_\infty^2}{(cd/abq, q; q)_\infty (a^2, q^2/a^2, b^2, q^2/b^2; q^2)_\infty}.$$

(Askey [1981])

5.16 Show that

$$\int_0^\infty \frac{(\alpha at, a/t, \alpha bt, b/t, \alpha ct, c/t, \alpha dt, d/t; q)_\infty}{(\alpha qt^2, q/\alpha t^2; q)_\infty} \frac{d_q t}{t}$$
$$= \frac{(1-q)(\alpha a, a, \alpha b, b, \alpha c, c, \alpha d, d; q)_\infty}{(\alpha q, q/\alpha; q)_\infty}$$
$$\times\ {}_6\psi_6 \left[\begin{matrix} q\sqrt{\alpha}, -q\sqrt{\alpha}, q/a, q/b, q/c, q/d \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha a, \alpha b, \alpha c, \alpha d \end{matrix} ; q, \frac{\alpha^2 abcd}{q^3} \right]$$
$$= \frac{(1-q)(q, \alpha ab/q, \alpha ac/q, \alpha ad/q, \alpha bc/q, \alpha bd/q, \alpha cd/q; q)_\infty}{(\alpha^2 abcd/q^3; q)_\infty}$$

when $|\alpha^2 abcd/q^3| < 1$.

5.17 Show that

$$\int_0^\infty \frac{(a_1 t, \ldots, a_r t, b_1/t, \ldots, b_s/t; q)_\infty}{(c_1 t, \ldots, c_r t, d_1/t, \ldots, d_s/t; q)_\infty} t^{\gamma-1} d_q t$$
$$= \frac{(1-q)(a_1, \ldots, a_r, b_1, \ldots, b_s; q)_\infty}{(c_1, \ldots, c_r, d_1, \ldots, d_s; q)_\infty}$$
$$\times\ {}_{r+s}\psi_{r+s} \left[\begin{matrix} c_1, \ldots, c_r, q/b_1, \ldots, q/b_s \\ a_1, \ldots, a_r, q/d_1, \ldots, q/d_s \end{matrix} ; q, \frac{b_1 \cdots b_s}{d_1 \cdots d_s} q^\gamma \right]$$

when

$$\left|\frac{b_1 \cdots b_s}{d_1 \cdots d_s} q^\gamma\right| < 1 < \left|\frac{c_1 \cdots c_r}{a_1 \cdots a_r} q^\gamma\right|.$$

5.18 Derive Bailey's [1950b] summation formulas:

(i) $${}_3\psi_3 \left[\begin{matrix} b, c, d \\ q/b, q/c, q/d \end{matrix} ; q, \frac{q}{bcd} \right] = \frac{(q, q/bc, q/bd, q/cd; q)_\infty}{(q/b, q/c, q/d, q/bcd; q)_\infty},$$

(ii) $${}_3\psi_3\left[\begin{matrix} b, c, d \\ q^2/b, q^2/c, q^2/d \end{matrix}; q, \frac{q^2}{bcd}\right] = \frac{(q, q^2/bc, q^2/bd, q^2/cd; q)_\infty}{(q^2/b, q^2/c, q^2/d, q^2/bcd; q)_\infty},$$

(iii) $${}_5\psi_5\left[\begin{matrix} b, c, d, e, q^{-n} \\ q/b, q/c, q/d, q/e, q^{n+1} \end{matrix}; q, q\right] = \frac{(q, q/bc, q/bd, q/cd; q)_n}{(q/b, q/c, q/d, q/bcd; q)_n} \text{ when } bcde = q^{n+1},$$

(iv) $${}_5\psi_5\left[\begin{matrix} b, c, d, e, q^{-n} \\ q^2/b, q^2/c, q^2/d, q^2/e, q^{n+2} \end{matrix}; q, q\right] = \frac{(1-q)(q^2, q^2/bc, q^2/bd, q^2/cd; q)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd; q)_n}$$

when $bcde = q^{n+3}$, where $n = 0, 1, \ldots$.

5.19 Show that

(i) $$\sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n}}{(q,q,q;q)_n}$$

and

(ii) $$\sum_{k=-n-1}^{n} (-1)^k \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n+1}}{(q,q,q;q)_n}$$

for $n = 0, 1, \ldots$. (Bailey [1950b])

5.20 Derive the ${}_2\psi_2$ transformation formulas

(i) $${}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] = \frac{(az, d/a, c/b, dq/abz; q)_\infty}{(z, d, q/b, cd/abz; q)_\infty} \times {}_2\psi_2\left[\begin{matrix} a, abz/d \\ az, c \end{matrix}; q, \frac{d}{a}\right],$$

(ii) $${}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] = \frac{(az, bz, cq/abz, dq/abz; q)_\infty}{(q/a, q/b, c, d; q)_\infty} \times {}_2\psi_2\left[\begin{matrix} abz/c, abz/d \\ az, bz \end{matrix}; q, \frac{cd}{abz}\right].$$

(Bailey [1950a])

5.21 Verify that Ex. 2.16 is equivalent to

$$\begin{aligned}(b/a, aq/b, df/a, aq/df, ef/a, aq/ef, bde/a, aq/bde; q)_\infty \\ = (f/a, aq/f, bd/a, aq/bd, be/a, aq/be, def/a, aq/def; q)_\infty \\ - \frac{b}{a}(d, q/d, e, q/e, f/b, qb/f, bdef/a^2, a^2q/bdef; q)_\infty.\end{aligned}$$

(See Bailey [1936])

5.22 Extend the above identity to

$$ab\left(\frac{bc}{a},\frac{aq}{bc},\frac{bd}{a},\frac{aq}{bd},\frac{be}{a},\frac{aq}{be},\frac{bf}{a},\frac{aq}{bf},\frac{g}{a},\frac{aq}{g},\frac{h}{a},\frac{aq}{h},\frac{g}{h},\frac{hq}{g};q\right)_\infty$$
$$-ab\left(\frac{ch}{a},\frac{aq}{ch},\frac{dh}{a},\frac{aq}{dh},\frac{eh}{a},\frac{aq}{eh},\frac{fh}{a},\frac{aq}{fh},\frac{b}{a},\frac{aq}{b},\frac{g}{a},\frac{aq}{g},\frac{g}{b},\frac{bq}{g};q\right)_\infty$$
$$=ag\left(\frac{cg}{a},\frac{aq}{cg},\frac{dg}{a},\frac{aq}{dg},\frac{eg}{a},\frac{aq}{eg},\frac{fg}{a},\frac{aq}{fg},\frac{b}{a},\frac{aq}{b},\frac{h}{a},\frac{aq}{h},\frac{b}{h},\frac{hq}{b};q\right)_\infty$$
$$-bh\left(c,\frac{q}{c},d,\frac{q}{d},e,\frac{q}{e},f,\frac{q}{f},\frac{b}{h},\frac{hq}{b},\frac{g}{h},\frac{hq}{g},\frac{g}{b},\frac{bq}{g};q\right)_\infty,$$

where $a^3q^2 = bcdefgh$. (See Slater [1954a])

5.23 More generally, show that it follows from the general formula for sigma functions in Whittaker and Watson [1965, p. 451, Example 3] and Tannery and Molk [1898, §400], and also from (5.4.3) that

$$\sum_{k=1}^{n}\frac{(a_k/b_1,a_k/b_2,\ldots,a_k/b_n;q)_\infty}{(a_k/a_1,a_k/a_2,\ldots,a_k/a_{k-1},a_k/a_{k+1},\ldots,a_k/a_n;q)_\infty}$$
$$\times\frac{(qb_1/a_k,qb_2/a_k,\ldots,qb_n/a_k;q)_\infty}{(qa_1/a_k,qa_2/a_k,\ldots,qa_{k-1}/a_k,qa_{k+1}/a_k,\ldots,qa_n/a_k;q)_\infty}=0,$$

where $a_1a_2\cdots a_n = b_1b_2\cdots b_n$.
(See Slater 1954a])

5.24 Extend the summation formula (1.9.6) to

$${}_{r+2}\psi_{r+2}\left[\begin{matrix}a,b,b_1q^{m_1},\ldots,b_rq^{m_r}\\ d,bq,b_1,\ldots,b_r\end{matrix};q,a^{-1}q^{1-n}\right]$$
$$=\frac{(q,q,bq/a,d/b;q)_\infty}{(bq,q/a,q/b,d;q)_\infty}\frac{(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}}b^n,$$

where $m_1,\ldots,m_r$ are nonnegative integers, n is an integer, and $|q/a| < |q^n| < |q^{m_1+\cdots+m_r}/d|$.
(W. Chu [1994a])

5.25 More generally, extend the transformation formula in Ex. 1.34(ii) to

$${}_{r+2}\psi_{r+2}\left[\begin{matrix}a,b,b_1q^{m_1},\ldots,b_rq^{m_r}\\ d,bcq,b_1,\ldots,b_r\end{matrix};q,a^{-1}q^{1-n}\right]$$
$$=\frac{(q,cq,bq/a,d/b;q)_\infty}{(bcq,q/a,q/b,d;q)_\infty}\frac{(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}}b^n$$
$$\times\ {}_{r+2}\phi_{r+1}\left[\begin{matrix}c^{-1},bq/d,bq/b_1,\ldots,bq/b_r\\ bq/a,bq^{1-m_1}/b_1,\ldots,bq^{1-m_r}/b_r\end{matrix};q,cdq^{n-(m_1+\cdots+m_r)}\right],$$

where $m_1,\ldots,m_r$ are nonnegative integers, n is an integer, and $|q/a| < |q^n| < |q^{m_1+\cdots+m_r}/cd|$.
(W. Chu [1994a])

5.26 Extend the summation formula in Ex. 2.33(i) to

$$
{}_{6+2k}\psi_{6+2k}\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, b, a/b, c, d, e_1, \ldots, e_k, \frac{aq^{n_1+1}}{e_1}, \ldots, \frac{aq^{n_k+1}}{e_k} \\ a^{1/2}, -a^{1/2}, \frac{aq}{b}, bq, \frac{aq}{c}, \frac{ad}{d}, \frac{aq}{e_1}, \ldots, \frac{aq}{e_k}, e_1 q^{-n_1}, \ldots, e_k q^{-n_k} \end{matrix}; q, \frac{aq^{1-N}}{cd}\right]
$$

$$
= \frac{(q, q, aq, q/a, aq/bc, aq/bd, bq/c, bq/d; q)_\infty}{(aq/b, aq/c, aq/d, bq, bq/a, q/b, q/c, q/d; q)_\infty}
\times \prod_{j=1}^{k} \frac{(aq/be_j, bq/e_j; q)_{n_j}}{(aq/e_j, q/e_j; q)_{n_j}},
$$

where $n_1, \ldots, n_k$ are nonnegative integers, $N = n_1 + \cdots + n_k$, and $|aq^{1-N}/cd| < 1$ when the series does not terminate.
(W. Chu [1998a])

5.27 Prove that

$$
\frac{S(a^{-2}, bc, bd, cd)}{S(b/a, c/a, d/a, abcd)} + \text{ idem } (a; b, c, d) = 2,
$$

where S is defined in Ex. 2.16. See Askey and Wilson [1985, pp. 10, 11], where it is used to evaluate the integral in (6.1.1).

Notes

§5.2 Andrews [1979c] used Ramanujan's sum (5.2.1) to prove a continued fraction identity that appeared in Ramanujan's [1988] "lost" notebook. Formal Laurent series and Ramanujan's sum are considered in Askey [1987]. A probabilistic proof of (5.2.1) can be found in Kadell [1987b]. Milne [1986, 1988a, 1989] derived multidimensional $U(n)$ generalizations of (5.2.1).

§5.3 Gustafson [1987b, 1989, 1990] derived a multilateral generalization of (5.2.1), (5.3.1) and related formulas by employing contour integration and Milne's [1985d, 1987, 1993, 1994a,b, 1997] work on $U(n)$ generalizations of the q-Gauss, q-Saalschütz, and very-well-poised ${}_6\phi_5$ summation formulas.

§5.4 M. Jackson [1954] employed (5.4.3) to derive transformation formulas for ${}_3\psi_3$ series.

§5.6 A transformation formula between certain ${}_4\phi_3$ and ${}_8\psi_8$ series was found by Jain [1980b], along with transformation formulas for particular ${}_7\psi_7$ series, and then used to deduce identities of Rogers–Ramanujan type with moduli 5, 6, 8, 12, 16, 20 and 24. Some recent results on bilateral basic hypergeometric series are given in Schlosser [2003a,b,c].

Ex. 5.6 Watson [1929b] derived this identity in an equivalent form. For various proofs of the quintuple product identity (and of its equivalent forms) and applications to number theory, Lie algebras, etc., see Adiga, Berndt, Bhargava and Watson [1985], Andrews [1974a], Atkin and Swinnerton-Dyer [1954], Bailey [1951], Carlitz and Subbarao [1972], Gordon [1961], Hirschhorn [1988], Kac [1978, 1985], Sears [1952], and Subbarao and Vidyasagar [1970].

Exercises 5.12 and 5.13 See Rahman and Suslov [1994a, 1998] for more general formulas.

Exercises 5.15 and 5.16 For integrals of Ramanujan-type that correspond to the summation formulas of basic bilateral series, see Rahman and Suslov [1994b, 1998] and Ismail and Rahman [1995].

Ex. 5.18 Using (i) and (ii) one can show that the formula in Ex. 2.31 holds even when a_1, a_2, a_3 are not nonnegative integers, provided that $|q| < 1$ and $|q^{a_1+a_2+a_3+1}| < 1$.

Exercises 5.21–5.23 Additional identities connecting sums of infinite products are given in Slater [1951, 1954b, 1966] and Watson [1929b].

6

THE ASKEY-WILSON q-BETA INTEGRAL AND SOME ASSOCIATED FORMULAS

6.1 The Askey-Wilson q-extension of the beta integral

It should be clear by now that the beta integral and extensions of it that can be evaluated compactly are important. A significant extension of the beta integral was found by Askey and Wilson [1985]. Since it has five degrees of freedom, four free parameters and the parameter q from basic hypergeometric functions, it has enough flexibility to be useful in many situations. This integral is

$$\int_{-1}^{1} \frac{h(x;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}})}{h(x;a,b,c,d)} \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{6.1.1}$$

where

$$\begin{aligned} h(x;a_1,a_2,\ldots,a_m) &= h(x;a_1,a_2,\ldots,a_m;q) \\ &= h(x;a_1)h(x;a_2)\cdots h(x;a_m), \\ h(x;a) = h(x;a;q) &= \prod_{n=0}^{\infty}(1-2axq^n+a^2q^{2n}) \\ &= \left(ae^{i\theta},ae^{-i\theta};q\right)_\infty, \quad x=\cos\theta, \end{aligned} \tag{6.1.2}$$

and

$$\max\left(|a|,|b|,|c|,|d|,|q|\right) < 1. \tag{6.1.3}$$

As in (6.1.1), we shall use the h notation without the base q displayed when the base is q.

Askey and Wilson deduced (6.1.1) from the contour integral

$$\frac{1}{2\pi i}\int_K \frac{(z^2,z^{-2};q)_\infty}{(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q)_\infty}\frac{dz}{z} = \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{6.1.4}$$

where the contour K is as defined in §4.9 and the parameters a,b,c,d are no longer restricted by (6.1.3), but by the milder restriction that their pairwise products are not of the form $q^{-j},\ j=0,1,2,\ldots$. Askey and Wilson's original proof of (6.1.4) required a number of interim assumptions that had to be removed by continuity and analytic continuation arguments. In their paper they also provided a direct evaluation of the reduced integral

$$\int_{-1}^{1} \frac{h(x;1,-1)}{h(x;a,b)} \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi(-abq;q)_\infty}{(q,-q,aq^{\frac{1}{2}},-aq^{\frac{1}{2}},bq^{\frac{1}{2}},-bq^{\frac{1}{2}},ab;q)_\infty} \tag{6.1.5}$$

by using summation formulas for ${}_1\psi_1$ and ${}_4\psi_4$ series. Simpler proofs of (6.1.1) were subsequently found by Rahman [1984] and Ismail and Stanton [1988]. In the following section we shall give Rahman's proof since it only uses formulas that we have already proved, whereas the Ismail and Stanton proof uses some results for certain orthogonal polynomials which will not be covered until Chapter 7.

We shall conclude this section by showing that the beta integral

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta dx = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{6.1.6}$$

is a limit case of (6.1.5).

Let $0 < q < 1,\ a = q^{\alpha+\frac{1}{2}},\ b = -q^{\beta+\frac{1}{2}}$ and use the notation

$$(z;q)_\alpha = \frac{(z;q)_\infty}{(zq^\alpha;q)_\infty} \tag{6.1.7}$$

and the definition (1.10.1) of the q-gamma function to express the right side of (6.1.5) in the form

$$2^{2\alpha+2\beta+2} \frac{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+2)} \frac{\pi}{\Gamma_q^2\left(\frac{1}{2}\right)} \frac{(-q;q)_{\alpha+\beta}(-q^{\frac{1}{2}};q)_{\alpha+\frac{1}{2}}(-q^{\frac{1}{2}};q)_{\beta+\frac{1}{2}}}{2^{2\alpha+2\beta+1}}.$$

By (1.10.3) this tends to $2^{2\alpha+2\beta+1}\ \Gamma(\alpha+1)\ \Gamma(\beta+1)/\Gamma(\alpha+\beta+2)$ as $q \to 1^-$, since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

For the integrand in (6.1.5) we have

$$\frac{h(x;1,-1)}{h(x;a,b)} = \left(e^{i\theta};q\right)_{\alpha+\frac{1}{2}} \left(e^{-i\theta};q\right)_{\alpha+\frac{1}{2}} \left(-e^{i\theta};q\right)_{\beta+\frac{1}{2}} \left(-e^{-i\theta};q\right)_{\beta+\frac{1}{2}}$$

and hence

$$\begin{aligned} &\lim_{q\to 1^-} \frac{h(x;1,-1)}{h(x;q^{\alpha+\frac{1}{2}},-q^{\beta+\frac{1}{2}})} \\ &= \left[\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)\right]^{\alpha+\frac{1}{2}} \left[\left(1+e^{i\theta}\right)\left(1+e^{-i\theta}\right)\right]^{\beta+\frac{1}{2}} \\ &= 2^{\alpha+\beta+1}(1-\cos\theta)^{\alpha+\frac{1}{2}}(1+\cos\theta)^{\beta+\frac{1}{2}}, \end{aligned}$$

which shows that (6.1.6) is a limit of (6.1.5).

Formula (6.1.1) is substantially more general than (6.1.5) since it contains two more parameters. It is the freedom provided by these extra parameters which will enable us to prove a number of important results in this and the subsequent chapters.

6.2 Proof of formula (6.1.1)

Denote the integral in (6.1.1) by $I(a,b,c,d)$. Since $x = \cos\theta$ is an even function of θ, one can write

$$I(a,b,c,d) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{h(x;1,-1,\sqrt{q},-\sqrt{q})}{h(x;a,b,c,d)}d\theta. \tag{6.2.1}$$

Let us assume, for the moment, that a,b,c,d and their pairwise products and quotients are not of the form q^{-j}, $j = 0,1,2,\ldots$. It is easy to check that, by (2.10.18),

$$\begin{aligned} &h(x;1)/h(x;a,b) \\ &= \frac{(a^{-1},b^{-1};q)_\infty}{b(1-q)(q,a/b,bq/a,ab;q)_\infty} \int_a^b \frac{(qu/a,qu/b,u;q)_\infty}{(u/ab;q)_\infty}\, \frac{d_qu}{h(x;u)}\,, \end{aligned} \tag{6.2.2}$$

$$\begin{aligned} &h(x;-1)/h(x;c,d) \\ &= \frac{(-c^{-1},-d^{-1};q)_\infty}{d(1-q)(q,c/d,dq/c,cd;q)_\infty} \int_c^d \frac{(qv/c,qv/d,-v;q)_\infty}{(-v/cd;q)_\infty}\, \frac{d_qv}{h(x;v)}\,, \end{aligned} \tag{6.2.3}$$

and

$$\begin{aligned} h(x;-q^{\frac{1}{2}})/h(x;u,v) &= \frac{q^{\frac{1}{2}}(-q^{\frac{1}{2}}u^{-1},-q^{\frac{1}{2}}v^{-1};q)_\infty}{v(1-q)(q,u/v,vq/u,uv;q)_\infty} \\ &\times \int_{uq^{-\frac{1}{2}}}^{vq^{-\frac{1}{2}}} \frac{(tq^{\frac{3}{2}}/u,tq^{\frac{3}{2}}/v,-qt;q)_\infty}{(-qt/uv;q)_\infty}\, \frac{d_qt}{h(x;tq^{\frac{1}{2}})}\,. \end{aligned} \tag{6.2.4}$$

Also,

$$\begin{aligned} &\frac{1}{2}\int_{-\pi}^{\pi} \frac{h(x;q^{\frac{1}{2}})}{h(x;tq^{\frac{1}{2}})}d\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{(q^{\frac{1}{2}}e^{i\theta},q^{\frac{1}{2}}e^{-i\theta};q)_\infty}{(tq^{\frac{1}{2}}e^{i\theta},tq^{\frac{1}{2}}e^{-i\theta};q)_\infty}d\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(t^{-1};q)_k(t^{-1};q)_\ell}{(q;q)_k(q;q)_\ell}\left(tq^{\frac{1}{2}}\right)^{k+\ell} e^{i(k-\ell)\theta}d\theta \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(t^{-1};q)_k(t^{-1};q)_\ell}{(q;q)_k(q;q)_\ell}\left(tq^{\frac{1}{2}}\right)^{k+\ell} \int_{-\pi}^{\pi} e^{i(k-\ell)\theta}d\theta \\ &= \pi\sum_{k=0}^{\infty} \frac{(t^{-1},t^{-1};q)_k}{(q,q;q)_k}(qt^2)^k = \pi\frac{(qt,qt;q)_\infty}{(q,qt^2;q)_\infty} \end{aligned} \tag{6.2.5}$$

for $|tq^{\frac{1}{2}}| < 1$, by (1.5.1). Since

$$(qt^2;q)_\infty = (qt^2,q^2t^2;q^2)_\infty = (tq^{\frac{1}{2}},-tq^{\frac{1}{2}},qt,-qt;q)_\infty, \tag{6.2.6}$$

we have

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{h(x;q^{\frac{1}{2}})}{h(x;tq^{\frac{1}{2}})}d\theta = \frac{\pi(qt;q)_\infty}{(q,tq^{\frac{1}{2}},-tq^{\frac{1}{2}},-tq;q)_\infty}\,. \tag{6.2.7}$$

Thus

$$\begin{aligned}
&I(a,b,c,d)\\
&= \frac{\pi q^{\frac{1}{2}}(a^{-1},b^{-1},-c^{-1},-d^{-1};q)_\infty}{bd(1-q)^3(q;q)_\infty^4(a/b,bq/a,ab,c/d,dq/c,cd;q)_\infty}\\
&\quad\times\int_a^b d_qu\frac{(qu/a,qu/b,u;q)_\infty}{(u/ab;q)_\infty}\int_c^d d_qv\frac{(qv/c,qv/d,-v,-q^{\frac{1}{2}}/u,-q^{\frac{1}{2}}/v;q)_\infty}{v(-v/cd,vq/u,u/v,uv;q)_\infty}\\
&\quad\times\int_{uq^{-\frac{1}{2}}}^{vq^{-\frac{1}{2}}} d_qt\frac{(tq^{\frac{3}{2}}/u,tq^{\frac{3}{2}}/v,qt;q)_\infty}{(tq^{\frac{1}{2}},-tq^{\frac{1}{2}},-qt/uv;q)_\infty}\\
&= \frac{\pi(a^{-1},b^{-1},-c^{-1},-d^{-1};q)_\infty}{bd(1-q)^2(q;q)_\infty^3(a/b,bq/a,ab,c/d,dq/c,cd;q)_\infty}\\
&\quad\times\int_a^b d_qu\frac{(qu/a,qu/b;q)_\infty}{(-u,u/ab;q)_\infty}\int_c^d d_qv\frac{(qv/c,qv/d,-uv;q)_\infty}{(v,uv,-v/cd;q)_\infty}\\
&= \frac{\pi(a^{-1},b^{-1},q^{\frac{1}{2}},-q^{\frac{1}{2}},-1;q)_\infty}{b(1-q)(q;q)_\infty^2(a/b,bq/a,ab,c,d,cd;q)_\infty}\int_a^b d_qu\frac{(qu/a,qu/b,cdu;q)_\infty}{(cu,du,u/ab;q)_\infty}\\
&= \frac{\pi(-1,q^{\frac{1}{2}},-q^{\frac{1}{2}},abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},
\end{aligned} \tag{6.2.8}$$

by repeated applications of (2.10.18).

Since $(-1,q^{\frac{1}{2}},-q^{\frac{1}{2}};q)_\infty = 2(q^{\frac{1}{2}},-q^{\frac{1}{2}},-q;q)_\infty = 2$, which follows from (6.2.6) by setting $t=1$, we get (6.1.1). By analytic continuation, the restrictions on a,b,c,d mentioned above may be removed.

6.3 Integral representations for very-well-poised ${}_8\phi_7$ series

Formulas (2.10.18) and (2.10.19) enable us to use the Askey-Wilson q-beta integral (6.1.1) to derive Riemann integral representations for very-well-poised ${}_8\phi_7$ series.

Let us first set

$$w(x;a,b,c,d) = (1-x^2)^{-\frac{1}{2}}\frac{h(x;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}})}{h(x;a,b,c,d)} \tag{6.3.1}$$

and

$$J(a,b,c,d,f,g) = \int_{-1}^1 w(x;a,b,c,d)\frac{h(x;g)}{h(x;f)}\,dx, \tag{6.3.2}$$

where $\max(|a|,|b|,|c|,|d|,|f|,|q|) < 1$ and g is arbitrary. Since, by (2.10.18),

$$\begin{aligned}
\frac{h(x;g)}{h(x;d,f)} &= \frac{(g/d,g/f;q)_\infty}{f(1-q)(q,d/f,qf/d,fd;q)_\infty}\\
&\quad\times\int_d^f d_qu\frac{(qu/d,qu/f,gu;q)_\infty}{(gu/df;q)_\infty h(x;u)},
\end{aligned} \tag{6.3.3}$$

we have

$$J(a,b,c,d,f,g) = \frac{(g/d, g/f; q)_\infty}{f(1-q)(q, d/f, qf/d, df; q)_\infty} \times \int_d^f d_q u \frac{(qu/d, qu/f, gu; q)_\infty}{(gu/df; q)_\infty} \int_{-1}^1 w(x; a, b, c, u)\, dx. \tag{6.3.4}$$

By (6.1.1),

$$\int_{-1}^1 w(x; a, b, c, u)\, dx = \frac{2\pi (abcu; q)_\infty}{(q, ab, ac, bc; q)_\infty (au, bu, cu; q)_\infty}. \tag{6.3.5}$$

Substituting this into (6.3.4), we obtain

$$J(a,b,c,d,f,g) = \frac{2\pi (g/d, g/f; q)_\infty}{f(1-q)(q;q)_\infty^2 (d/f, qf/d, df, ab, ac, bc; q)_\infty} \times \int_d^f d_q u \frac{(qu/d, qu/f, gu, abcu; q)_\infty}{(au, bu, cu, gu/df; q)_\infty}. \tag{6.3.6}$$

The parameters in this q-integral are such that (2.10.19) can be applied to obtain

$$J(a,b,c,d,f,g) = \frac{2\pi (g/f, fg, abcf, bcdf, cdaf, dabf; q)_\infty}{(q, ad, bd, cd, af, bf, cf, df, ab, ac, bc, abcdf^2; q)_\infty} \times {}_8W_7(abcdf^2 q^{-1}; af, bf, cf, df, abcdfg^{-1}; q, g/f), \tag{6.3.7}$$

provided $|g/f| < 1$, if the series does not terminate. By virtue of the transformation formula (2.10.1) many different forms of (6.3.7) can be written down. Two particularly useful ones are

$$J(a,b,c,d,f,g) = \frac{2\pi (ag, bg, cg, abcd, abcf; q)_\infty}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, abcg; q)_\infty} \times {}_8W_7(abcgq^{-1}; ab, ac, bc, g/d, g/f; q, df), \tag{6.3.8}$$

which was derived in Nassrallah and Rahman [1985], and

$$J(a,b,c,d,f,g) = \frac{2\pi (ag, bg, cg, dg, fg, abcdf/g; q)_\infty}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, df, g^2; q)_\infty} \times {}_8W_7(g^2 q^{-1}; g/a, g/b, g/c, g/d, g/f; q, abcdfg^{-1}). \tag{6.3.9}$$

If the series in (6.3.9) does not terminate, then we must impose the condition $|abcdfg^{-1}| < 1$ so that it converges.

Note that, if in (6.3.8) we let $0 < q < 1$, $a = -b = q^{\frac{1}{2}}$, $c = q^{\alpha + \frac{1}{2}}$, $d = z$, $f = -q^{\beta + \frac{1}{2}}$, $g = zq^\gamma$ with $\mathrm{Re}(\alpha, \beta) > -\frac{1}{2}$ and then take the limit $q \to 1^-$, we obtain, after some simplification,

$${}_2F_1(\gamma, \alpha+1; \alpha+\beta+2; z) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 x^\alpha (1-x)^\beta (1-xz)^{-\gamma} dx. \tag{6.3.10}$$

This shows that (6.3.8) is a q-analogue of Euler's integral representation (1.11.10).

Another limiting case of (6.3.8) was pointed out in Rahman [1986b]. To derive it, replace a,b,c,d,f,g in (6.3.8) by q^a,q^b,q^c,q^d,q^f,q^g, respectively. Also, replace x in the integral (6.3.2) by $\cos(t\log q)$, which corresponds to replacing $e^{i\theta}$ by q^{it}. Now let $q\to 1^-$ and use (1.10.1) and (1.10.3) to get the formula

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma(a+it)\Gamma(a-it)\Gamma(b+it)\Gamma(b-it)\Gamma(c+it)\Gamma(c-it)}{\Gamma(2it)\Gamma(-2it)}$$
$$\times\frac{\Gamma(d+it)\Gamma(d-it)\Gamma(f+it)\Gamma(f-it)}{\Gamma(g+it)\Gamma(g-it)}\,dt$$
$$=\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(a+f)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+g)\Gamma(b+g)\Gamma(c+g)}$$
$$\times\frac{\Gamma(b+f)\Gamma(c+d)\Gamma(c+f)\Gamma(a+b+c+g)}{\Gamma(a+b+c+d)\Gamma(a+b+c+f)}$$
$$\times{}_7F_6\left[\begin{matrix}a+b+c+g-1,\ \frac{1}{2}(a+b+c+g+1),\ a+b,\ a+c,\ b+c\\ \frac{1}{2}(a+b+c+g-1),\ c+g,\ b+g,\ a+g,\end{matrix}\right.$$
$$\left.\begin{matrix}g-d, & g-f,\\ a+b+c+d, & a+b+c+f\end{matrix};1\right], \tag{6.3.11}$$

where Re $(a,b,c,d,f)>0$.

6.4 Integral representations for very-well-poised ${}_{10}\phi_9$ series

If we set $g=abcdf$ in (6.3.7), then the ${}_8W_7$ series collapses to one term with value 1 and so we have the formula

$$\int_{-1}^{1}\frac{h(x;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}},abcdf)}{h(x;a,b,c,d,f)}\frac{dx}{\sqrt{1-x^2}}$$
$$=\frac{2\pi(abcd,abcf,bcdf,abdf,acdf;q)_\infty}{(q,ab,ac,ad,af,bc,bd,bf,cd,cf,df;q)_\infty}$$
$$=g_0(a,b,c,d,f),\quad \text{say,} \tag{6.4.1}$$

where $(\max|a|,|b|,|c|,|d|,|f|,|q|)<1$. This is a q-analogue of the formula

$$\int_0^1 x^{a-1}(1-x)^{b-1}(1-tx)^{-a-b}\,dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(1-t)^{-a},\ \text{Re}(a,b)>0.$$

Replace f by fq^n in (6.4.1), where n is a nonnegative integer, to get

$$\int_{-1}^{1}v(x;a,b,c,d,f)\frac{(fe^{i\theta},fe^{-i\theta};q)_n}{(abcdfe^{-i\theta},abcdfe^{i\theta};q)_n}\,dx$$
$$=g_0(a,b,c,d,f)\frac{(af,bf,cf,df;q)_n}{(bcdf,acdf,abdf,abcf;q)_n}, \tag{6.4.2}$$

where

$$v(x;a,b,c,d,f)=\left(1-x^2\right)^{-\frac{1}{2}}\frac{h(x;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}},abcdf)}{h(x;a,b,c,d,f)}. \tag{6.4.3}$$

Let $\sigma = abcdf$. If $|z| < 1$, then (6.4.2) gives the formula

$$\int_{-1}^{1} v(x;a,b,c,d,f)\ {}_{r+5}W_{r+4}(\sigma f q^{-1};a_1,\ldots,a_r,fe^{i\theta},fe^{-i\theta};q,z)\,dx$$
$$= g_0(a,b,c,d,f)\ {}_{r+7}W_{r+6}(\sigma f q^{-1};a_1,\ldots,a_r,af,bf,cf,df;q,z). \quad (6.4.4)$$

In particular, for $r=3$ and $z=\sigma^2/a_1a_2a_3$, we have the formula

$$\int_{-1}^{1} v(x;a,b,c,d,f)\ {}_8W_7(\sigma f q^{-1};a_1,a_2,a_3,fe^{i\theta},fe^{-i\theta};q,\sigma^2/a_1a_2a_3)\,dx$$
$$= g_0(a,b,c,d,f)\ {}_{10}W_9(\sigma f q^{-1};a_1,a_2,a_3,af,bf,cf,df;q,\sigma^2/a_1a_2a_3), \quad (6.4.5)$$

where $|\sigma^2/a_1a_2a_3| < 1$, if the series do not terminate.

Let us assume that

$$a_1a_2a_3q = \sigma^2 \quad (6.4.6)$$

which ensures that the very-well-poised series on either side of (6.4.5) are balanced. Then, by (2.11.7)

$$\begin{aligned}
&{}_8W_7(\sigma f q^{-1};a_1,a_2,a_3,fe^{i\theta},fe^{-i\theta};q,q)\\
&\quad + \frac{(\sigma f, qa_1/\sigma f, a_2, qa_1/a_2, a_3, qa_1/a_3;q)_\infty h(x;f,qa_1/f)}{(\sigma f/qa_1, \sigma f/a_2, \sigma f/a_3, qa_1a_2/\sigma f, qa_1a_3/\sigma f, q^2a_1^2/\sigma f;q)_\infty h(x;\sigma,qa_1/\sigma)}\\
&\quad \times\ {}_8W_7(qa_1^2/\sigma f;a_1,qa_1a_2/\sigma f,qa_1a_3/\sigma f,qa_1e^{i\theta}/\sigma,qa_1e^{-i\theta}/\sigma;q,q)\\
&= \frac{(\sigma f,\sigma f/a_2a_3,qa_1/\sigma f,\sigma/f;q)_\infty h(x;\sigma/a_2,\sigma/a_3)}{(\sigma f/a_2,\sigma f/a_3,qa_1a_2/\sigma f,qa_1a_3/\sigma f;q)_\infty h(x;\sigma,\sigma/a_2a_3)},
\end{aligned} \quad (6.4.7)$$

and hence

$$\begin{aligned}
&\int_{-1}^{1} \frac{h(x;1,-1,q^{\frac12},-q^{\frac12},\sigma/a_2,\sigma/a_3)}{h(x;a,b,c,d,f,\sigma/a_2a_3)}\frac{dx}{\sqrt{1-x^2}}\\
&= g_0(a,b,c,d,f)\left\{\frac{(\sigma f/a_2,\sigma f/a_3,\sigma/fa_2,\sigma/fa_3;q)_\infty}{(\sigma f,\sigma f/a_2a_3,\sigma/f,\sigma/fa_2a_3;q)_\infty}\right.\\
&\quad \times\ {}_{10}W_9(\sigma f q^{-1};a_1,a_2,a_3,af,bf,cf,df;q,q)\\
&\quad + \frac{(af,bf,cf,df,a_2,a_3,qa_1/a_2,qa_1/a_3;q)_\infty}{(\sigma/a,\sigma/b,\sigma/c,\sigma/d,qaa_1/\sigma,qba_1/\sigma,qca_1/\sigma,qda_1/\sigma;q)_\infty}\\
&\quad \times \frac{(qa_1/af,qa_1/bf,qa_1/cf,qa_1/df,qa_1/\sigma f;q)_\infty}{(\sigma/f,\sigma f/qa_1,\sigma f/a_2a_3,\sigma/fa_2a_3,q^2a_1^2/\sigma f;q)_\infty}\\
&\quad \left.\times\ {}_{10}W_9(qa_1^2/\sigma f;a_1,\sigma/fa_2,\sigma/fa_3,qaa_1/\sigma,qba_1/\sigma,qca_1/\sigma,qda_1/\sigma;q,q)\right\},
\end{aligned} \quad (6.4.8)$$

where $\sigma = abcdf$ and $a_1a_2a_3q = \sigma^2$.

Since the integrand on the left side of (6.4.8) is symmetric in a,b,c,d and f, the expression on the right side must have the same property. This provides an alternate proof of Bailey's four-term transformation formula (2.12.9) for VWP-balanced ${}_{10}\phi_9$ series which are balanced and nonterminating.

If we set $a_3 = q^{-n},\ n = 0, 1, 2, \ldots$, then the coefficient of the second ${}_{10}W_9$ on the right side of (6.4.8) vanishes and we obtain

$$\int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \sigma)}{h(x; a, b, c, d, f)} \frac{(\sigma e^{i\theta}/g, \sigma e^{-i\theta}/g; q)_n}{(\sigma e^{i\theta}, \sigma e^{-i\theta}; q)_n} \frac{dx}{\sqrt{1-x^2}}$$
$$= g_0(a, b, c, d, f) \frac{(\sigma f/g, \sigma/fg; q)_n}{(\sigma f, \sigma/f; q)_n}$$
$$\times\ {}_{10}\phi_9 \left[\begin{matrix} \nu, q\sqrt{\nu}, -q\sqrt{\nu}, & g, & af, & bf, & cf, & df, & \sigma^2 q^{n-1}/g, & q^{-n} \\ \sqrt{\nu}, & -\sqrt{\nu}, & \sigma f/g, & \sigma/a, & \sigma/b, & \sigma/c, & \sigma/d, & fgq^{1-n}/\sigma, \sigma f q^n \end{matrix} ; q, q \right], \tag{6.4.9}$$

where $\nu = \sigma f/q$. By applying the iteration of the transformation formula (2.9.1) given in Exercise 2.19, this can be written in the more symmetric form

$$\int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \sigma q^n, \tau)}{h(x; a, b, c, d, f, \tau q^n)} \frac{dx}{\sqrt{1-x^2}}$$
$$= g_0(a, b, c, d, f) \frac{(\tau a, \tau b, \tau c, \tau d, \tau f, \sigma/\tau; q)_n}{(\sigma/a, \sigma/b, \sigma/c, \sigma/d, \sigma/f, \tau^2; q)_n}$$
$$\times\ {}_{10}\phi_9 \left[\begin{matrix} \tau^2 q^{-1}, \tau q^{\frac{1}{2}}, -\tau q^{\frac{1}{2}}, \tau/a, \tau/b, \tau/c, \tau/d, \tau/f, \sigma\tau q^{n-1}, q^{-n} \\ \tau q^{-\frac{1}{2}}, -\tau q^{-\frac{1}{2}}, \tau a, \tau b, \tau c, \tau d, \tau f, \tau q^{1-n}/\sigma, \tau^2 q^n \end{matrix} ; q, q \right], \tag{6.4.10}$$

where $\sigma = abcdf$ and τ is arbitrary. Similarly, by applying (2.12.9) twice one can rewrite (6.4.8) in the form

$$\int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \lambda, \mu)}{h(x; a, b, c, d, f, g)} \frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{2\pi(\lambda\mu/af, \lambda\mu/bf, \lambda\mu/cf, \lambda\mu/df, \lambda g, \mu g, \lambda/g, \mu/g; q)_\infty}{(q, ab, ac, ad, ag, bc, bd, bg, cd, cg, dg, fg, f/g, \lambda\mu g/f; q)_\infty}$$
$$\times {}_{10}\phi_9 \left[\begin{matrix} \nu_1, q\sqrt{\nu_1}, -q\sqrt{\nu_1}, & ag, & bg, & cg, & dg, & \lambda/f, \mu/f, \lambda\mu/q \\ \sqrt{\nu_1}, & -\sqrt{\nu_1}, & \lambda\mu/af, & \lambda\mu/bf, & \lambda\mu/cf, & \lambda\mu/df, \ \mu g, \ \lambda g, \ gq/f \end{matrix} ; q, q \right]$$
$$+ \frac{2\pi(\lambda\mu/ag, \lambda\mu/bg, \lambda\mu/cg, \lambda\mu/dg, \lambda f, \mu f, \lambda/f, \mu/f; q)_\infty}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, df, gf, g/f, \lambda\mu f/g; q)_\infty}$$
$$\times {}_{10}\phi_9 \left[\begin{matrix} \nu_2, q\sqrt{\nu_2}, -q\sqrt{\nu_2}, & af, & bf, & cf, & df, & \lambda/g, \mu/g, \lambda\mu/q \\ \sqrt{\nu_2}, & -\sqrt{\nu_2}, & \lambda\mu/ag, & \lambda\mu/bg, & \lambda\mu/cg, & \lambda\mu/dg, \ \mu f, \ \lambda f, \ fq/g \end{matrix} ; q, q \right] \tag{6.4.11}$$

where $\nu_1 = \lambda\mu g/fq$, $\nu_2 = \lambda\mu f/gq$, $\lambda\mu = abcdfg$ and $\max(|a|, |b|, |c|, |d|, |f|, |g|, |q|) < 1$. For these and other results see Rahman [1986b].

If λ or μ equals a, b, c or d, then the right side of (6.4.11) is summable by (2.11.7), while the integral on the left side gives the sum by (6.4.1). For this reason the integral in (6.4.1) may be considered as an integral analogue of Bailey's formula (2.11.7) for the sum of two balanced very-well-poised ${}_8\phi_7$ series. Likewise, the integral in (6.4.2) is an integral analogue of Jackson's sum (2.6.2). Indeed, a basic integral may be called well-poised if it can be written in the form

$$\int_{-\pi}^{\pi} \frac{\prod_{j=1}^{n}(a_je^{i\theta}, a_je^{-i\theta}; q)_\infty}{\prod_{j=1}^{m}(b_je^{i\theta}, b_je^{-i\theta}; q)_\infty} d\theta. \tag{6.4.12}$$

The integrals in (6.1.1), (6.3.2), (6.4.1) and (6.4.2) are all well-poised.

6.5 A quadratic transformation formula for very-well-poised balanced ${}_{10}\phi_9$ series

In (6.4.11) let us set $\mu = -\lambda,\ g = -f$ and $b = -a$ so that $\lambda^2 = -a^2cdf^2$. Then the expression on the right side of (6.4.11) becomes

$$\begin{aligned}
&\frac{2\pi(\lambda^2/cf, \lambda^2/df; q)_\infty(\lambda^4/a^2f^2, \lambda^2f^2, \lambda^2/f^2; q^2)_\infty}{(q, -1, -a^2, -f^2, \lambda^2; q)_\infty(a^2c^2, a^2d^2, a^2f^2; q^2)_\infty(cd, cf, df; q)_\infty}\\
&\quad\times {}_{10}W_9(\lambda^2q^{-1}; -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, cf, df; q, q)\\
&\quad+\frac{2\pi(-\lambda^2/cf, -\lambda^2/df; q)_\infty(\lambda^4/a^2f^2, \lambda^2f^2, \lambda^2/f^2; q^2)_\infty}{(q, -1, -a^2, -f^2, \lambda^2; q)_\infty(a^2c^2, a^2d^2, a^2f^2; q^2)_\infty(cd, -cf, -df; q)_\infty}\\
&\quad\times {}_{10}W_9(\lambda^2q^{-1}; -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, -cf, -df; q, q).
\end{aligned} \tag{6.5.1}$$

We now turn to the integral on the left side of (6.4.11). Observing that

$$h(x; a, -a) = (a^2e^{2i\theta}, a^2e^{-2i\theta}; q^2)_\infty = h(\xi; a^2; q^2), \tag{6.5.2}$$

where $x = \cos\theta$ and $\xi = \cos 2\theta = 2x^2 - 1$, it follows from (2.10.18) that

$$\begin{aligned}
\frac{h(x; \lambda - \lambda)}{h(x; a, -a, f, -f)} &= \frac{h(\xi; \lambda^2; q^2)}{h(\xi; a^2, f^2; q^2)}\\
&= \frac{(\lambda^2/a^2, \lambda^2/f^2; q^2)_\infty}{f^2(1-q^2)(q^2, a^2/f^2, q^2f^2/a^2, a^2f^2; q^2)_\infty}\\
&\quad\times \int_{a^2}^{f^2} \frac{(q^2u/a^2, q^2u/f^2, \lambda^2u; q^2)_\infty}{(\lambda^2u/a^2f^2; q^2)_\infty h(\xi; u; q^2)} d_{q^2}u.
\end{aligned} \tag{6.5.3}$$

Hence the integral on the left side of (6.4.11) can be expressed as

$$\begin{aligned}
&\frac{(\lambda^2/a^2, \lambda^2/f^2; q^2)_\infty}{f^2(1-q^2)(q^2, a^2/f^2, q^2f^2/a^2, a^2f^2; q^2)_\infty}\\
&\quad\times \int_{a^2}^{f^2} \frac{(q^2u/a^2, q^2u/f^2, \lambda^2u; q^2)_\infty}{(\lambda^2u/a^2f^2; q^2)_\infty} d_{q^2}u \int_{-1}^{1} w(x; c, d, u^{\frac12}, -u^{\frac12})\, dx\\
&= \frac{2\pi(\lambda^2/a^2, \lambda^2/f^2; q^2)_\infty}{f^2(1-q^2)(q^2, a^2/f^2, q^2f^2/a^2, a^2f^2; q^2)_\infty(q, cd; q)_\infty}\\
&\quad\times \int_{a^2}^{f^2} \frac{(q^2u/a^2, q^2u/f^2, \lambda^2u, -cdu, -cdqu; q^2)_\infty}{(\lambda^2u/a^2f^2, c^2u, d^2u, -u, -uq; q^2)_\infty} d_{q^2}u
\end{aligned} \tag{6.5.4}$$

by (6.1.1). Since $\lambda^2 = -a^2cdf^2$, the q-integral on the right side of (6.5.4) reduces to

$$\int_{a^2}^{f^2} \frac{(q^2u/a^2, q^2u/f^2, -cdqu, -a^2cdf^2u; q^2)_\infty}{(c^2u, d^2u, -u, -uq; q^2)_\infty} d_{q^2}u$$
$$= \frac{f^2(1-q^2)(q^2, a^2/f^2, q^2f^2/a^2; q^2)_\infty}{(a^2c^2, a^2d^2, c^2f^2, d^2f^2; q^2)_\infty}$$
$$\times \frac{(-a^2c^2f^2, -a^2d^2f^2, a^2c^2d^2f^2, -cdqf^2, -a^2cdf^4; q^2)_\infty}{(-a^2, -f^2, -qf^2, -a^2c^2d^2f^4; q^2)_\infty}$$
$$\times {}_8W_7(-a^2c^2d^2f^4q^{-2}; c^2f^2, d^2f^2, -f^2, cd, cda^2f^2q^{-1}; q^2, -qa^2), \tag{6.5.5}$$

by (2.10.19).

Using this in (6.5.4) and equating with (6.5.1) we obtain the desired quadratic transformation formula

$$\frac{(-a^2cf, -a^2df; q)_\infty}{(df, cf; q)_\infty} {}_{10}W_9(\lambda^2q^{-1}; -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, cf, df; q, q)$$
$$+ \frac{(a^2cf, a^2df; q)_\infty}{(-df, -cf; q)_\infty} {}_{10}W_9(\lambda^2q^{-1}; -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, -cf, -df; q, q)$$
$$= (-1, -cdf^2, -cda^2f^2; q)_\infty \frac{(-qa^2, -a^2c^2f^2, -a^2d^2f^2; q^2)_\infty}{(c^2f^2, d^2f^2, -a^2c^2d^2f^4; q^2)_\infty}$$
$$\times {}_8W_7(-a^2c^2d^2f^4q^{-2}; c^2f^2, d^2f^2, -f^2, cd, cda^2f^2q^{-1}; q^2, -qa^2), \tag{6.5.6}$$

with $\lambda^2 = -cda^2f^2$. This gives a nonterminating extension of the transformation formula (3.10.3) when the bibasic Φ series is a balanced ${}_{10}\phi_9$. For an extension of (3.10.3) when the Φ series is not balanced, see Nassrallah and Rahman [1986].

6.6 The Askey-Wilson integral when max $(|a|,|b|,|c|,|d|) \geq 1$

Our aim in this section is to extend the Askey-Wilson formula (6.1.1) to cases in which $|q| < 1$ and the absolute value of at least one of the parameters is greater than or equal to 1. Since the integral in (6.1.1), which we have already denoted by $I(a,b,c,d)$, is symmetric in a, b, c, d, without loss of generality we may assume that $|a| \geq 1$.

Let us first consider

$$|a| \geq 1 > \max(|b|,|c|,|d|). \tag{6.6.1}$$

If $a = \pm 1$, then the functions $h(x;\pm 1)$ and $h(x;a)$ in the integrand in (6.1.1) cancel and by continuity it follows that

$$I(\pm 1, b, c, d) = \frac{2\pi(\pm bcd; q)_\infty}{(q, \pm b, \pm c, \pm d, bc, bd, cd; q)_\infty}. \tag{6.6.2}$$

However, if $|a| = 1$ and $a \neq \pm 1$, then $h(x;a) = 0$ for some x in the interval $(-1,1)$ and so the integral in (6.1.1) does not converge. Similarly, this integral does not converge if $|aq^n| = 1$ and $aq^n \neq \pm 1$ for some positive integer n.

If there is a nonnegative integer m such that

$$|aq^{m+1}| < 1 < |aq^m| \tag{6.6.3}$$

and if ab, ac and ad are not of the form q^{-n} for any nonnegative integer n, then the integral in (6.1.1) converges and we can evaluate it by the following technique.

Observe that, since

$$\begin{aligned} h(x;a) &= (ae^{i\theta}, ae^{-i\theta};q)_{m+1}\, h(x;aq^{m+1}) \\ &= a^{2m+2}q^{m+m^2} h(x;aq^{m+1}, q^{-m}/a)/h(x;q/a), \end{aligned} \tag{6.6.4}$$

we have

$$\begin{aligned} I(a,b,c,d) &= a^{-2m-2}q^{-m-m^2} \\ &\times \int_{-1}^{1} \frac{h(x;1,-1,q^{\frac{1}{2}}-q^{\frac{1}{2}},q/a)}{h(x;b,c,d,aq^{m+1},q^{-m}/a)} \frac{dx}{\sqrt{1-x^2}}, \end{aligned} \tag{6.6.5}$$

where the parameters $b, c, d, aq^{m+1}, q^{-m}/a$ in the denominator of the integrand are now all less than 1 in absolute value. By (6.3.8),

$$\begin{aligned} I(a,b,c,d) = &\frac{2\pi(qb/a, qc/a, qd/a, abcdq^{m+1}, bcdq^{-m}/a;q)_\infty}{(q,bc,bd,cd,abq^{m+1},acq^{m+1},adq^{m+1},bcdq/a;q)_\infty} \\ &\times \frac{a^{-2m-2}q^{-m-m^2}}{(bq^{-m}/a, cq^{-m}/a, dq^{-m}/a;q)_\infty} \\ &\times {}_8W_7(bcda^{-1}; bc, bd, cd, q^{-m}a^{-2}, q^{m+1}; q, q). \end{aligned} \tag{6.6.6}$$

The series in (6.6.6) is balanced and so we can apply Bailey's summation formula (2.11.7). After some simplification we find that

$$I(a,b,c,d) = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty} + L_m(a;b,c,d), \tag{6.6.7}$$

where

$$\begin{aligned} L_m(a;b,c,d) = &\frac{2\pi(aq/d, bq/d, cq/d, q^{m+1}, a^2bcq^{m+1}, q^{-m}/a^2, bcq^{-m};q)_\infty}{(q,ab,ac,bc,abq^{m+1},acq^{m+1},adq^{m+1},aq^{m+1}/d,abcq/d;q)_\infty} \\ &\times \frac{a^{-2m-1}d^{-1}q^{-m-m^2}}{(bq^{-m}/a, cq^{-m}/a, dq^{-m}/a, q^{-m}/ad;q)_\infty} \\ &\times {}_8W_7(abcd^{-1}; bc, q^{-m}/ad, aq^{m+1}/d, ab, ac; q, q). \end{aligned} \tag{6.6.8}$$

By (2.10.1),

$$\begin{aligned} &{}_8W_7(abcd^{-1}; bc, q^{-m}/ad, aq^{m+1}/d, ab, ac; q, q) \\ &= \frac{(abcq/d, q/ad, abq, acq;q)_\infty}{(bq/d, cq/d, q, a^2bcq;q)_\infty} \\ &\times {}_8W_7(a^2bc; abcd, ab, ac, a^2q^{m+1}, q^{-m}; q, q/ad). \end{aligned} \tag{6.6.9}$$

Since m is a nonnegative integer, the series on the right side of (6.6.9) terminates and hence, by Watson's formula (2.5.1),

$$
\begin{aligned}
&{}_8W_7(a^2bc; abcd, ab, ac, a^2q^{m+1}, q^{-m}; q, q/ad) = \frac{(a^2bcq, q; q)_m}{(abq, acq; q)_m} \\
&\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-m}, \ ab, \ ac, \ q^{-m}/ad \\ bcq^{-m}, \ aq/d, \ q^{-m} \end{matrix} ; q, q \right] = \frac{(a^2bcq, q, aq/b, aq/c; q)_m}{(qa^2, abq, acq, q/bc; q)_m} \\
&\quad \times {}_8\phi_7 \left[\begin{matrix} a^2, qa, -qa, ab, ac, ad, a^2q^{m+1}, q^{-m} \\ a, -a, aq/b, aq/c, aq/d, q^{-m}, a^2q^{m+1} \end{matrix} ; q, \frac{q}{abcd} \right] \\
&= \frac{(a^2bcq, q, aq/b, aq/c; q)_m}{(qa^2, abq, acq, q/bc; q)_m} \\
&\quad \times \sum_{k=0}^{m} \frac{(a^2; q)_k (1 - a^2q^{2k})(ab, ac, ad; q)_k}{(q; q)_k (1 - a^2)(aq/b, aq/c, aq/d; q)_k} \left(\frac{q}{abcd} \right)^k . \qquad (6.6.10)
\end{aligned}
$$

Using (6.6.10) and (6.6.9) in (6.6.8), we obtain

$$
\begin{aligned}
L_m(a; b, c, d) &= -\frac{2\pi (a^{-2}; q)_\infty}{(q, ab, ac, ad, b/a, c/a, d/a; q)_\infty} \\
&\quad \times \sum_{k=0}^{m} \frac{(a^2; q)_k (1 - a^2q^{2k})(ab, ac, ad; q)_k}{(q; q)_k (1 - a^2)(aq/b, aq/c, aq/d; q)_k} \left(\frac{q}{abcd} \right)^k . \qquad (6.6.11)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{h(x; a, b, c, d)} \frac{dx}{\sqrt{1 - x^2}} + \frac{2\pi (a^{-2}; q)_\infty}{(q, ab, ac, ad, b/a, c/a, d/a; q)_\infty} \\
&\quad \times \sum_{k=0}^{m} \frac{(a^2; q)_k (1 - a^2q^{2k})(ab, ac, ad; q)_k}{(q; q)_k (1 - a^2)(aq/b, aq/c, aq/d; q)_k} \left(\frac{q}{abcd} \right)^k \\
&= \frac{2\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \qquad (6.6.12)
\end{aligned}
$$

where $\max(|b|, |c|, |d|, |q|) < 1$, $|aq^{m+1}| < 1 < |aq^m|$ for some nonnegative integer m, and the products ab, ac, ad are not of the form $q^{-n}, n = 1, 2, \ldots$. Askey and Wilson [1985] proved this formula by using contour integration. By continuity, formula (6.6.12) also holds if the restriction (6.6.3) is replaced by $aq^m = \pm 1$.

Note that, if one of the products ab, ac or ad is of the form q^{-n} for some nonnegative integer n, the integral in (6.6.12) converges even though the denominator on the right side of (6.6.12) equals zero as does the denominator in the coefficient of the sum in (6.6.12). If we let ab tend to q^{-n} then, since $|b| < 1$ and $|aq^{m+1}| < 1 < |aq^m|$, we must have $n \leq m$. We may then multiply (6.6.12) by $1 - abq^n$ and take the limit $ab \to q^{-n}$. The result is a terminating ${}_6\phi_5$ series on the left side and its sum on the right, giving the summation formula (2.4.2).

If $\max(|c|, |d|, |q|) < 1$ and there are nonnegative integers m and r such that

$$
|aq^{m+1}| < 1 < |aq^m|, \ |bq^{r+1}| < 1 < |bq^r|, \qquad (6.6.13)
$$

then the above technique can be extended to evaluate $I(a,b,c,d)$ provided the products ab, ac, ad, bc, bd are not of the form q^{-n} and $a/b \neq q^{\pm n}$ for any nonnegative integer n. Splitting $h(x;b)$ in the same way as in (6.6.4) and using (6.3.4) we get

$$\begin{aligned}
&I(a,b,c,d)\\
&= b^{-2r-2}q^{-r-r^2}\ J(a, bq^{r+1}, q^{-r}/b, c, d, q/b)\\
&= b^{-2r-2}q^{-r-r^2}\ \frac{(q/bc, q/bd;q)_\infty}{d(1-q)(q, c/d, dq/c, cd;q)_\infty}\\
&\quad \times \int_c^d d_qu \frac{(qu/c, qu/d, qu/b;q)_\infty}{(qu/bcd;q)_\infty} \int_{-1}^1 w(x;a, bq^{r+1}, q^{-r}/b, u)\ dx.
\end{aligned} \tag{6.6.14}$$

However, by (6.6.12),

$$\begin{aligned}
&\int_{-1}^1 w(x;q, bq^{r+1}, q^{-r}/b, u)\ dx\\
&= \frac{2\pi(aqu;q)_\infty}{(q, q, abq^{r+1}, aq^{-r}/b;q)_\infty(au, buq^{r+1}, uq^{-r}/b;q)_\infty}\\
&\quad - \frac{2\pi(a^{-2};q)_\infty}{(q, abq^{r+1}, q^{-r}/ab, bq^{r+1}/a;q)_\infty(au, u/a;q)_\infty}\\
&\quad \times \sum_{k=0}^m \frac{(a^2;q)_k(1-a^2q^{2k})(au;q)_k}{(q;q)_k(1-a^2)(aq/u;q)_k}\ (au)^{-k}.
\end{aligned} \tag{6.6.15}$$

Since by (2.10.18),

$$\begin{aligned}
&\int_c^d\ d_qu \frac{(qu/c, qu/d, qu/b;q)_\infty}{(qu/bcd, au, u/a;q)_\infty}\ \frac{(au;q)_k}{(aq/u;q)_k}(qu)^{-k}\\
&= (-1)^k a^{-2k}q^{-k(k+1)/2} \int_c^d\ d_qu \frac{(qu/c, qu/d, qu/b;q)_\infty}{(qu/bcd, auq^k, uq^{-k}/a;q)_\infty}\\
&= \frac{d(1-q)(q, c/d, dq/c, cd, q/ab, aq/b;q)_\infty}{(q/bc, q/bd, ac, ad, c/a, d/a;q)_\infty}\\
&\quad \times \frac{(ab, ac, ad;q)_k}{(aq/b, aq/c, aq/d;q)_k}\ \left(\frac{q}{abcd}\right)^k,
\end{aligned} \tag{6.6.16}$$

we find that

$$\begin{aligned}
I(a,b,c,d) &= L_m(a;b,c,d)\\
&+ \frac{2\pi(adq, cdq, dq/b, q^{r+2}, q^{1-r}/b^2;q)_\infty b^{-2r-2}q^{-r-r^2}}{(q, q, ad, cd, dq^2/b, abq^{r+1}, bcq^{r+1}, bdq^{r+1}, aq^{-r}/b, cq^{-r}/b, dq^{-r}/b;q)_\infty}\\
&\times {}_8W_7(dq/b; bdq^{r+1}, q, dq^{-r}/b, q/bc, q/ab;q, ac),
\end{aligned} \tag{6.6.17}$$

where $L_m(a;b,c,d)$ is as defined in (6.6.11). The reduction of this ${}_8W_7$ will be done in two stages. First we use (2.10.1) twice to reduce it to a balanced ${}_8W_7$ and then apply (2.11.7) to obtain

$${}_8W_7(dq/b; bdq^{r+1}, q, dq^{-r}/b, q/bc, q/ab;q, ac)$$

$$
\begin{aligned}
&= \frac{(dq^2/b, q/bd, abcdq^{r+1}, acdq^{-r}/b; q)_\infty}{(q^{1-r}/b^2, q^{r+2}, ac, qacd^2; q)_\infty} \\
&\quad \times {}_8W_7(acd^2; abcd, bdq^{r+1}, dq^{-r}/b, ad, cd; q, q/bd) \\
&= \frac{(dq^2/b, aq/b, cq/b, q, abcdq^{r+1}, acdq^{-r}/b; q)_\infty}{(ac, adq, cdq, acdq/b, q^{r+2}, q^{1-r}/b^2; q)_\infty} \\
&\quad \times {}_8W_7(acdb^{-1}; cd, ad, ac, q^{r+1}, q^{-r}/b^2; q, q) \\
&= \frac{(dq^2/b, aq/b, b/a, q, q/ab, abcd, bcq^{r+1}, bdq^{r+1}, cq^{-r}/b, dq^{-r}/b; q)_\infty}{(ac, adq, cdq, dq/b, bd, bc, q^{r+2}, bq^{r+1}/a, q^{-r}/ab, q^{1-r}/b^2; q)_\infty} \\
&\quad + \frac{b}{a} \frac{(dq^2/b, q, bq/a, cq/a, dq/a, ad, q^{r+1}, b^2cdq^{r+1}, cdq^{-r}, q^{-r}/b^2; q)_\infty}{(bcdq/a, dq/b, bc, bd, adq, cdq, q^{r+2}, bq^{r+1}/a, q^{-r}/ab, q^{1-r}/b^2; q)_\infty} \\
&\quad \times {}_8W_7(bcda^{-1}; bc, bd, cd, bq^{r+1}/a, q^{-r}/ab; q, q). \qquad (6.6.18)
\end{aligned}
$$

Substituting (6.6.18) into (6.6.17) and simplifying the coefficients we get

$$
\begin{aligned}
&I(a,b,c,d) - L_m(a;b,c,d) - \frac{2\pi(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \\
&= \frac{2\pi(bq/a, cq/a, dq/a, q^{r+1}, b^2cdq^{r+1}, cdq^{-r}, q^{-r}/b^2; q)_\infty}{(q, bc, bd, cd, abq^{r+1}, bcq^{r+1}, bq^{r+1}/a, aq^{-r}/b, cq^{-r}/b, dq^{-r}; q)_\infty} \\
&\quad \times \frac{b^{-2r-1}a^{-1}q^{-r-r^2}}{(q^{-r}/ab, bcdq/a; q)_\infty} {}_8W_7(bcda^{-1}; bc, bd, cd, bq^{r+1}/a, q^{1-r}/ab; q, q).
\end{aligned}
\qquad (6.6.19)
$$

The expression on the right side of (6.6.19) is the same as that in (6.6.8) with a, b, c, d, m replaced by b, d, c, a and r, respectively, and so has the value

$$
\begin{aligned}
&- \frac{2\pi(b^{-2};q)_\infty}{(q, ba, bc, bd, a/b, c/b, d/b; q)_\infty} \\
&\quad \times \sum_{k=0}^{r} \frac{(b^2;q)_k(1-b^2q^{2k})(ba, bc, bd; q)_k}{(q;q)_k(1-b^2)(bq/a, bq/c, bq/d; q)_k} \left(\frac{q}{abcd}\right)^k \\
&= L_r(b;c,d,a), \qquad (6.6.20)
\end{aligned}
$$

by (6.6.11). So we find that

$$
\begin{aligned}
&I(a,b,c,d) - L_m(a;b,c,d) - L_r(b;c,d,a) \\
&\quad = \frac{2\pi(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \qquad (6.6.21)
\end{aligned}
$$

where the parameters satisfy the conditions stated earlier.

It is now clear that we can handle the cases of three or all four of the parameters a, b, c, d exceeding 1 in absolute value in exactly the same way. For example, in the extreme case when $\min(|a|,|b|,|c|,|d|) > 1 > |q|$ with

$$
\begin{aligned}
&|aq^{m+1}| < 1 < |aq^m|, \quad |bq^{r+1}| < 1 < |bq^r|, \\
&|cq^{s+1}| < 1 < |cq^s|, \quad |dq^{t+1}| < 1 < |dq^t|, \qquad (6.6.22)
\end{aligned}
$$

for some nonnegative integers m, r, s, t such that the pairwise products of a, b, c, d are not of the form q^{-n} and the pairwise ratios of a, b, c, d are not of the form $q^{\pm n}$ for $n = 0, 1, 2, \ldots$, we have the formula

$$I(a,b,c,d) - L_m(a;b,c,d) - L_r(b;c,d,a) - L_s(c;d,a,b) - L_t(d;a,b,c)$$
$$= \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{6.6.23}$$

where $L_s(c;d,a,b)$ and $L_t(d;a,b,c)$ are the same type of finite series as those in (6.6.11) and (6.6.20), and can be written down by obvious replacement of the parameters.

Exercises

6.1 Prove that
$$\int_0^\pi \frac{\sin^2\theta\, d\theta}{\prod_{j=1}^{4}(1-2a_j\cos\theta + a_j^2)} = \frac{\pi(1-a_1a_2a_3a_4)}{2\prod_{1\le j<k\le 4}(1-a_ja_k)}$$
when $\max(|a_1|,|a_2|,|a_3|,|a_4|) < 1$.

6.2 Use the q-binomial theorem and an appropriate transformation formula for the ${}_2\phi_1$ series to show that
$$\int_{-1}^1 \frac{h(x;1,-1)}{h(x;a,-a)}\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{2\pi(qa^2;q)_\infty}{(q,-q,aq^{\frac12},-aq^{\frac12},aq^{\frac12},-aq^{\frac12},-a^2;q)_\infty}, \quad |a|<1.$$

6.3 Prove that
$$\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta};q^2)_\infty}{(qe^{2i\theta},qe^{-2i\theta};q^2)_\infty}\frac{d\theta}{1-2z\cos\theta+z^2}$$
$$= \frac{2\pi(q,qz^2,q^3z^2;q^2)_\infty}{(q^2,q^2,q^2z^2;q^2)_\infty}\sum_{n=0}^\infty \frac{1-q}{1-q^{2n+1}}(qz^2)^n,$$
where $0<q<1$ and $|zq^{\frac12}|<1$.

6.4 If $0<q<1$ and $\max(|a|,|b|,|c|,|d|)<1$, show that
$$\int_{-1}^1 \frac{h(x;1,-1,q^{\frac12},f)}{h(x;a,b,c,d)}\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{2\pi(af,bf,cf,-abcq^{\frac12},abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,-aq^{\frac12},-bq^{\frac12},-cq^{\frac12},abcf;q)_\infty}$$
$$\times\, {}_8W_7(abcfq^{-1};ab,ac,bc,-fq^{-\frac12},f/d;q,-dq^{\frac12}).$$

6.5 If $\max(|a|,|b|,|c|,|d|,|q|)<1$, show that
$$\int_{-1}^1 \frac{h(x;1,-1,q^{\frac12},-q^{\frac12},-q/c)}{h(x;a,-a,b,-b,c)}\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{2\pi(-q;q)_\infty(-a^2b^2c^2,-q^2/c^2;q^2)_\infty}{(q;q)_\infty(-a^2,-b^2,a^2b^2,a^2c^2,b^2c^2;q^2)_\infty}.$$

6.6 If $b = a^{-1}, |c| < 1, |d| < 1$ and $|aq^{m+1}| < 1 < |aq^m|$ for some nonnegative integer m such that ac, ad are not of the form $q^{-n}, n = 0, 1, 2, \ldots$, show that

$$
\begin{aligned}
& I(a, a^{-1}, c, d) + \frac{2\pi (qa^2)^{-1}}{(q, q, acq, adq, c/aq, d/aq; q)_\infty} \\
& \quad \times \sum_{k=0}^{m-1} \frac{(1 - a^2 q^{2k+2})(acq, adq; q)_k (q/cd)^k}{(1 - q^{k+1})(1 - a^2 q^{k+1})(aq^2/c, aq^2/d; q)_k} \\
& = \frac{2\pi}{(q, q, ac, ad, c/a, d/a; q)_\infty} \\
& \quad \times \sum_{r=0}^{\infty} \left\{ \frac{1}{a^2 - q^r} + \frac{1}{(cd)^{-1} - q^r} - \frac{1}{ac^{-1} - q^r} - \frac{1}{ad^{-1} - q^r} \right\} q^r,
\end{aligned}
$$

where $I(a, b, c, d)$ is as defined in (6.2.1). If $m = 0$, the series on the left side is to be interpreted as zero.

6.7 Applying (2.12.9) twice deduce from(6.4.8) that

$$
\begin{aligned}
& \int_0^\pi \frac{h(\cos\theta; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \lambda, \mu)}{h(\cos\theta; a_1, a_2, a_3, a_4, a_5, a_6)} d\theta \\
& = A \frac{(\mu^2; q)_\infty}{(\mu/\lambda; q)_\infty} \prod_{j=1}^{6} \frac{(\lambda a_j; q)_\infty}{(\lambda/a_j; q)_\infty} \\
& \quad \times {}_{10}W_9(\lambda^2 q^{-1}; \lambda\mu/q, \lambda/a_1, \lambda/a_2, \lambda/a_3, \lambda/a_4, \lambda/a_5, \lambda/a_6; q, q) \\
& \quad + A \frac{(\lambda^2; q)_\infty}{(\lambda/\mu; q)_\infty} \prod_{j=1}^{6} \frac{(\mu a_j; q)_\infty}{(\mu/a_j; q)_\infty} \\
& \quad \times {}_{10}W_9(\mu^2 q^{-1}; \lambda\mu/q, \mu/a_1, \mu/a_2, \mu/a_3, \mu/a_4, \mu/a_5, \mu/a_6; q, q),
\end{aligned}
$$

where $\lambda\mu = \prod_{j=1}^{6} a_j$, $\max(|q|, |a_1|, \ldots, |a_6|) < 1$, and

$$
A = \frac{2\pi \prod_{j=1}^{6} (\lambda/a_j, \mu/a_j; q)_\infty}{(q, \lambda^2, \mu^2; q)_\infty \prod_{1 \le j < k \le 6} (a_j a_k; q)_\infty}.
$$

(Rahman [1986b, 1988b])

6.8 Prove that

$$
\begin{aligned}
& {}_8W_7\left(abczq^{-1}; az, bz, cz, abcdf/q, 1/f; q, \frac{q}{dz}\right) \\
& = \frac{(q, az, a/z, bz, b/z, cz, abcz, abc/\lambda z, \lambda\mu^2 q/abdz, \mu^2, ab/\mu^2; q)_\infty}{2\pi(ab, ac, bc, q/dz, zq/fd, abczf, q\mu^2/dz, aq/fd\mu^2; q)_\infty}
\end{aligned}
$$

$$\times \left(\frac{\lambda q}{bd}, \frac{a^2bc}{\lambda\mu^2}, bcf; q\right)_\infty \int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \frac{\mu q}{d}, \frac{q}{fd\mu})}{h(x; \frac{a}{\mu}, \frac{b}{\mu}, \mu z, \frac{\mu}{z}, \frac{abc}{\lambda\mu}, \frac{\lambda\mu q}{abd})}$$
$$\times {}_8W_7\left(\frac{a}{df\mu^2}; \frac{ae^{i\theta}}{\mu}, \frac{ae^{-i\theta}}{\mu}, \frac{q}{bdf}, \frac{ab}{f\lambda\mu^2}, \frac{\lambda q}{abcdf}; q, bcf\right)$$
$$\times {}_8W_7\left(\frac{\mu^2}{dz}; \frac{\mu e^{i\theta}}{z}, \frac{\mu e^{-i\theta}}{z}, \frac{q}{dz}, \frac{ab}{\lambda}, \frac{\lambda\mu^2 q}{abcd}; q, cz\right) \frac{dx}{\sqrt{1-x^2}},$$

where $|z| = 1$ and $\max(|a|, |b|, |c|, |d|, |f|, |\mu|, |a/\mu|, |b/\mu|, |abc/\lambda\mu|, |\lambda\mu q/abd|, |q/dz|, |q|) < 1$.

6.9 Choosing $\lambda = abcdq^{-1}$ and $f = q^n$ deduce from Ex. 6.8 that

$${}_4\phi_3\left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & a\mu e^{i\psi}, & a\mu e^{-i\psi} \\ & ab\mu^2, & ac, & ad \end{matrix}; q, q\right]$$
$$= \frac{(q, ab, \mu^2; q)_\infty}{2\pi(ab\mu^2; q)_\infty} h(z; a\mu, b\mu) \int_{-1}^{1} \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{h(x; a, b, \mu e^{i\psi}, \mu e^{-i\psi})}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\theta}, & ae^{-i\theta} \\ & ab, & ac, & ad \end{matrix}; q, q\right] \frac{dx}{\sqrt{1-x^2}},$$

where $z = \cos\psi$ and $\max(|a|, |b|, |\mu|, |q|) < 1$.

6.10 Show that

$$\int_0^\infty \frac{(iax, -ia/x, ibx, -ib/x, icx, -ic/x, idx, -id/x; q)_\infty}{(-qx^2, -q/x^2; q)_\infty} \frac{dx}{x}$$
$$= \frac{(\log q^{-1})(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q, q; q)_\infty}{(abcd/q^3; q)_\infty}$$

when $|abcd/q^3| < 1$ and $|q| < 1$.
(Askey [1988b])

6.11 Show that

$$\int_0^\infty \int_0^\infty t_1^{x-1} t_2^{x-1} \frac{(-t_1 q^{x+y+2k}, -t_2 q^{x+y+2k}; q)_\infty}{(-t_1, -t_2; q)_\infty} t_1^{2k} (t_2 q^{1-k}/t_1; q)_{2k} dt_1 dt_2$$
$$= \frac{\Gamma_q(x)\Gamma_q(x+k)\Gamma_q(y)\Gamma_q(y+k)\Gamma_q(k+1)\Gamma_q(2k+1)}{\Gamma_q(x+y+k)\Gamma_q(x+y+2k)\Gamma_q(k+1)\Gamma_q(k+1)}$$
$$\times \frac{[\Gamma(x)\Gamma(1-x)]^2}{\Gamma_q(x)\Gamma_q(1-x)\Gamma_q(x+2k)\Gamma_q(1-x-2k)}$$

when Re $x > 0$, Re $y > 0$, $|q| < 1$, and $k = 0, 1, \ldots$.
(Askey [1980b])

6.12 Show that

$$\int_0^\infty \int_0^\infty t_1^{x-1} t_2^{x-1} \frac{(-at_1 q^{x+y+2k}, -at_2 q^{x+y+2k}; q)_\infty}{(-at_1, -at_2; q)_\infty} \times t_1^{2k} (t_2 q^{1-k}/t_1; q)_{2k} \, d_q t_1 d_q t_2$$
$$= \frac{\Gamma_q(x)\Gamma_q(x+k)\Gamma_q(y)\Gamma_q(y+k)\Gamma_q(k+1)\Gamma_q(2k+1)}{\Gamma_q(x+y+k)\Gamma_q(x+y+2k)\Gamma_q(k+1)\Gamma_q(k+1)} \times \frac{(-aq^x, -aq^{x+2k}, -q^{1-x}/a, -q^{1-x-2k}/a; q)_\infty}{(-a, -a, -q/a, -q/a; q)_\infty}$$

when Re $x > 0$, Re $y > 0$, $|q| < 1$, and $k = 0, 1, \ldots$.
(Askey [1980b])

6.13 With $w(x; a, b, c, d)$ defined as in (6.3.1), prove that

$$\int_{-1}^1 \int_{-1}^1 w(x; a, aq^{\frac12}, b, bq^{\frac12}) \; w(y; a, aq^{\frac12}, b, bq^{\frac12}) \times \left|(q^{\frac12} e^{i(\theta+\phi)}, q^{\frac12} e^{i(\theta-\phi)}; q)_k\right|^2 dxdy$$
$$= \prod_{j=1}^{2} \frac{2\pi}{[(1-ab)(q, abq^{(j-1)k+1/2}, abq^{(j-1)k+1}; q)_\infty]^2} \times \prod_{j=1}^{2} \frac{(a^2b^2q^{jk+1}, q^{k+1}, q; q)_\infty}{(a^2q^{(j-1)k+1/2}, b^2q^{(j-1)k+1/2}, q^{jk+1}; q)_\infty},$$

where $x = \cos\theta$, $y = \cos\phi$, $|q| < 1$, and $k = 0, 1, \ldots$.
(Rahman [1986a])

6.14 Let $C.T.\ f(x)$ denote the constant term in the Laurent expansion of a function $f(x)$. Prove that if j and k are nonnegative integers, then

$$C.T.\ (qx, x^{-1}; q)_j (qx^2, qx^{-2}; q^2)_k$$
$$= \frac{1}{2\pi} \int_0^\pi (qe^{2i\theta}, e^{-2i\theta}; q)_j (qe^{4i\theta}, qe^{-4i\theta}; q^2)_k \, d\theta$$
$$= \frac{(q;q)_{2j} (q^2;q^2)_{2k} (q^{2j+1}; q^2)_k}{(q;q)_j (q^2;q^2)_k (q;q)_{j+2k}}.$$

(Askey [1982b])

6.15 Verify Ramanujan's identities

(i)
$$\int_{-\infty}^\infty e^{-x^2+2mx} (-aqe^{2kx}, -bqe^{-2kx}; q)_\infty \, dx = \frac{\sqrt{\pi}(abq;q)_\infty e^{m^2}}{(ae^{2mk}\sqrt{q}, be^{-2mk}\sqrt{q}; q)_\infty},$$

(ii)
$$\int_{-\infty}^\infty \frac{e^{-x^2+2mx} dx}{(ae^{2ikx}\sqrt{q}, be^{-2ikx}\sqrt{q}; q)_\infty} = \frac{\sqrt{\pi}\; e^{m^2} (-aqe^{2imk}, -bqe^{-2imk}; q)_\infty}{(abq;q)_\infty},$$

where $q = e^{-2k^2}$ and $|q| < 1$.
(See Ramanujan [1988] and Askey [1982a])

6.16 Derive the q-beta integral formulas

$$\text{(i)} \qquad \int_0^\infty \frac{(-tq^b, -q^{a+1}/t;q)_\infty}{(-t,-q/t;q)_\infty} \frac{d_qt}{t} = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}$$

and

$$\text{(ii)} \qquad \int_0^\infty \frac{(-tq^b, -q^{a+1}/t;q)_\infty}{(-t,-q/t;q)_\infty} \frac{dt}{t} = \frac{-\log q}{1-q} \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)},$$

where $0 < q < 1$, Re $a > 0$ and Re $b > 0$.
(Askey and Roy [1986], Gasper [1987])

6.17 Extend the above q-beta integral formulas to

$$\text{(i)} \qquad \int_0^\infty t^{c-1} \frac{(-tq^b, -q^{a+1}/t;q)_\infty}{(-t,-q/t;q)_\infty} d_qt = \frac{(-q^c, -q^{1-c};q)_\infty}{(-1,-q;q)_\infty} \frac{\Gamma_q(a+c)\Gamma_q(b-c)}{\Gamma_q(a+b)}$$

and

$$\text{(ii)} \qquad \int_0^\infty t^{c-1} \frac{(-tq^b, -q^{a+1}/t;q)_\infty}{(-t,-q/t;q)_\infty} dt = \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a+c)\Gamma_q(b-c)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+b)},$$

where $0 < q < 1$, Re $(a+c) > 0$ and Re $(b-c) > 0$.
(See Ramanujan [1915], Askey and Roy [1986], Gasper [1987], Askey [1988b], and Koornwinder [2003])

6.18 Use Ex. 2.16(ii) to show that, for $y = \cos\phi$,

$$\lim_{q\to 1^-} \frac{(q^{1/2}, q^{1/2};q)_\infty h(x; -q^{1/4}, -q^{3/4}) h(y; q^{1/4}, q^{3/4})}{2h(x; q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})} = \begin{cases} 1, & \text{if } x > y, \\ 0, & \text{if } x < y, \\ 1/2, & \text{if } x = y. \end{cases}$$

(Ismail and Rahman [2002b])

6.19 Denoting $S(a_1,\ldots,a_m) = S(a_1,\ldots,a_m;q) = \prod_{k=1}^m (a_k, q/a_k;q)_\infty$, use Ex. 2.16(i) to prove that

$$\prod_{k=1}^n h(x;\lambda_k, q/\lambda_k) = \sum_{j=1}^{n+1} \prod_{k=1}^n S(\lambda_k/a_j, \lambda_k a_j) \prod_{\substack{r=1 \\ r\neq j}}^{n+1} \frac{h(x;a_r,q/a_r)}{S(a_r a_j, a_r/a_j)}.$$

Deduce that

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{h(\cos\theta; a, b)} \frac{\prod_{k=1}^{n} h(\cos\theta; \lambda_k, q/\lambda_k)}{\prod_{m=1}^{n+1} h(\cos\theta; a_m, q/a_m)} d\theta$$

$$= \frac{2\pi}{(1-ab)(q,q;q)_\infty} \sum_{j=1}^{n+1} \prod_{k=1}^{n} S(\lambda_k/a_j, \lambda_k a_j)$$

$$\times \prod_{\substack{r=1 \\ r\neq j}}^{n+1} [S(a_r a_j, a_r/a_j)(aa_j, ba_j, aq/a_j, bq/a_j; q)_\infty]^{-1}.$$

(Ismail and Rahman [2002b])

6.20 Show that

$$\int_{-1}^1 \frac{h(x; 1, -1, q^{1/2}, -q^{1/2}, \alpha q^{1/2}, q^{1/2}/\alpha)}{h(x; a_1, a_2, \ldots, a_6)} \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{(a_1a_2a_3a_4a_5a_6/q; q)_\infty \prod_{j=1}^{6} (\alpha a_j q^{1/2}, a_j q^{1/2}/\alpha; q)_\infty}{(q, q, q\alpha^2, q/\alpha^2; q)_\infty \prod_{1\le j<k\le 6} (a_j a_k; q)_\infty}$$

$$\times {}_8\psi_8 \left[\begin{matrix} q\alpha, -q\alpha, \alpha q^{1/2}/a_1, \ldots, \alpha q^{1/2}/a_6 \\ \alpha, -\alpha, \alpha a_1 q^{1/2}, \ldots, \alpha a_6 q^{1/2} \end{matrix} ; q, a_1a_2a_3a_4a_5a_6/q \right],$$

provided $\alpha \neq q^{\pm n}$, $n = 0, 1, 2, \ldots$, $|a_1a_2a_3a_4a_5a_6/q| < 1$, and $|a_k| < 1$, $k = 1, \ldots, 6$.

(See Gustafson [1987b] and Rahman [1996a])

6.21 Prove that

$$\int_{-\infty}^{\infty} \frac{\prod_{j=1}^{4} (-t_j e^u, t_j e^{-u}; q)_\infty}{|(e^{u+v}, -e^{v-u}, -qe^{u-v}, qe^{-u-v}; q)_\infty|^2} \cosh u \, du$$

$$= \frac{\pi e^{-2v_1} \prod_{1\le j<k\le 4} (-t_j t_k/q; q)_\infty}{\sin v_2 \cosh v_1 (q, -qe^{2v_1}, -qe^{-2v_1}, t_1t_2t_3t_4/q^3; q)_\infty |(qe^{2iv_2}; q)_\infty|^2},$$

where $v = v_1 + iv_2$, $0 < v_2 < \pi/2$ or $v_2 = \pi/2$ and $v_1 \le 0$, $|t_j| < 1$, $|t_1t_2t_3t_4/q^3| < 1$.

(Ismail and Masson [1994])

Notes

Ex. 6.7 Setting $a_1 = -a_2 = a$, $a_3 = -a_4 = aq^{\frac{1}{2}}$, $a_5 = b$, $a_6 = c, \lambda = a^2bq$, $\mu = a^2c$ and using Ex. 2.25, Rahman [1988b] evaluated this integral in closed form and found the corresponding systems of biorthogonal rational functions.

Ex. 6.8 This may be regarded as a q-extension of the terminating case of Erdélyi's [1953, 2.4(3)] formula

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\mu)\Gamma(c-\mu)} \int_0^1 t^{\mu-1}(1-t)^{c-\mu-1}(1-tx)^{\lambda-a-b}$$
$$\times F(\lambda-a,\lambda-b;\mu;tx)\ F(a+b-\lambda,\lambda-\mu;c-\mu;(1-t)x/(1-tx))\, dt,$$

where $|x|<1,\ 0<\operatorname{Re}\mu<\operatorname{Re}c$. Also see Gasper [1975c, 2000].

Ex. 6.10 Askey [1988b] found the ${}_4\phi_3$ polynomials which are orthogonal with respect to the weight function associated with this integral.

Exercises 6.11–6.13 The double integrals in these exercises are q-analogues of the $n=2$ case of Selberg's [1944] important multivariable extension of the beta integral:

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{x-1}(1-t_i)^{y-1} \left| \prod_{1\le i<j\le n} (t_i-t_j) \right|^{2z} dt_1\cdots dt_n$$
$$= \prod_{j=1}^n \frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)},$$

where $\operatorname{Re} x>0$, $\operatorname{Re} y>0$, $\operatorname{Re} z>-\min(1/n, \operatorname{Re} x/(n-1), \operatorname{Re} y/(n-1))$.

Aomoto [1987] considered a generalization of Selberg's integral and utilized the extra freedom that he had in his integral to give a short elegant proof of it. Habsieger [1988] and Kadell [1988b] proved a q-analogue of Selberg's integral that was conjectured in Askey [1980b]. For conjectured multivariable extensions of the integrals in Exercises 6.11–6.13, other conjectured q-analogues of Selberg's integral, and related constant term identities that come from root systems associated with Lie algebras, see Andrews [1986, 1988], Askey [1980b, 1982b, 1985a, 1989f, 1990], Evans, Ismail and Stanton [1982], Garvan [1990], Garvan and Gonnet [1992], Habsieger [1988], Kadell [1988a–1994], Macdonald [1972–1995], Milne [1985a, 1989], Morris [1982], Rahman [1986a], Stanton [1986b, 1989], and Zeilberger [1987, 1988, 1990a]. Also see the extension of Ex. 6.11, the q-Selberg integrals, and the identities in Aomoto [1998], Evans [1994], Ito [2002], Kaneko [1996–1998], Lassalle and Schlosser [2003], Macdonald [1998a,b], Rains [2003a], Stokman [2002] and Tarasov and Varchenko [1997].

7

APPLICATIONS TO ORTHOGONAL POLYNOMIALS

7.1 Orthogonality

Let $\alpha(x)$ be a non-constant, non-decreasing, real-valued bounded function defined on $(-\infty, \infty)$ such that its *moments*

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \quad n = 0, 1, 2, \ldots, \tag{7.1.1}$$

are finite. A finite or infinite sequence $p_0(x), p_1(x), \ldots$ of polynomials, where $p_n(x)$ is of degree n in x, is said to be *orthogonal with respect to the measure* $d\alpha(x)$ and called an *orthogonal system of polynomials* if

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\, d\alpha(x) = 0, \quad m \neq n. \tag{7.1.2}$$

In view of the definition of $\alpha(x)$ the integrals in (7.1.1) and (7.1.2) exist in the Lebesgue-Stieltjes sense. If $\alpha(x)$ is absolutely continuous and $d\alpha(x) = w(x)dx$, then the orthogonality relation reduces to

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)w(x)\, dx = 0, \quad m \neq n, \tag{7.1.3}$$

and the sequence $\{p_n(x)\}$ is said to be *orthogonal with respect to the weight function* $w(x)$.

If $\alpha(x)$ is a step function (usually taken to be right-continuous) with jumps w_j at $x = x_j, j = 0, 1, 2, \ldots$, then (7.1.2) reduces to

$$\sum_j p_m(x_j)p_n(x_j)w_j = 0, \quad m \neq n. \tag{7.1.4}$$

In this case the polynomials are said to be *orthogonal with respect to a jump function* and are usually referred to as *orthogonal polynomials of a discrete variable.*

Every orthogonal system of real valued polynomials $\{p_n(x)\}$ satisfies a *three-term recurrence relation* of the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \tag{7.1.5}$$

with $p_{-1}(x) \equiv 0, p_0(x) \equiv 1$, where A_n, B_n, C_n are real and $A_n C_{n+1} > 0$. Conversely, if (7.1.5) holds for a sequence of polynomials $\{p_n(x)\}$ such that $p_{-1}(x) \equiv 0, p_0(x) \equiv 1$ and A_n, B_n, C_n are real with $A_n C_{n+1} > 0$, then there exists a positive measure $d\alpha(x)$ such that

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\, d\alpha(x) = \Big(v_n^{-1} \int_{-\infty}^{\infty} d\alpha(x)\Big)\delta_{m,n} \tag{7.1.6}$$

where

$$v_n = \prod_{k=1}^{n} \frac{A_{k-1}}{C_k}, \quad v_0 = 1. \tag{7.1.7}$$

If $\{p_n(x)\} = \{p_n(x)\}_{n=0}^{\infty}$ and $A_n C_{n+1} > 0$ for $n = 0, 1, 2, \ldots$, then the measure has infinitely many points of support, (7.1.5) holds for $n = 0, 1, 2, \ldots$, and (7.1.6) holds for $m, n = 0, 1, 2, \ldots$. If $\{p_n(x)\} = \{p_n(x)\}_{n=0}^{N}$ and $A_n C_{n+1} > 0$ for $n = 0, 1, 2, \ldots, N-1$, where N is a fixed positive integer, then the measure can be taken to have support on $N+1$ points $x_0, x_1, \ldots, x_N$, (7.1.5) holds for $n = 0, 1, \ldots, N-1$, and (7.1.6) holds for $m, n = 0, 1, 2, \ldots, N$.

This characterization theorem of orthogonal polynomials is usually attributed to Favard [1935], but it appeared earlier in published works of Perron [1929], Wintner [1929] and Stone [1932]. For a detailed discussion of this theorem see, for example, Atkinson [1964], Chihara [1978], Freud [1971] and Szegő [1975].

In the finite discrete case the recurrence relation (7.1.5) is a discrete analogue of a Sturm-Liouville two-point boundary-value problem with boundary conditions $p_{-1}(x) = 0, p_{N+1}(x) = 0$. If $x_0, x_1, \ldots, x_N$ are the zeros of $p_{N+1}(x)$, which can be easily proved to be real and distinct (see e.g., Atkinson [1964] for a complete proof), then the orthogonality relation (7.1.6) can be written in the form

$$\sum_{j=0}^{N} p_m(x_j) p_n(x_j) w_j = v_n^{-1} \sum_{j=0}^{N} w_j \; \delta_{m,n}, \tag{7.1.8}$$

$m, n = 0, 1, \ldots, N$, where w_j is the positive jump at x_j and v_n is as defined in (7.1.7). The *dual orthogonality* relation

$$\sum_{n=0}^{N} p_n(x_j) p_n(x_k) v_n = w_j^{-1} \sum_{n=0}^{N} w_n \; \delta_{j,k}, \tag{7.1.9}$$

$j, k = 0, 1, \ldots, N$, follows from the fact that a matrix that is orthogonal by rows is also orthogonal by columns. It can be shown that

$$w_j = \left[A_N v_N p_N(x_j) p'_{N+1}(x_j)\right]^{-1} \sum_{n=0}^{N} w_n, \quad j = 0, 1, \ldots, N, \tag{7.1.10}$$

where the prime indicates the first derivative.

In general, the measure in (7.1.6) is not unique and, given a recurrence relation, it may not be possible to find an explicit formula for $\alpha(x)$. Even though the classical orthogonal polynomials, which include the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \tag{7.1.11}$$

and the ultraspherical polynomials

$$\begin{aligned} C_n^{\lambda}(x) &= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})}(x) \\ &= \sum_{k=0}^{n} \frac{(\lambda)_k (\lambda)_{n-k}}{k!(n-k)!} e^{i(n-2k)\theta}, \quad x = \cos\theta, \end{aligned} \tag{7.1.12}$$

are orthogonal with respect to unique measures (see Szegő [1975]), it is not easy to discover these measures from the corresponding recurrence relations (see e.g., Askey and Ismail [1984]). However, for a wide class of discrete orthogonal polynomials it is possible to use the recurrence relation (7.1.5) and the formulas (7.1.8)–(7.1.10) to compute the jumps w_j and hence the measure. We shall illustrate this in the next section by considering the q-Racah polynomials (Askey and Wilson [1979]).

7.2 The finite discrete case: the q-Racah polynomials and some special cases

Suppose $\{p_n(x)\}$ is a finite discrete orthogonal polynomial sequence which satisfies a three-term recurrence relation of the form (7.1.5) and the orthogonality relations (7.1.8) and (7.1.9) with the weights w_j and the normalization constants v_n given by (7.1.10) and (7.1.7), respectively. We shall now assume, without any loss of generality, that $p_n(x_0) = 1$ for $n = 0, 1, \ldots, N$. This enables us to rewrite (7.1.5) in the form

$$(x - x_0)p_n(x) = A_n\left[p_{n+1}(x) - p_n(x)\right] - C_n\left[p_n(x) - p_{n-1}(x)\right], \tag{7.2.1}$$

where $n = 0, 1, \ldots, N$. Setting $j = k = 0$ in (7.1.9) we find that

$$\sum_{n=0}^{N} v_n = w_o^{-1} \sum_{n=0}^{N} w_n. \tag{7.2.2}$$

It is clear that in order to obtain solutions of (7.2.1) which are representable in terms of basic series it would be helpful if v_n and $\sum\limits_{n=0}^{N} v_n$ were equal to quotients of products of q-shifted factorials. Therefore, with the ${}_6\phi_5$ sum (2.4.2) in mind, let us take

$$\begin{aligned} v_n &= \frac{(abq;q)_n \left(1 - abq^{2n+1}\right)(aq, cq, bdq; q)_n}{(q;q)_n(1-abq)(bq, abq/c, aq/d; q)_n}(cdq)^{-n} \\ &= \prod_{k=1}^{n} \frac{(1-abq^k)\left(1-abq^{2k+1}\right)(1-aq^k)(1-cq^k)(1-bdq^k)}{(1-q^k)\left(1-abq^{2k-1}\right)(1-bq^k)(1-abq^k/c)(1-aq^k/d)cdq}, \end{aligned} \tag{7.2.3}$$

where $bdq = q^{-N}, 0 < q < 1$, so that

$$\begin{aligned} \sum_{n=0}^{N} v_n &= {}_6\phi_5\left[\begin{matrix} abq, q(abq)^{\frac{1}{2}}, -q(abq)^{\frac{1}{2}}, aq, cq, q^{-N} \\ (abq)^{\frac{1}{2}}, -(abq)^{\frac{1}{2}}, bq, abq/c, abq^{N+2} \end{matrix}; q, \frac{bq^N}{c}\right] \\ &= \frac{(abq^2, b/c; q)_N}{(bq, abq/c; q)_N}, \end{aligned} \tag{7.2.4}$$

where it is assumed that a, b, c, d are such that $v_n > 0$ for $n = 0, 1, \ldots, N$. In view of (7.1.7) we can take

$$A_{k-1} = \frac{(1-abq^k)(1-aq^k)(1-cq^k)(1-bdq^k)}{(1-abq^{2k-1})} r_k, \tag{7.2.5}$$

$$C_k = \frac{cdq(1-q^k)(1-bq^k)\left(1-abq^k/c\right)\left(1-aq^k/d\right)}{(1-abq^{2k+1})} r_k, \tag{7.2.6}$$

where $\{r_k\}_{k=1}^N$ is an arbitrary sequence with $r_k \neq 0, 1 \leq k \leq N$. Since $C_0 = 0$ and $A_0 = (1-aq)(1-cq)(1-bdq)r_1$, we have from the $n = 0$ case of (7.2.1) that

$$p_1(x_j) = 1 - \frac{(1-q^{-1})(x_j - x_0)qr_1^{-1}}{(1-q)(1-aq)(1-cq)(1-bdq)}. \tag{7.2.7}$$

This suggests that we should look for a basic series representation of $p_n(x_j)$ whose $(k+1)$-th term has $(q, aq, cq, bdq; q)_k$ as its denominator, which in turn suggests considering a terminating ${}_4\phi_3$ series. In view of the product $(-n)_k(n+\alpha+\beta+1)_k$ in the numerator of the $(k+1)$-th term in the hypergeometric series representation of $P_n^{(\alpha,\beta)}(x)$ in (7.1.11) and the dual orthogonality relations (7.1.8) and (7.1.9) required for $p_n(x_j)$ it is natural to look for a ${}_4\phi_3$ series whose $(k+1)$-th term has the numerator

$$\left(q^{-n}, q^{n+\alpha+\beta+1}, q^{-j}, q^{j+\gamma+\delta+1}; q\right)_k q^k.$$

Replacing $q^\alpha, q^\beta, q^\gamma, q^\delta$ by a, b, c, d, respectively, we are then led to consider a ${}_4\phi_3$ series of the form

$${}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-j}, cdq^{j+1} \\ aq, cq, bdq \end{matrix}; q, q\right]. \tag{7.2.8}$$

Observing that $\left(q^{-j}, cdq^{j+1}; q\right)_k$ is a polynomial of degree k in the variable $q^{-j} + cdq^{j+1}$, we find that if we take

$$x_j = q^{-j} + cdq^{j+1} \tag{7.2.9}$$

then $x_j - x_0 = -(1-q^{-j})(1-cdq^{j+1})$, and so (7.2.7) is satisfied with

$$r_k = \left(1 - abq^{2k}\right)^{-1}. \tag{7.2.10}$$

Then $A_k C_{k+1} > 0$ for $0 \leq k \leq N-1$ if, for example, a, b, c are real, $d = b^{-1}q^{-N-1}, bc < 0$, $\max(|aq|, |bq|, |cq|, |ab/c|) < 1$ and $0 < q < 1$.

We shall now verify that

$$p_n(x_j) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-j}, cdq^{j+1} \\ aq, cq, bdq \end{matrix}; q, q\right] \tag{7.2.11}$$

satisfies (7.2.1) with $x = x_j$. A straightforward calculation gives

$$\begin{aligned} &p_n(x_j) - p_{n-1}(x_j) \\ &= \frac{-q^{1-n}\left(1-abq^{2n}\right)\left(1-q^{-j}\right)\left(1-cdq^{j+1}\right)}{(1-aq)(1-cq)(1-bdq)} \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{1-n}, abq^{n+1}, q^{1-j}, cdq^{j+2} \\ aq^2, cq^2, bdq^2 \end{matrix}; q, q\right], \end{aligned} \tag{7.2.12}$$

$0 \le n \le N$. So we need to verify that

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-j}, cdq^{j+1} \\ aq, cq, bdq \end{matrix} ; q, q\right] \\
&= \frac{A_n q^{-n}\left(1-abq^{2n+2}\right)}{(1-aq)(1-cq)(1-bdq)}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+2}, q^{1-j}, cdq^{j+2} \\ aq^2, cq^2, bdq^2 \end{matrix} ; q, q\right] \\
&\quad - \frac{C_n q^{1-n}\left(1-abq^{2n}\right)}{(1-aq)(1-cq)(1-bdq)}\, {}_4\phi_3\left[\begin{matrix} q^{1-n}, abq^{n+1}, q^{1-j}, cdq^{j+2} \\ aq^2, cq^2, bdq^2 \end{matrix} ; q, q\right],
\end{aligned} \tag{7.2.13}$$

where A_n and C_n are given by (7.2.5) and (7.2.6) with r_k as defined in (7.2.10). Use of (2.10.4) on both sides of (7.2.13) reduces the problem to verifying that

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, bq^{-j}/c, bdq^{j+1} \\ bq, abq/c, bdq \end{matrix} ; q, q\right] \\
&= \frac{\left(1-abq^{n+1}\right)\left(1-bdq^{n+1}\right)}{\left(1-abq^{2n+1}\right)(1-bdq)} \\
&\quad \times {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+2}, bq^{-j}/c, bdq^{j+1} \\ bq, abq/c, bdq^2 \end{matrix} ; q, q\right] \\
&\quad - \frac{(1-q^n)\left(1-aq^n/d\right)}{\left(1-abq^{2n+1}\right)(1-bdq)} bdq \\
&\quad \times {}_4\phi_3\left[\begin{matrix} q^{1-n}, abq^{n+1}, bq^{-j}/c, bdq^{j+1} \\ bq, abq/c, bdq^2 \end{matrix} ; q, q\right],
\end{aligned} \tag{7.2.14}$$

which follows immediately from the fact that

$$\begin{aligned}
&\frac{(q^{-n};q)_k\left(abq^{n+1};q\right)_{k+1}\left(1-bdq^{n+1}\right)}{\left(1-abq^{2n+1}\right)\left(1-bdq^{k+1}\right)} \\
&+ \frac{(q^{-n};q)_{k+1}\left(abq^{n+1};q\right)_k\left(1-\frac{aq^n}{d}\right)bdq^{n+1}}{\left(1-abq^{2n+1}\right)\left(1-bdq^{k+1}\right)} = \left(q^{-n}, abq^{n+1};q\right)_k .
\end{aligned}$$

The verification that (7.2.11) satisfies the boundary condition $p_{N+1}(x_j) = 0$ for $0 \le j \le N$ is left as an exercise (Ex. 7.3).

Note that the ${}_4\phi_3$ series in (7.2.11) remains unchanged if we switch n, a, b, respectively, with j, c, d. This implies that the polynomials $p_n(x_j)$ are self-dual in the sense that they are of degree n in $x_j = q^{-j} + cdq^{j+1}$ and of degree j in $y_n = q^{-n} + abq^{n+1}$ and that the weights w_j are obtained from the v_n in (7.2.3) by replacing n, a, b, c, d, by j, c, d, a, b, respectively, i.e.,

$$w_j = \frac{(cdq;q)_j\left(1-cdq^{2j+1}\right)(cq, aq, bdq;q)_j}{(q;q)_j(1-cdq)\left(dq, cdq/a, cq/b;q\right)_j}(abq)^{-j}, \tag{7.2.15}$$

$0 \le j \le N$. Thus $\{p_n(x_j)\}_{n=0}^N$ is orthogonal with respect to the weights w_j, while $\{p_j(y_n)\}_{j=0}^N$ is orthogonal with respect to the weights v_n (see Ex. 7.4). The calculations have so far been done with the assumption that $bdq = q^{-N}$, but the same results will hold if we assume that one of aq, cq and bdq is q^{-N}.

The polynomials $p_n(x_j)$ were discovered by Askey and Wilson [1979] as q-analogues of the *Racah polynomials*

$$p_n(\lambda_j) = {}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, -j, j+\gamma+\delta+1\\ \alpha+1, \gamma+1, \beta+\delta+1\end{matrix}; 1\right], \tag{7.2.16}$$

where $\lambda_j = j(j+\gamma+\delta+1)$, which were named after the physicist Racah, who worked out their orthogonality without apparently being aware of the polynomial orthogonality. In the physicist's language the $p_n(\lambda_j)$'s are known as Racah coefficients or $6j$-symbols. The connection between the $6j$-symbols and the ${}_4F_3$ polynomials (7.2.16) was first made in Wilson's [1978] thesis. The q-analogues in (7.2.11) are called the *q-Racah polynomials.*

We shall now point out some important special cases of the q-Racah polynomials. First, let us set $d = b^{-1}q^{-N-1}$ in (7.2.11) and replace c by bc to rewrite the q-Racah polynomials in the more standard notation

$$W_n(x; a, b, c, N; q) = {}_4\phi_3\left[\begin{matrix}q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N}\\ aq, q^{-N}, bcq\end{matrix}; q, q\right]. \tag{7.2.17}$$

Then, by (7.1.8), (7.2.3) and (7.2.15),

$$\sum_{x=0}^{N} \rho(x; q) W_m(x; q) W_n(x; q) = \frac{\delta_{m,n}}{h_n(q)}, \tag{7.2.18}$$

where

$$\begin{aligned} W_n(x; q) &\equiv W_n(x; a, b, c, N; q),\\ \rho(x; q) &\equiv \rho(x; a, b, c, N; q)\\ &= \frac{(cq^{-N}; q)_x \left(1-cq^{2x-N}\right)\left(aq, bcq, q^{-N}; q\right)_x}{(q; q)_x (1-cq^{-N})(ca^{-1}q^{-N}, b^{-1}q^{-N}, cq; q)_x}(abq)^{-x} \end{aligned} \tag{7.2.19}$$

and

$$\begin{aligned} h_n(q) &\equiv h_n(a, b, c, N; q)\\ &= \frac{(bq, aq/c; q)_N}{(abq^2, 1/c; q)_N}\, \frac{(abq; q)_n \left(1-abq^{2n+1}\right)\left(aq, bcq, q^{-N}; q\right)_n}{(q; q)_n (1-abq)(bq, aq/c, abq^{N+2}; q)_n} \left(q^N/c\right)^n. \end{aligned} \tag{7.2.20}$$

Setting $c = 0$ in (7.2.17) gives the *q-Hahn polynomials*

$$Q_n(x) \equiv Q_n(x; a, b, N; q) = {}_3\phi_2\left[\begin{matrix}q^{-n}, abq^{n+1}, q^{-x}\\ aq, q^{-N}\end{matrix}; q, q\right], \tag{7.2.21}$$

which were introduced by Hahn [1949a] and, by (7.2.18), satisfy the orthogonality relation

$$\begin{aligned} &\sum_{x=0}^{N} Q_m(x) Q_n(x) \frac{(aq; q)_x (bq; q)_{N-x}}{(q; q)_x (q; q)_{N-x}} (aq)^{-x}\\ &= \frac{(abq^2; q)_N (aq)^{-N}}{(q; q)_N}\, \frac{(q; q)_n (1-abq)(bq, abq^{N+2}; q)_n}{(abq; q)_n (1-abq^{2n+1})(aq, q^{-N}; q)_n} (-aq)^n q^{\binom{n}{2}-Nn} \delta_{m,n} \end{aligned} \tag{7.2.22}$$

for $m, n = 0, 1, \ldots, N$.

Setting $a = 0$ in (7.2.17) we obtain the *dual q-Hahn polynomials* (Hahn [1949b])

$$R_n(\mu(x)) \equiv R_n(\mu(x); b, c, N; q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{-x}, cq^{x-N} \\ q^{-N}, bcq \end{matrix}; q, q\right], \tag{7.2.23}$$

where $\mu(x) = q^{-x} + cq^{x-N}$, which satisfy the orthogonality relation

$$\sum_{x=0}^{N} R_m(\mu(x))R_n(\mu(x)) \frac{\left(cq^{-N}; q\right)_x \left(1 - cq^{2x-N}\right)\left(bcq, q^{-N}; q\right)_x}{(q;q)_x(1-cq^{-N})(b^{-1}q^{-N}, cq; q)_x} q^{Nx-\binom{x}{2}}(-bcq)^{-x}$$
$$= \frac{(1/c;q)_N}{(bq;q)_N} \frac{(q, bq; q)_n}{(bcq, q^{-N}; q)_n} \left(cq^{-N}\right)^n \delta_{m,n}, \tag{7.2.24}$$

$m, n = 0, 1, \ldots, N$. For some applications of q-Hahn, dual q-Hahn polynomials, and their limit cases, see Delsarte [1976a,b, 1978], Delsarte and Goethals [1975], Dunkl [1977–1980] and Stanton [1977–1986c].

7.3 The infinite discrete case: the little and big q-Jacobi polynomials

As a q-analogue of the Jacobi polynomials (7.1.11), Hahn [1949a] (also see Andrews and Askey [1977]) introduced the polynomials

$$p_n(x; a, b; q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, xq). \tag{7.3.1}$$

It can be easily verified that

$$\lim_{q\to 1} p_n\left(\frac{1-x}{2}; q^\alpha, q^\beta; q\right) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}. \tag{7.3.2}$$

He proved that

$$\sum_{x=0}^{\infty} p_m(q^x; a, b; q)p_n(q^x; a, b; q)\frac{(bq;q)_x}{(q;q)_x}(aq)^x = \frac{\delta_{m,n}}{h_n(a,b;q)}, \tag{7.3.3}$$

where $0 < q, aq < 1$ and

$$h_n(a, b; q) = \frac{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(aq;q)_\infty}{(q;q)_n(1-abq)(bq;q)_n(abq^2;q)_\infty}(aq)^{-n}. \tag{7.3.4}$$

Observe that (7.3.3) and (7.3.4) also follow from (7.2.22) when we replace x by $N - x$ and then let $N \to \infty$. To prove (7.3.3), assume, as we may, that $0 \le m \le n$, and observe that

$$\sum_{x=0}^{\infty} \frac{(bq;q)_x}{(q;q)_x}(aq)^x q^{xk} p_m(q^x; a, b; q)$$
$$= \sum_{j=0}^{m} \frac{\left(q^{-m}, abq^{m+1}; q\right)_j}{(q, aq; q)_j} q^j \, {}_1\phi_0\left(bq; -; q, aq^{1+k+j}\right)$$

$$= \frac{\left(abq^{k+2};q\right)_\infty}{\left(aq^{k+1};q\right)_\infty} {}_3\phi_2\left[\begin{matrix} q^{-m}, & abq^{m+1}, & aq^{k+1} \\ & aq, & abq^{k+2} \end{matrix};q,q\right],$$

$$= \frac{(q,bq;q)_m(abq^2;q)_\infty}{(abq^2;q)_{2m}(aq;q)_\infty}(-aq)^m q^{\binom{m}{2}}\delta_{k,m}, \quad 0 \le k \le m, \tag{7.3.5}$$

by the q-binomial and q-Saalschütz formulas. Then the orthogonality relation (7.3.3) follows immediately by using (1.5.2) and (7.3.5).

It is easy to verify that $p_n(x;a,b;q)$ satisfies the three-term recurrence relation

$$xp_n(x) = A_n[p_{n+1}(x) - p_n(x)] - C_n[p_n(x) - p_{n-1}(x)], \tag{7.3.6}$$

for $n \ge 0$, where

$$A_n = \frac{\left(1-abq^{n+1}\right)\left(1-aq^{n+1}\right)}{\left(1-abq^{2n+1}\right)\left(1-abq^{2n+2}\right)}(-q^n), \tag{7.3.7}$$

$$C_n = \frac{\left(1-q^n\right)\left(1-bq^n\right)}{\left(1-abq^{2n}\right)\left(1-abq^{2n+1}\right)}(-aq^n), \tag{7.3.8}$$

so the condition that $A_nC_{n+1} > 0$ for $n = 0, 1, \ldots$, is satisfied if $0 < q, aq < 1$ and $bq < 1$. When $b < 0$ the polynomials $p_n(x;a,b;q)$ give a q-analogue of the Laguerre polynomials $L_n^{(\alpha)}(x)$ since

$$\lim_{q\to 1} p_n\left((1-q)x;q^\alpha,-q^\beta;q\right) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0). \tag{7.3.9}$$

Andrews and Askey [1985] introduced a second q-analogue of the Jacobi polynomials,

$$P_n(x;a,b,c;q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix};q,q\right], \tag{7.3.10}$$

which has the property that

$$\lim_{q\to 1} P_n(x;q^\alpha,q^\beta,-q^\gamma;q) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \tag{7.3.11}$$

where γ is real. In view of the third free parameter in (7.3.10) they called the $P_n(x;a,b,c;q)$ the *big q-Jacobi* and the $p_n(x;a,b;q)$ the *little q-Jacobi polynomials.*

We shall now prove that the big q-Jacobi polynomials satisfy the orthogonality relation

$$\int_{cq}^{aq} P_m(x;a,b,c;q)P_n(x;a,b,c;q)\frac{(x/a,x/c;q)_\infty}{(x,bx/c;q)_\infty}d_qx$$

$$= \frac{\delta_{m,n}}{h_n(a,b,c;q)}, \tag{7.3.12}$$

where

$$h_n(a,b,c;q) = M^{-1}\frac{(abq;q)_n\left(1-abq^{2n+1}\right)(aq,cq;q)_n}{(q;q)_n(1-abq)(bq,abq/c;q)_n}(-acq^2)^{-n}q^{-\binom{n}{2}} \tag{7.3.13}$$

and

$$M = \int_{cq}^{aq} \frac{(x/a, x/c; q)_\infty}{(x, bx/c; q)_\infty} d_q x$$
$$= \frac{aq(1-q)(q, c/a, aq/c, abq^2; q)_\infty}{(aq, bq, cq, abq/c; q)_\infty} \tag{7.3.14}$$

by (2.10.20). Since

$$\int_{cq}^{aq} (bx/c; q)_j (x; q)_k \frac{(x/a, x/c; q)_\infty}{(x, bx/c; q)_\infty} d_q x$$
$$= \int_{cq}^{aq} \frac{(x/a, x/c; q)_\infty}{(xq^k, bxq^j/c; q)_\infty} d_q x$$
$$= M \frac{(bq, abq/c; q)_j (aq, cq; q)_k}{(abq^2; q)_{j+k}},$$

the left side of (7.3.12) becomes

$$M \frac{(bq, abq/c; q)_m}{(aq, cq; q)_m} (c/b)^m$$
$$\times \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{\left(q^{-m}, abq^{m+1}; q\right)_j \left(q^{-n}, abq^{n+1}; q\right)_k}{(q; q)_j (q; q)_k (abq^2; q)_{j+k}} q^{j+k}, \tag{7.3.15}$$

where we used (7.3.10) and the observation that, by (3.2.2) and (3.2.5),

$$P_m(x; a, b, c; q)$$
$$= \frac{(bq, abq/c; q)_m}{(aq, cq; q)_m} (c/b)^m \, {}_3\phi_2 \left[\begin{matrix} q^{-m}, abq^{m+1}, bx/c \\ bq, abq/c \end{matrix} ; q, q \right]. \tag{7.3.16}$$

Assume that $0 \le m \le n$. Since, by (1.5.3)

$$\sum_{j=0}^{m} \frac{\left(q^{-m}, abq^{m+1}; q\right)_j}{\left(q, abq^{k+2}; q\right)_j} q^j = \frac{\left(q^{1+k-m}; q\right)_m}{\left(abq^{k+2}; q\right)_m} \left(abq^{m+1}\right)^m,$$

the double sum in (7.3.15) equals

$$\frac{\left(q^{-n}, abq^{n+1}; q\right)_m}{(abq^2; q)_{2m}} \left(abq^{m+2}\right)^m \sum_{k=0}^{n-m} \frac{\left(q^{m-n}, abq^{m+n+1}; q\right)_k}{\left(q, abq^{2m+2}; q\right)_k} q^k$$
$$= \frac{\left(q^{-n}, abq^{n+1}; q\right)_n}{\left(abq^2; q\right)_{2n}} \left(abq^{n+2}\right)^n \delta_{m,n}. \tag{7.3.17}$$

Substituting this into (7.3.15), we obtain (7.3.12).

7.4 An absolutely continuous measure: the continuous q-ultraspherical polynomials

In this and the following section we shall give two important examples of orthogonal polynomials which are orthogonal with respect to an absolutely continuous measure $d\alpha(x) = w(x)dx$.

In his work of the 1890's, in which he discovered the now-famous Rogers–Ramanujan identities, Rogers [1893b, 1894, 1895] introduced a set of orthogonal polynomials that are representable in terms of basic hypergeometric series and have the ultraspherical polynomials (7.1.12) as limits when $q \to 1$. Following Askey and Ismail [1983], we shall call these polynomials the *continuous q-ultraspherical polynomials* and define them by the generating function

$$\frac{\left(\beta te^{i\theta}, \beta te^{-i\theta}; q\right)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} C_n(x; \beta|q)t^n, \tag{7.4.1}$$

where $x = \cos\theta$, $0 \le \theta \le \pi$ and $\max(|q|, |t|) < 1$. Using the q-binomial theorem, it follows from (7.4.1) that

$$\begin{aligned} C_n(x; \beta|q) &= \sum_{k=0}^{n} \frac{(\beta; q)_k(\beta; q)_{n-k}}{(q; q)_k(q; q)_{n-k}} e^{i(n-2k)\theta} \\ &= \frac{(\beta; q)_n}{(q; q)_n} e^{in\theta}\ {}_2\phi_1\left(q^{-n}, \beta; \beta^{-1}q^{1-n}; q, q\beta^{-1}e^{-2i\theta}\right). \end{aligned} \tag{7.4.2}$$

Note that

$$\begin{aligned} \lim_{q\to 1} C_n(x; q^\lambda|q) &= \sum_{k=0}^{n} \frac{(\lambda)_k(\lambda)_{n-k}}{k!(n-k)!} e^{i(n-2k)\theta} \\ &= C_n^\lambda(x). \end{aligned} \tag{7.4.3}$$

Before considering the orthogonality relation for $C_n(x; \beta|q)$, we shall first derive some important formulas for these polynomials. For $0 < \theta < \pi, |\beta| < 1$, set $a = q\beta^{-1}e^{2i\theta}$, $b = q\beta^{-1}$, $c = qe^{2i\theta}$ and $z = \beta^2q^n$ in (3.3.5) to obtain

$$\begin{aligned} &{}_2\phi_1\left(q^{-n}, \beta; \beta^{-1}q^{1-n}; q, q\beta^{-1}e^{-2i\theta}\right) \\ &= \frac{\left(\beta q^n, \beta e^{-2i\theta}; q\right)_\infty}{(q^{n+1}, e^{-2i\theta}; q)_\infty}\ {}_2\phi_1\left(q\beta^{-1}, q\beta^{-1}e^{2i\theta}; qe^{2i\theta}; q, \beta^2q^n\right) \\ &\quad + \frac{\left(\beta, q\beta^{-1}, \beta q^n e^{2i\theta}, \beta^{-1}q^{1-n}e^{-2i\theta}; q\right)_\infty}{\left(q^{n+1}, e^{2i\theta}, q\beta^{-1}e^{-2i\theta}, \beta^{-1}q^{1-n}; q\right)_\infty} \\ &\quad \times {}_2\phi_1\left(q\beta^{-1}, q\beta^{-1}e^{-2i\theta}; qe^{-2i\theta}; q, \beta^2q^n\right). \end{aligned} \tag{7.4.4}$$

Then, application of the transformation formula (1.4.3) to the two ${}_2\phi_1$ series on the right side of (7.4.4) gives

$$\begin{aligned} &C_n(x; \beta|q) \\ &= \frac{(\beta; q)_\infty}{(\beta^2; q)_\infty} W_\beta^{-1}(x|q) \frac{(\beta^2; q)_n}{(q; q)_n} \end{aligned}$$

$$\times \left\{ \frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty} e^{in\theta} \ {}_2\phi_1\left(\beta, \beta e^{2i\theta}; qe^{2i\theta}; q, q^{n+1}\right) \right.$$
$$\left. + \frac{(e^{-2i\theta};q)_\infty}{(\beta e^{-2i\theta};q)_\infty} e^{-in\theta} \ {}_2\phi_1\left(\beta, \beta e^{-2i\theta}; qe^{-2i\theta}; q, q^{n+1}\right) \right\}, \tag{7.4.5}$$

where

$$W_\beta(x|q) = \frac{\left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{\left(\beta e^{2i\theta}, \beta e^{-2i\theta}; q\right)_\infty}, \quad x = \cos\theta. \tag{7.4.6}$$

Rewriting the right side of (7.4.5) as a q-integral we obtain the formula

$$C_n(x;\beta|q) = \frac{2i\sin\theta}{(1-q)W_\beta(x|q)} \frac{(\beta,\beta;q)_\infty}{(q,\beta^2;q)_\infty} \frac{\left(\beta^2;q\right)_n}{(q;q)_n}$$
$$\times \int_{e^{i\theta}}^{e^{-i\theta}} \frac{\left(que^{i\theta}, que^{-i\theta};q\right)_\infty}{\left(\beta ue^{i\theta}, \beta ue^{-i\theta};q\right)_\infty} u^n \, d_qu, \tag{7.4.7}$$

which was found by Rahman and Verma [1986a].

Now use (1.4.1) to obtain from (7.4.5) that

$$C_n(x;\beta|q) = \frac{(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty} W_\beta^{-1}(x|q) \frac{(\beta^2;q)_n}{(\beta q;q)_n}$$
$$\times \left\{ \left(1-e^{2i\theta}\right) e^{in\theta} \ {}_2\phi_1\left(q\beta^{-1}, q^{n+1}; \beta q^{n+1}; q, \beta e^{2i\theta}\right) \right.$$
$$\left. + \left(1-e^{-2i\theta}\right) e^{-in\theta} \ {}_2\phi_1\left(q\beta^{-1}, q^{n+1}; \beta q^{n+1}; q, \beta e^{-2i\theta}\right) \right\}$$
$$= 4\sin\theta \, W_\beta^{-1}(x|q) \sum_{k=0}^{\infty} b(k,n;\beta) \sin(n+2k+1)\theta, \tag{7.4.8}$$

where $0 < \theta < \pi, |\beta| < 1$ and

$$b(k,n;\beta) = \frac{(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty} \frac{(\beta^2;q)_n (q\beta^{-1};q)_k (q;q)_{n+k}}{(q;q)_n (q;q)_k (\beta q;q)_{n+k}} \beta^k. \tag{7.4.9}$$

The series on the right side of (7.4.8) is absolutely convergent if $|\beta| < 1$. For $|x| < 1$, $|q| < 1$ and large n it is clear from (7.4.5) that the leading term in the asymptotic expansion of $C_n(\cos\theta;\beta|q)$ is given by

$$C_n(\cos\theta;\beta|q) \sim \frac{(\beta;q)_\infty}{(q;q)_\infty} \left\{ \frac{\left(\beta e^{2i\theta};q\right)_\infty}{(e^{2i\theta};q)_\infty} e^{-in\theta} + \frac{\left(\beta e^{-2i\theta};q\right)_\infty}{(e^{-2i\theta};q)_\infty} e^{in\theta} \right\}$$
$$= 2\frac{(\beta;q)_\infty}{(q;q)_\infty} \left|A(e^{i\theta})\right| \cos(n\theta - \alpha), \tag{7.4.10}$$

where

$$A(z) = \frac{\left(\beta z^2;q\right)_\infty}{(z^2;q)_\infty}, \qquad \alpha = \arg A(e^{i\theta}), \tag{7.4.11}$$

and, as elsewhere, $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$. For further results on the asymptotics of $C_n(x;\beta|q)$, see Askey and Ismail [1980] and Rahman and Verma [1986a].

If we use (3.5.4) to express ${}_2\phi_1\left(q^{-n},\beta;\beta^{-1}q^{1-n};q,q\beta^{-1}e^{-2i\theta}\right)$ as a terminating VWP-balanced ${}_8\phi_7$ series in base $q^{\frac{1}{2}}$ and then apply (2.5.1), we obtain

$$
\begin{aligned}
&{}_2\phi_1\left(q^{-n},\beta;\beta^{-1}q^{1-n};q,q\beta^{-1}e^{-2i\theta}\right) = \frac{(q^{\frac{1}{2}-n}e^{-2i\theta};q)_n(-q^{(1-n)/2}\beta^{-1};q^{\frac{1}{2}})_n}{(\beta^{-1}q^{\frac{1}{2}-n};q^{\frac{1}{2}})_n} \\
&\quad\times {}_4\phi_3\left[\begin{matrix} q^{-n/2},\beta^{\frac{1}{2}}e^{-i\theta},-\beta^{\frac{1}{2}}e^{-i\theta},-q^{-n/2} \\ -\beta,q^{\frac{1}{4}-n/2}e^{-i\theta},-q^{\frac{1}{4}-n/2}e^{-i\theta}\end{matrix};q^{\frac{1}{2}},q^{\frac{1}{2}}\right] \\
&= \frac{(\beta^2;q)_n}{(\beta;q)_n}\beta^{-n/2}e^{-in\theta}\,{}_4\phi_3\left[\begin{matrix} q^{-n/2},\beta q^{n/2},\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta} \\ -\beta,\beta^{\frac{1}{2}}q^{\frac{1}{4}},-\beta^{\frac{1}{2}}q^{\frac{1}{4}}\end{matrix};q^{\frac{1}{2}},q^{\frac{1}{2}}\right] \qquad (7.4.12)
\end{aligned}
$$

by (2.10.4). However, by (3.10.13),

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n/2},\beta q^{n/2},\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta} \\ -\beta,\beta^{\frac{1}{2}}q^{\frac{1}{4}},-\beta^{\frac{1}{2}}q^{\frac{1}{4}}\end{matrix};q^{\frac{1}{2}},q^{\frac{1}{2}}\right] \\
&\quad = {}_4\phi_3\left[\begin{matrix} q^{-n},\beta^2q^n,\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}},-\beta,-\beta q^{\frac{1}{2}}\end{matrix};q,q\right], \qquad (7.4.13)
\end{aligned}
$$

and hence, from (7.4.2), (7.4.12) and (7.4.13), we have

$$
\begin{aligned}
&C_n(\cos\theta;\beta|q) \\
&\quad = \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-n/2}\,{}_4\phi_3\left[\begin{matrix} q^{-n},\beta^2q^n,\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}},-\beta,-\beta q^{\frac{1}{2}}\end{matrix};q,q\right]. \qquad (7.4.14)
\end{aligned}
$$

Since $W_\beta(\cos\theta|q) = |A(e^{i\theta})|^{-2}$ for real β, it follows from Theorem 40 in Nevai [1979] and the asymptotic formula (7.4.10) that the polynomials $C_n(\cos\theta;\beta|q)$ are orthogonal on $[0,\pi]$ with respect to the measure $W_\beta(\cos\theta|q)d\theta$, $-1<\beta<1$. One can also guess the weight function by setting $\beta = q^\lambda$ and comparing the generating function (7.4.1) and the expansion (7.4.8) with the $q\to 1$ limit cases and the weight function $(1-e^{2i\theta})^\lambda(1-e^{-2i\theta})^\lambda$ for the ultraspherical polynomials $C_n^\lambda(\cos\theta)$.

We shall now give a direct proof of the orthogonality relation

$$
\int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)W_\beta(\cos\theta|q)\,d\theta = \frac{\delta_{m,n}}{h_n(\beta|q)}, \qquad (7.4.15)
$$

where $|q|<1, |\beta|<1$ and

$$
h_n(\beta|q) = \frac{(q,\beta^2;q)_\infty}{2\pi(\beta,\beta q;q)_\infty}\,\frac{(q;q)_n(1-\beta q^n)}{(\beta^2;q)_n(1-\beta)}. \qquad (7.4.16)
$$

As we shall see in the next section, (7.4.15) can be proved by using (7.4.14) and the Askey-Wilson q-beta integral (6.1.1); but here we shall give a direct proof by using (7.4.2), (1.9.10) and (1.9.11), as in Gasper [1981b], to evaluate the integral. Since the integrand in (7.4.15) is even in θ, it suffices to prove that

$$
\int_{-\pi}^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)W_\beta(\cos\theta|q)\,d\theta = \frac{2\delta_{m,n}}{h_n(\beta|q)}, \qquad (7.4.17)
$$

when $0 \le m \le n$. We first show that, for any integer k,

$$\int_{-\pi}^{\pi} e^{ik\theta} W_\beta(\cos\theta|q)\, d\theta = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ c_{k/2}(\beta|q), & \text{if } k \text{ is even}, \end{cases} \tag{7.4.18}$$

where

$$c_j(\beta|q) = \frac{2\pi(\beta, \beta q; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(\beta^{-1}; q)_j}{(\beta q; q)_j} (1 + q^j)\beta^j. \tag{7.4.19}$$

By the q-binomial theorem,

$$\begin{aligned} &\int_{-\pi}^{\pi} e^{ik\theta} W_\beta(\cos\theta|q)\, d\theta \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\beta^{-1}; q)_r (\beta^{-1}; q)_s}{(q; q)_r (q; q)_s} \beta^{r+s} \int_{-\pi}^{\pi} e^{i(k+2r-2s)\theta} d\theta, \end{aligned}$$

which equals zero when k is odd and equals

$$\begin{aligned} &2\pi \sum_{s=0}^{\infty} \frac{(\beta^{-1}; q)_s (\beta^{-1}; q)_{s+j}}{(q; q)_s (q; q)_{s+j}} \beta^{j+2s} \\ &= 2\pi \frac{(\beta^{-1}; q)_j}{(q; q)_j} \beta^j \; {}_2\phi_1\left(\beta^{-1}, \beta^{-1} q^j; q^{j+1}; q, \beta^2\right) \end{aligned}$$

when $k = 2j$, $j = 0, 1, \ldots$. By (1.4.5), the above ${}_2\phi_1$ series equals

$$\begin{aligned} &\frac{(\beta, \beta q^{j+1}; q)_\infty}{(\beta^2, q^{j+1}; q)_\infty} \; {}_2\phi_1\left(q^{-1}, \beta^{-1}; \beta; q, \beta q^{j+1}\right) \\ &= \frac{(\beta, \beta q; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(q; q)_j}{(\beta q; q)_j} (1 + q^j). \end{aligned}$$

From this and the fact that $W_\beta(\cos\theta|q)$ is symmetric in θ, so that we can handle negative k's, we get (7.4.18). Hence, from (7.4.2),

$$\int_{-\pi}^{\pi} e^{ik\theta} C_n(\cos\theta; \beta|q) W_\beta(\cos\theta|q)\, d\theta \tag{7.4.20}$$

equals zero when $n - k$ is odd and equals

$$\begin{aligned} &\frac{2\pi(\beta, \beta q; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(\beta; q)_n (\beta^{-1}; q)_j}{(q; q)_n (\beta q; q)_j} (1 + q^j)\beta^j \\ &\times \; {}_4\phi_3\left[\begin{matrix} q^{-n}, \beta, \beta^{-1} q^{-j}, -q^{1-j} \\ \beta q^{1-j}, \beta^{-1} q^{1-n}, -q^{-j} \end{matrix}; q, q\right] \end{aligned} \tag{7.4.21}$$

when $n - k = 2j$ is even. From (1.9.11) it follows that this ${}_4\phi_3$ series and hence the integral (7.4.20) are equal to zero when $n > |k|$. Hence (7.4.17) holds when $m \ne n$. If $k = \pm n \ne 0$, then (1.9.10) gives

$$\begin{aligned} &\int_{-\pi}^{\pi} e^{ik\theta} C_n(\cos\theta; \beta|q) W_\beta(\cos\theta|q)\, d\theta \\ &= \frac{2\pi(\beta, \beta q; q)_\infty (\beta^2; q)_n}{(q, \beta^2; q)_\infty (\beta q; q)_n} \end{aligned} \tag{7.4.22}$$

from which it follows that (7.4.17) also holds when $m = n$.

7.5 The Askey-Wilson polynomials

In view of the ${}_4\phi_3$ series representation (7.4.14) for the continuous q-ultraspherical polynomials it is natural to consider the more general polynomials

$$ {}_4\phi_3 \left[\begin{matrix} q^{-n}, \alpha q^n, \beta e^{i\theta}, \beta e^{-i\theta} \\ \gamma, \delta, \epsilon \end{matrix} ; q, q \right] \tag{7.5.1} $$

which are of degree n in $x = \cos\theta$, and to try to determine the values of $\alpha, \beta, \gamma, \delta, \epsilon$ for which these polynomials are orthogonal. Because terminating balanced ${}_4\phi_3$ series can be transformed to other balanced ${}_4\phi_3$ series and to VWP-balanced ${}_8\phi_7$ series which satisfy three-term transformation formulas (see, e.g., (7.2.13), (2.11.1), Exercise 2.15 and the three-term recurrence relation for the q-Racah polynomials), one is led to consider balanced ${}_4\phi_3$ series. From Sears' transformation formula (2.10.4) it follows that if we set $\alpha = abcdq^{-1}, \beta = a, \gamma = ab, \delta = ac$ and $\epsilon = ad$, then the polynomials

$$ \begin{aligned} p_n(x) &\equiv p_n(x; a, b, c, d|q) \\ &= (ab, ac, ad; q)_n a^{-n} \, {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right] \end{aligned} \tag{7.5.2} $$

are symmetric in a, b, c, d. In addition, for real θ these polynomials are analytic functions of a, b, c, d and are, in view of the coefficient $(ab, ac, ad; q)_n a^{-n}$, real-valued when a, b, c, d are real or, if complex, occur in conjugate pairs.

Askey and Wilson [1985] introduced these polynomials as q-analogues of the ${}_4F_3$ polynomials of Wilson [1978, 1980]. Since they derived the orthogonality relation, three-term recurrence relation, difference equation and other properties of $p_n(x; a, b, c, d|q)$, these polynomials are now called the *Askey-Wilson polynomials.*

Since the three-term recurrence relation (7.2.1) for the q-Racah polynomials continues to hold without the restriction $bdq = q^{-N}$, by translating it into the notation for $p_n(x; a, b, c, d|q)$ we find, as in Askey and Wilson [1985], that the recurrence relation for these polynomials can be written in the form

$$ 2xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \quad n \geq 0, \tag{7.5.3} $$

with $p_{-1}(x) = 0, p_0(x) = 1$, where

$$ A_n = \frac{1 - abcdq^{n-1}}{(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \tag{7.5.4} $$

$$ \begin{aligned} C_n = {} & \frac{(1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})} \\ & \times (1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1}), \end{aligned} \tag{7.5.5} $$

and

$$B_n = a + a^{-1} - A_n a^{-1}(1-abq^n)(1-acq^n)(1-adq^n) \\ - C_n a/(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1}). \quad (7.5.6)$$

It is clear that A_n, B_n, C_n are real if a, b, c, d are real or, if complex, occur in conjugate pairs. Also $A_n C_{n+1} > 0, n = 0, 1, \ldots,$ if the pairwise products of a, b, c, d are less than 1 in absolute value. So by Favard's theorem, there exists a measure $d\alpha(x)$ with respect to which $p_n(x; a, b, c, d|q)$ are orthogonal. In order to determine this measure let us assume that $\max(|a|, |b|, |c|, |d|, |q|) < 1$. Then, by (2.5.1),

$$p_n(\cos\theta; a, b, c, d|q) \\ = \frac{\left(ab, ac, bc, de^{-i\theta}, ; q\right)_n}{(abce^{i\theta}; q)_n} e^{in\theta} \\ \times {}_8W_7\left(abce^{i\theta}q^{-1}; ae^{i\theta}, be^{i\theta}, ce^{i\theta}, abcdq^{n-1}, q^{-n}; q, qd^{-1}e^{-i\theta}\right), \quad (7.5.7)$$

and, by (2.11.1),

$$ {}_8W_7\left(abce^{i\theta}q^{-1}; ae^{i\theta}, be^{i\theta}, ce^{i\theta}, abcdq^{n-1}, q^{-n}; qd^{-1}e^{-i\theta}\right) \\ = \frac{\left(abce^{i\theta}, bcde^{i\theta}q^n, be^{i\theta}q^{n+1}, ce^{i\theta}q^{n+1}, ae^{-i\theta}, be^{-i\theta}, ce^{-i\theta}, de^{-i\theta}q^n; q\right)_\infty}{(ab, ac, bc, q^{n+1}, bdq^n, cdq^n, bcq^{n+1}e^{2i\theta}, e^{-2i\theta}; q)_\infty} \\ \times {}_8W_7\left(bcq^n e^{2i\theta}; bcq^n, be^{i\theta}, ce^{i\theta}, qa^{-1}e^{i\theta}, qd^{-1}e^{i\theta}; q, adq^n\right) \\ + \frac{\left(abce^{i\theta}, abce^{-i\theta}q^n, bcde^{-i\theta}q^n, be^{-i\theta}q^{n+1}, ce^{-i\theta}q^{n+1}; q\right)_\infty}{(q^{n+1}, bdq^n, cdq^n, abce^{i\theta}q^n, bcq^{n+1}e^{-2i\theta}; q)_\infty} \\ \times \frac{\left(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta}; q\right)_\infty}{(ab, ac, ad, e^{2i\theta}; q)_\infty} e^{-2in\theta} \\ \times {}_8W_7\left(bcq^n e^{-2i\theta}; bcq^n, be^{-i\theta}, ce^{-i\theta}, qa^{-1}e^{-i\theta}, qd^{-1}e^{-i\theta}; q, adq^n\right), \quad (7.5.8)$$

where $0 < \theta < \pi$. Hence

$$p_n(\cos\theta; a, b, c, d|q) \\ = (bc, bd, cd; q)_n \left\{Q_n\left(e^{i\theta}; a, b, c, d|q\right) + Q_n\left(e^{-i\theta}; a, b, c, d|q\right)\right\}, \quad (7.5.9)$$

where

$$Q_n(z; a, b, c, d|q) \\ = \frac{\left(abczq^n, bcdzq^n, bzq^{n+1}, czq^{n+1}, a/z, b/z, c/z, d/z; q\right)_\infty}{(bc, bd, cd, abq^n, acq^n, q^{n+1}, bcz^2q^{n+1}, z^{-2}; q)_\infty} z^n \\ \times {}_8W_7\left(bcq^n z^2; bcq^n, bz, cz, qa^{-1}z, qd^{-1}z; q, adq^n\right). \quad (7.5.10)$$

It is clear from (7.5.10) that

$$Q_n(z; a, b, c, d|q) \sim z^n B(z^{-1})/(bc, bd, cd; q)_\infty, \quad (7.5.11)$$

where

$$B(z) = (az, bz, cz, dz; q)_\infty/(z^2; q)_\infty \quad (7.5.12)$$

as $n \to \infty$, uniformly for z, a, b, c, d in compact sets avoiding the poles $z^2 = q^{-k}$, $k = 0, 1, \ldots$. Using (7.5.9) we find that

$$\begin{aligned} &p_n(\cos\theta; a, b, c, d|q) \\ &\sim e^{in\theta} B(e^{-i\theta}) + e^{-in\theta} B(e^{i\theta}) \\ &= 2|B(e^{i\theta})| \cos(n\theta - \beta), \end{aligned} \tag{7.5.13}$$

where $\beta = \arg B(e^{i\theta})$ and $0 < \theta < \pi$ (see Rahman [1986c]). Then

$$\left|B(e^{i\theta})\right|^2 = [\sin\theta \; w(\cos\theta; a, b, c, d|q)]^{-1}, \tag{7.5.14}$$

where, in order to be consistent with the $p_n(x; a, b, c, d|q)$ notation, we have used $w(x; a, b, c, d|q)$ to denote the weight function $w(x; a, b, c, d)$ defined in (6.3.1). It follows from Theorem 40 in Nevai [1979] that the polynomials $p_n(x; a, b, c, d|q)$ are orthogonal on $[-1, 1]$ with respect to the measure $w(x; a, b, c, d|q)dx$ when $\max(|a|, |b|, |c|, |d|, |q|) < 1$.

We shall now give a direct proof of the orthogonality relation

$$\int_{-1}^{1} p_m(x) p_n(x) w(x) \, dx = \frac{\delta_{m,n}}{h_n}, \tag{7.5.15}$$

where $w(x) \equiv w(x; a, b, c, d|q)$ and

$$\begin{aligned} h_n &\equiv h_n(a, b, c, d|q) \\ &= \kappa^{-1}(a, b, c, d|q) \frac{\left(abcdq^{-1}; q\right)_n \left(1 - abcdq^{2n-1}\right)}{(q; q)_n \left(1 - abcdq^{-1}\right) (ab, ac, ad, bc, bd, cd; q)_n}, \end{aligned} \tag{7.5.16}$$

with

$$\begin{aligned} \kappa(a, b, c, d|q) &= \int_{-1}^{1} w(x; a, b, c, d|q) \, dx \\ &= \frac{2\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \end{aligned} \tag{7.5.17}$$

by (6.1.1). First observe that, by (7.5.17),

$$\begin{aligned} &\int_{-1}^{1} \left(ae^{i\theta}, ae^{-i\theta}; q\right)_j \left(be^{i\theta}, be^{-i\theta}; q\right)_k w(x; a, b, c, d|q) \, dx \\ &= \int_{-1}^{1} w(x; aq^j, bq^k, c, d|q) \, dx \\ &= \kappa(aq^j, bq^k, c, d|q). \end{aligned} \tag{7.5.18}$$

By using (7.5.2) and the fact that $p_n(x; a, b, c, d|q) = p_n(x; b, a, c, d|q)$ we find that the left side of (7.5.15) equals

$$\begin{aligned} &\kappa(a, b, c, d|q)(ab, ac, ad; q)_m (ba, bc, bd; q)_n a^{-m} b^{-n} \\ &\times \sum_{j=0}^{m} \frac{\left(q^{-m}, abcdq^{m-1}; q\right)_j}{\left(q, abcd; q\right)_j} q^j \; {}_3\phi_2 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, abq^j \\ abcdq^j, ab \end{matrix} ; q, q \right]. \end{aligned} \tag{7.5.19}$$

Assuming that $0 \le n \le m$ and using the q-Saalschütz formula to sum the ${}_3\phi_2$ series, the sum over j in (7.5.19) gives

$$\begin{aligned}
&\frac{\left(cd, abcdq^{m-1}, q^{-m}; q\right)_n}{\left(q^{1-n}/ab; q\right)_n (abcd; q)_{2n}} q^n \ {}_2\phi_1\left(q^{n-m}, abcdq^{n+m-1}; abcdq^{2n}; q, q\right) \\
&= \frac{\left(cd, abcdq^{m-1}, q^{-m}; q\right)_n}{\left(q^{1-n}/ab; q\right)_n (abcd; q)_{2n}} \frac{\left(q^{1+n-m}; q\right)_{m-n}}{\left(abcdq^{2n}; q\right)_{m-n}} \\
&= \frac{(q, cd; q)_n \left(1 - abcdq^{-1}\right)}{\left(abcdq^{-1}, ab; q\right)_n \left(1 - abcdq^{2n-1}\right)} (ab)^n \delta_{m,n}. \qquad (7.5.20)
\end{aligned}$$

Combining (7.5.19) and (7.5.20) completes the proof of (7.5.15).

Askey and Wilson proved a more general orthogonality relation by using contour integration. They showed that if $|q| < 1$ and the pairwise products and quotients of a, b, c, d are not of the form q^{-k}, $k = 0, 1, \ldots$, then

$$\begin{aligned}
&\int_{-1}^{1} p_m(x) p_n(x) w(x)\, dx + 2\pi \sum_k p_m(x_k) p_n(x_k) w_k \\
&= \frac{\delta_{m,n}}{h_n(a, b, c, d|q)}, \qquad (7.5.21)
\end{aligned}$$

where x_k are the points $\frac{1}{2}\left(fq^k + f^{-1}q^{-k}\right)$ with f equal to any of the parameters a, b, c, d whose absolute value is greater than 1, the sum is over the k with $|fq^k| > 1$, and

$$\begin{aligned}
w_k &\equiv w_k(a, b, c, d|q) \\
&= \frac{\left(a^{-2}; q\right)_\infty}{(q, ab, ac, ad, b/a, c/a, d/a; q)_\infty} \\
&\quad \times \frac{(a^2; q)_k (1 - a^2 q^{2k})(ab, ac, ad; q)_k}{(q; q)_k (1 - a^2)(aq/b, aq/c, aq/d; q)_k} \left(\frac{q}{abcd}\right)^k \qquad (7.5.22)
\end{aligned}$$

when $x_k = \frac{1}{2}\left(aq^k + a^{-1}q^{-k}\right)$. For a proof and complete discussion, see Askey and Wilson [1985]. Also, see Ex. 7.31.

In order to get a q-analogue of Jacobi polynomials, Askey and Wilson set

$$a = q^{(2\alpha+1)/4}, b = q^{(2\alpha+3)/4}, c = -q^{(2\beta+1)/4}, d = -q^{(2\beta+3)/4} \qquad (7.5.23)$$

and defined the continuous q-Jacobi polynomials by

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x|q) &= \frac{\left(q^{\alpha+1}; q\right)_n}{(q; q)_n} \\
&\times {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{(2\alpha+1)/4}e^{i\theta}, q^{(2\alpha+1)/4}e^{-i\theta} \\ q^{\alpha+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2} \end{matrix}; q, q\right]. \qquad (7.5.24)
\end{aligned}$$

On the other hand, Rahman [1981] found it convenient to work with an apparently different q-analogue, namely,

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x; q) &= \frac{\left(q^{\alpha+1}, -q^{\beta+1}; q\right)_n}{(q, -q; q)_n} \\
&\times {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}}e^{i\theta}, q^{\frac{1}{2}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\beta+1}, -q \end{matrix}; q, q\right]. \qquad (7.5.25)
\end{aligned}$$

However, as Askey and Wilson pointed out, these two q-analogues are not really different since, by the quadratic transformation (3.10.13),

$$P_n^{(\alpha,\beta)}(x|q^2) = \frac{(-q;q)_n}{(-q^{\alpha+\beta+1};q)_n} q^{\alpha n} P_n^{(\alpha,\beta)}(x;q). \tag{7.5.26}$$

Note that

$$\lim_{q\to 1} P_n^{(\alpha,\beta)}(x|q) = \lim_{q\to 1} P_n^{(\alpha,\beta)}(x;q) = P_n^{(\alpha,\beta)}(x). \tag{7.5.27}$$

The orthogonality relations for these q-analogues are

$$\int_0^\pi P_m^{(\alpha,\beta)}(\cos\theta|q) P_n^{(\alpha,\beta)}(\cos\theta|q) w(\theta;q^{\frac{1}{2}})\, d\theta = \frac{\delta_{m,n}}{a_n(\alpha,\beta|q)}, \tag{7.5.28}$$

and

$$\int_0^\pi P_m^{(\alpha,\beta)}(\cos\theta;q) P_n^{(\alpha,\beta)}(\cos\theta;q) w(\theta;q)\, d\theta = \frac{\delta_{m,n}}{b_n(\alpha,\beta;q)}, \tag{7.5.29}$$

where $0 < q < 1$, $\alpha \geq -\frac{1}{2}, \beta \geq -\frac{1}{2}$,

$$w(\theta;q) = \left| \frac{\left(e^{i\theta}, -e^{i\theta};q\right)_\infty}{(q^{\alpha+\frac{1}{2}}e^{i\theta}, -q^{\beta+\frac{1}{2}}e^{i\theta};q)_\infty} \right|^2, \tag{7.5.30}$$

$$\begin{aligned} a_n(\alpha,\beta|q) = {} & \frac{\left(q, q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2};q\right)_\infty}{2\pi\left(q^{(\alpha+\beta+2)/2}, q^{(\alpha+\beta+3)/2};q\right)_\infty} \\ & \times \frac{\left(1-q^{2n+\alpha+\beta+1}\right)\left(q, q^{\alpha+\beta+1}, -q^{(\alpha+\beta+1)/2};q\right)_n}{\left(1-q^{\alpha+\beta+1}\right)\left(q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+3)/2};q\right)_n} q^{-n(2\alpha+1)/2} \end{aligned} \tag{7.5.31}$$

and

$$\begin{aligned} b_n(\alpha,\beta;q) = {} & \frac{\left(q, q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+1}, -q^{\beta+1}, -q^{\alpha+\beta+1}, -q;q\right)_\infty}{2\pi\,(q^{\alpha+\beta+2};q)_\infty} \\ & \times \frac{\left(1-q^{2n+\alpha+\beta+1}\right)\left(q^{\alpha+\beta+1}, q, -q, -q, -q;q\right)_n q^{-n}}{\left(1-q^{\alpha+\beta+1}\right)\left(q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+1}, -q^{\beta+1}, -q^{\alpha+\beta+1};q\right)_n}. \end{aligned} \tag{7.5.32}$$

From (7.4.14), (7.5.24) and (7.5.25) it is obvious that

$$C_n(\cos\theta;q^\lambda|q) = \frac{\left(q^{2\lambda};q\right)_n}{\left(q^{\lambda+\frac{1}{2}};q\right)_n} q^{-n\lambda/2} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(\cos\theta|q) \tag{7.5.33}$$

$$= \frac{\left(q^\lambda, -q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n}{\left(q^{(2\lambda+1)/4}, -q^{(2\lambda+1)/4};q^{\frac{1}{2}}\right)_n} q^{-n/4} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(\cos\theta;q^{\frac{1}{2}}). \tag{7.5.34}$$

It can also be shown that

$$C_{2n}(x;q^\lambda|q) = \frac{(q^\lambda,-q;q)_n}{\left(q^{\frac{1}{2}},-q^{\frac{1}{2}};q\right)_n} q^{-n/2} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1;q), \tag{7.5.35}$$

$$C_{2n+1}(x;q^\lambda|q) = x\frac{(q^\lambda,-1;q)_{n+1}}{\left(q^{\frac{1}{2}},-q^{\frac{1}{2}};q\right)_{n+1}} q^{-n/2} P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1;q), \tag{7.5.36}$$

which are q-analogues of the quadratic transformations

$$C_{2n}^\lambda(x) = \frac{(\lambda)_n}{(\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1) \tag{7.5.37}$$

and

$$C_{2n+1}^\lambda(x) = x\frac{(\lambda)_{n+1}}{(\frac{1}{2})_{n+1}} P_n^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1), \tag{7.5.38}$$

respectively. To prove (7.5.35), observe that from (7.4.2)

$$\begin{aligned} &C_{2n}(\cos\theta;q^\lambda|q) \\ &= \frac{(q^\lambda;q)_{2n}}{(q;q)_{2n}} e^{2in\theta}\ {}_2\phi_1\left(q^{-2n},q^\lambda;q^{1-\lambda-2n};q,q^{1-\lambda}e^{-2i\theta}\right), \end{aligned}$$

and hence, by the Sears-Carlitz formula (Ex. 2.26),

$$\begin{aligned} &C_{2n}(\cos\theta;q^\lambda|q) \\ &= \frac{(q^\lambda;q)_{2n}}{(q;q)_{2n}} \left(q^{\frac{1}{2}}e^{-2i\theta},q^{\frac{1}{2}-n}e^{-2i\theta};q\right)_n e^{2in\theta} \\ &\times {}_4\phi_3\left[\begin{matrix} q^{-n},-q^{-n},-q^{\frac{1}{2}-n},q^{\frac{1}{2}-\lambda-n} \\ q^{1-\lambda-2n},q^{\frac{1}{2}-n}e^{2i\theta},q^{\frac{1}{2}-n}e^{-2i\theta} \end{matrix};q,q\right]. \end{aligned} \tag{7.5.39}$$

Reversing the ${}_4\phi_3$ series, we obtain

$$\begin{aligned} &C_{2n}(\cos\theta;q^\lambda|q) \\ &= \frac{\left(q^\lambda,q^{\lambda+\frac{1}{2}};q\right)_n}{\left(q,q^{\frac{1}{2}};q\right)_n} q^{-n/2}\ {}_4\phi_3\left[\begin{matrix} q^{-n},q^{n+\lambda},q^{\frac{1}{2}}e^{2i\theta},q^{\frac{1}{2}}e^{-2i\theta} \\ q^{\lambda+\frac{1}{2}},-q^{\frac{1}{2}},-q \end{matrix};q,q\right]. \end{aligned} \tag{7.5.40}$$

This, together with (7.5.25), yields (7.5.35). The proof of (7.5.36) is left as an exercise.

Following Askey and Wilson [1985] we shall obtain what are now called the *continuous q-Hahn polynomials*. First note that the orthogonality relation (7.5.15) can be written in the form

$$\begin{aligned} &\int_{-\pi}^{\pi} p_m(\cos\theta;a,b,c,d|q)p_n(\cos\theta;a,b,c,d|q)w(\cos\theta;a,b,c,d|q)\sin\theta\ d\theta \\ &= \frac{2\delta_{m,n}}{h_n(a,b,c,d|q)}. \end{aligned} \tag{7.5.41}$$

Replace θ by $\theta+\phi$, a by $ae^{i\phi}, b$ by $ae^{-i\phi}$ and then set $c = be^{i\phi}, d = be^{-i\phi}$ to find by periodicity that if $-1 < a, b < 1$ or if $b = \bar{a}$ and $|a| < 1$, then

$$\int_{-\pi}^{\pi} p_m(\cos(\theta+\phi); a, b|q) p_n(\cos(\theta+\phi); a, b|q) W(\theta)\, d\theta = \frac{\delta_{m,n}}{\rho_n(a, b|q)}, \tag{7.5.42}$$

where

$$p_n(\cos(\theta+\phi); a, b|q) = (a^2, ab, abe^{2i\phi}; q)_n (ae^{i\phi})^{-n} \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, a^2b^2q^{n-1}, ae^{2i\phi+i\theta}, ae^{-i\theta} \\ a^2, ab, abe^{2i\phi} \end{matrix} ; q, q \right], \tag{7.5.43}$$

$$W(\theta) = \left| \frac{\left(e^{2i(\theta+\phi)}; q\right)_\infty}{\left(ae^{i\theta}, be^{i\theta}, ae^{i(\theta+2\phi)}, be^{i(\theta+2\phi)}; q\right)_\infty} \right|^2, \tag{7.5.44}$$

and

$$\rho_n(a, b|q) = \frac{\left(q, a^2, b^2, ab, ab, abe^{2i\phi}, abe^{-2i\phi}; q\right)_\infty}{4\pi(a^2b^2; q)_\infty} \times \frac{\left(1 - a^2b^2q^{2n-1}\right)\left(a^2b^2q^{-1}; q\right)_n}{\left(1 - a^2b^2q^{-1}\right)\left(q, a^2, b^2, ab, ab, abe^{2i\phi}, abe^{-2i\phi}; q\right)_n}. \tag{7.5.45}$$

The recurrence relation for these polynomials is

$$\begin{aligned} &2\cos(\theta+\phi) p_n(\cos(\theta+\phi); a, b|q) \\ &= A_n p_{n+1}(\cos(\theta+\phi); a, b|q) + B_n p_n(\cos(\theta+\phi); a, b|q) \\ &\quad + C_n p_{n-1}(\cos(\theta+\phi); a, b|q) \end{aligned} \tag{7.5.46}$$

for $n = 0, 1, \ldots$, where $p_{-1}(x; a, b|q) = 0$,

$$A_n = \frac{1 - a^2b^2q^{n-1}}{\left(1 - a^2b^2q^{2n-1}\right)\left(1 - a^2b^2q^{2n}\right)}, \tag{7.5.47}$$

$$C_n = \frac{(1-q^n)(1-a^2q^{n-1})(1-b^2q^{n-1})(1-abq^{n-1})(1-abq^{n-1})}{\left(1 - a^2b^2q^{2n-2}\right)\left(1 - a^2b^2q^{2n-1}\right)} \times (1 - abe^{2i\phi}q^{n-1})(1 - abe^{-2i\phi}q^{n-1}) \tag{7.5.48}$$

and

$$\begin{aligned} B_n = {} & ae^{i\phi} + a^{-1}e^{-i\phi} - A_n a^{-1} e^{-i\phi}(1 - a^2q^n)(1 - abq^n)(1 - abe^{2i\phi}q^n) \\ & - C_n ae^{i\phi}/(1 - a^2q^{n-1})(1 - abe^{2i\phi}q^{n-1})(1 - abq^{n-1}). \end{aligned} \tag{7.5.49}$$

If we set $a = q^\alpha, b = q^\beta, \theta = x \log q$ in (7.5.42) and then let $q \to 1$ we obtain

$$\int_{-\infty}^{\infty} p_m(x; \alpha, \beta) p_n(x; \alpha, \beta) |\Gamma(\alpha + ix)\Gamma(\beta + ix)|^2 dx = 0, \quad m \neq n, \tag{7.5.50}$$

where

$$p_n(x;\alpha,\beta) = i^n\ {}_3F_2\left[\begin{matrix} -n, n+2\alpha+2\beta-1, \alpha-ix \\ \alpha+\beta, 2\alpha \end{matrix}; 1\right]. \tag{7.5.51}$$

For further details and an extension to one more parameter, see Askey [1985b], Askey and Wilson [1982, 1985], Atakishiyev and Suslov [1985], and Suslov [1982, 1987].

7.6 Connection coefficients

Suppose $f_n(x)$ and $g_n(x), n = 0, 1, \ldots$, are polynomials of exact degree n in x. Sometimes it is of interest to express one of these sequences as a linear combination of the polynomials in the other sequence, say,

$$g_n(x) = \sum_{k=0}^{n} c_{k,n} f_k(x). \tag{7.6.1}$$

The numbers $c_{k,n}$ are called the *connection coefficients*. If the polynomials $f_n(x)$ happen to be orthogonal on an interval I with respect to a measure $d\alpha(x)$, then $c_{k,n}$ is the k-th Fourier coefficient of $g_n(x)$ with respect to the orthogonal polynomials $f_k(x)$ and hence can be expressed as a multiple of the integral $\int_I f_k(x) g_n(x) d\alpha(x)$.

A particularly interesting problem is to determine the conditions under which the connection coefficients are nonnegative for particular systems of orthogonal polynomials. Formula (7.6.1) is sometimes called a *projection formula* when all of the coefficients are nonnegative. See the applications to positive definite functions, isometric embeddings of metric spaces, and inequalities in Askey [1970, 1975], Askey and Gasper [1971], Gangolli [1967] and Gasper [1975a]. As an illustration we shall consider the coefficients $c_{k,n}$ in the relation

$$p_n(x;\alpha,\beta,\gamma,d|q) = \sum_{k=0}^{n} c_{k,n} p_k(x;a,b,c,d|q). \tag{7.6.2}$$

Askey and Wilson [1985] showed that

$$\begin{aligned} c_{k,n} = {} & \frac{(\alpha d, \beta d, \gamma d, q; q)_n \left(\alpha\beta\gamma d q^{n-1}; q\right)_k}{(\alpha d, \beta d, \gamma d, q, abcdq^{k-1}; q)_k\, (q;q)_{n-k}} q^{k^2-nk} d^{k-n} \\ & \times {}_5\phi_4\left[\begin{matrix} q^{k-n}, \alpha\beta\gamma d q^{n+k-1}, adq^k, bdq^k, cdq^k \\ abcdq^{2k}, \alpha dq^k, \beta dq^k, \gamma dq^k \end{matrix}; q, q\right]. \end{aligned} \tag{7.6.3}$$

To prove (7.6.3), temporarily assume that max $(|a|, |b|, |c|, |d|, |q|) < 1$, and observe that, by orthogonality,

$$b_{k,j} = \int_{-1}^{1} w(x;a,b,c,d|q) p_k(x;a,b,c,d|q) (de^{i\theta}, de^{-i\theta}; q)_j\ dx \tag{7.6.4}$$

vanishes if $j < k$, and that

$$b_{k,j} = \kappa(a,b,c,d|q)(ab,ac,ad;q)_k a^{-k}$$

$$\times \frac{(ad,bd,cd;q)_j}{(abcd;q)_j} {}_3\phi_2\left[\begin{matrix} q^{-k}, abcdq^{k-1}, adq^j \\ abcdq^j, ad \end{matrix}; q, q\right]$$
$$= \kappa(a,b,c,d|q)(ab,ac,bc,q^{-j};q)_k(ad,bd,cd;q)_j(dq^j)^k/(abcd;q)_{j+k} \quad (7.6.5)$$

if $j \geq k$. Since

$$p_n(x;\alpha,\beta,\gamma,d|q)$$
$$= (\alpha d,\beta d,\gamma d;q)_n d^{-n} \sum_{j=0}^{n} \frac{\left(q^{-n}, \alpha\beta\gamma dq^{n-1}, de^{i\theta}, de^{-i\theta};q\right)_j}{(q,\alpha d,\beta d,\gamma d;q)_j} q^j, \quad (7.6.6)$$

we find that

$$c_{k,n} = h_k(a,b,c,d|q)\int_{-1}^{1} w(x;a,b,c,d|q)p_k(x;a,b,c,d|q)p_n(x;\alpha,\beta,\gamma,d|q)\,dx$$
$$= h_k(a,b,c,d|q)(\alpha d,\beta d,\gamma d;q)_n d^{-n}\sum_{j=0}^{n}\frac{\left(q^{-n},\alpha\beta\gamma dq^{n-1};q\right)_j}{(q,\alpha d,\beta d,\gamma d;q)_j} q^j b_{k,j}$$
$$(7.6.7)$$

and hence (7.6.3) follows from (7.5.16), (7.5.17) and (7.6.5).

The ${}_5\phi_4$ series in (7.6.3) is balanced but, in general, cannot be transformed in a simple way, so one cannot hope to say much about the nonnegativity of $c_{k,n}$ unless the parameters are related in some way. One of the simplest cases is when the ${}_5\phi_4$ series reduces to a ${}_3\phi_2$, which can be summed by the q-Saalschütz formula. Thus for $\beta = b$ and $\gamma = c$ we get

$$p_n(x;\alpha,b,c,d|q) = \sum_{k=0}^{n} c_{k,n} p_k(x;a,b,c,d|q) \quad (7.6.8)$$

with

$$c_{k,n} = \frac{(\alpha/a;q)_{n-k}\,(\alpha bcdq^{n-1};q)_k(q,bc,bd,cd;q)_n a^{n-k}}{(q,bc,bd,cd;q)_k(abcdq^{k-1};q)_k(q,abcdq^{2k};q)_{n-k}}. \quad (7.6.9)$$

It is clear that $c_{k,n} > 0$ when $0 < \alpha < a < 1$, $0 < q < 1$ and $\max\,(bc,bd,cd,abcd) < 1$.

Another simple case is when the ${}_5\phi_4$ series in (7.6.3) reduces to a summable ${}_4\phi_3$ series. For example, this happens when we set $d = q^{\frac{1}{2}}, c = -q^{\frac{1}{2}} = \gamma, b = -a$ and $\beta = -\alpha$. The ${}_5\phi_4$ series then reduces to

$${}_4\phi_3\left[\begin{matrix} q^{k-n}, \alpha^2 q^{n+k}, aq^{k+\frac{1}{2}}, -aq^{k+\frac{1}{2}} \\ \alpha q^{k+\frac{1}{2}}, -\alpha q^{k+\frac{1}{2}}, a^2q^{2k+1} \end{matrix}; q, q\right] \quad (7.6.10)$$

which equals

$$\frac{\left(\alpha^2q^{n+k+1}, q^{k-n+1}, a^2q^{k+n+2}, a^2q^{k-n+2}/\alpha^2;q^2\right)_\infty}{(q,\alpha^2q^{2k+1},a^2q^{2k+2},a^2q^2/\alpha^2;q^2)_\infty}\left(\alpha^2q^{n+k}\right)^{(n-k)/2} \quad (7.6.11)$$

by Andrews' formula (Ex. 2.8). Clearly, this vanishes when $n-k$ is an odd integer and equals

$$\frac{\left(q\alpha^2,q^2a^2;q^2\right)_{n-2j}\left(q,\alpha^2/a^2;q^2\right)_j}{(q\alpha^2,q^2a^2;q^2)_{n-j}}\left(a^2q^{2n+1-4j}\right)^j \quad (7.6.12)$$

when $n-k=2j, j=0,1,\ldots,[\frac{n}{2}]$. Since by (3.10.12) and (7.4.14)

$$p_n(x;a,-a,-q^{\frac{1}{2}},q^{\frac{1}{2}}|q) = \frac{(q^2,qa^2;q^2)_n}{(a^2;q)_n} C_n(x;a^2|q^2), \tag{7.6.13}$$

we obtain, after some simplification, Rogers' formula

$$C_n(x;\gamma|q) = \sum_{k=0}^{[n/2]} \frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}(1-\beta q^{n-2k})}{(q;q)_k(\beta q;q)_{n-k}(1-\beta)} C_{n-2k}(x;\beta|q) \tag{7.6.14}$$

after replacing α^2, a^2, q^2 by γ, β and q, respectively.

It is left as an exercise (Ex. 7.15) to show that formulas (7.6.2) and (7.6.14) are special cases of the q-analogue of the Fields-Wimp formula given in (3.7.9).

For other applications of the connection coefficient formula (7.6.2), see Askey and Wilson [1985].

7.7 A difference equation and a Rodrigues-type formula for the Askey-Wilson polynomials

The polynomials $p_n(x;a,b,c,d|q)$, unlike the Jacobi polynomials, do not satisfy a differential equation; but, as Askey and Wilson [1985] showed, they satisfy a second-order difference equation. Define

$$E_q^{\pm} f(e^{i\theta}) = f\left(q^{\pm\frac{1}{2}}e^{i\theta}\right), \tag{7.7.1}$$

$$\delta_q f\left(e^{i\theta}\right) = \left(E_q^+ - E_q^-\right) f\left(e^{i\theta}\right) \tag{7.7.2}$$

$$\text{and} \quad D_q f(x) = \frac{\delta_q f(x)}{\delta_q x}, \quad x=\cos\theta. \tag{7.7.3}$$

Clearly,

$$\delta_q(\cos\theta) = \frac{1}{2}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\left(e^{i\theta}-e^{-i\theta}\right) = -iq^{-\frac{1}{2}}(1-q)\sin\theta, \tag{7.7.4}$$

$$\text{and } \delta_q\left(ae^{i\theta}, ae^{-i\theta};q\right)_n = 2aiq^{-\frac{1}{2}}(1-q^n)\sin\theta\ (aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta};q)_{n-1}, \tag{7.7.5}$$

so that

$$D_q\left(ae^{i\theta}, ae^{-i\theta};q\right)_n = -\frac{2a(1-q^n)}{(1-q)}(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta};q)_{n-1}, \tag{7.7.6}$$

which implies that the divided difference operator D_q plays the same role for $\left(ae^{i\theta}, ae^{-i\theta};q\right)_n$ as d/dx does for x^n. When $q\to 1$, formula (7.7.6) becomes

$$\frac{d}{dx}\left(1-2ax+a^2\right)^n = -2an(1-2ax+a^2)^{n-1}.$$

Generally, for a differentiable function $f(x)$ we have

$$\lim_{q\to 1} D_q f(x) = \frac{d}{dx} f(x).$$

Following Askey and Wilson [1985], we shall now use the operator D_q and the recurrence relation (7.5.3) to derive a Rodrigues-type formula for $p_n(x;a,b,c,d|q)$. First note that by (7.5.2) and (7.7.5)

$$\begin{aligned}&\delta_q p_n(x;a,b,c,d|q)\\&= -2iq^{\frac{1}{2}-n}\sin\theta\ (1-q^n)(1-q^{n-1}abcd)p_{n-1}(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q).\end{aligned} \tag{7.7.7}$$

If we define

$$A(\theta) = \frac{\left(1-ae^{i\theta}\right)\left(1-be^{i\theta}\right)\left(1-ce^{i\theta}\right)\left(1-de^{i\theta}\right)}{\left(1-e^{2i\theta}\right)\left(1-qe^{2i\theta}\right)} \tag{7.7.8}$$

and

$$r_n(e^{i\theta}) = {}_4\phi_3\left[\begin{matrix}q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}\\ ab, ac, ad\end{matrix}; q, q\right], \tag{7.7.9}$$

then the recurrence relation (7.5.3) can be written as

$$\begin{aligned}&q^{-n}(1-q^n)(1-abcdq^{n-1})r_n(e^{i\theta})\\&= A(-\theta)\left[r_n(q^{-1}e^{i\theta})-r_n(e^{i\theta})\right] + A(\theta)\left[r_n(qe^{i\theta})-r_n(e^{i\theta})\right].\end{aligned} \tag{7.7.10}$$

Also, setting

$$V(e^{i\theta}) = \frac{\left(e^{2i\theta};q\right)_\infty}{\left(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q\right)_\infty} \tag{7.7.11}$$

$$\text{and}\quad W(e^{i\theta}) \equiv W(e^{i\theta};a,b,c,d|q) = V(e^{i\theta})V(e^{-i\theta}) \tag{7.7.12}$$

we find that (7.7.10) can be expressed in the form

$$\begin{aligned}&q^{-n}(1-q^n)(1-abcdq^{n-1})V(e^{i\theta})V(e^{-i\theta})p_n(x)\\&= \delta_q\left[\left\{E_q^+V(e^{i\theta})\right\}\left\{E_q^-V(e^{-i\theta})\right\}\left\{\delta_q p_n(x)\right\}\right].\end{aligned} \tag{7.7.13}$$

Combining (7.7.7) and (7.7.13) we have

$$\begin{aligned}&-q^{-n/2}V(e^{i\theta})V(e^{-i\theta})p_n(x;a,b,c,d|q)\\&= \delta_q\left[\left\{E_q^+V(e^{i\theta})\right\}\left\{E_q^-V(e^{-i\theta})\right\}\left(e^{i\theta}-e^{-i\theta}\right)\right.\\&\qquad\left.\times\ p_{n-1}\left(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q\right)\right].\end{aligned} \tag{7.7.14}$$

Since

$$\begin{aligned}&\left(e^{i\theta}-e^{-i\theta}\right)E_q^+V(e^{i\theta})E_q^-V(e^{-i\theta})\\&= \frac{\left(e^{i\theta}-e^{-i\theta}\right)\left(qe^{2i\theta},qe^{-2i\theta};q\right)_\infty}{h(\cos\theta;aq^{\frac{1}{2}})h(\cos\theta;bq^{\frac{1}{2}})h(\cos\theta;cq^{\frac{1}{2}})h(\cos\theta;dq^{\frac{1}{2}})}\\&= -\frac{W\left(e^{i\theta};aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q\right)}{e^{i\theta}-e^{-i\theta}},\end{aligned}$$

(7.7.14) can be written in a slightly better form

$$
\begin{aligned}
&q^{-n/2}W(e^{i\theta};a,b,c,d|q)p_n(x;a,b,c,d|q)\\
&=\delta_q\left[(e^{i\theta}-e^{-i\theta})^{-1}W(e^{i\theta};aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q)\right.\\
&\qquad\left.\times\, p_{n-1}(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q)\right].
\end{aligned}
\tag{7.7.15}
$$

Observing that, by (6.3.1),

$$
\sqrt{1-x^2}w(x;a,b,c,d|q)=W(e^{i\theta};a,b,c,d|q)
$$

we find by iterating (7.7.15) that

$$
\begin{aligned}
&w(x;a,b,c,d|q)p_n(x;a,b,c,d|q)\\
&=(-1)^k\left(\frac{1-q}{2}\right)^k q^{nk/2-k(k+1)/4}\\
&\quad\times D_q^k\left[w(x;aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}|q)p_{n-k}(x;aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}|q)\right]\\
&=(-1)^n\left(\frac{1-q}{2}\right)^n q^{(n^2-n)/4}D_q^n\left[w(x;aq^{n/2},bq^{n/2},cq^{n/2},dq^{n/2}|q)\right].
\end{aligned}
\tag{7.7.16}
$$

This gives a Rodrigues-type formula for the Askey-Wilson polynomials.

By combining (7.7.7) and (7.7.15) it can be easily seen that the polynomials $p_n(x)=p_n(x;a,b,c,d|q)$ satisfy the second-order difference equation

$$
\begin{aligned}
&D_q\left[w(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}|q)D_qp_n(x)\right]\\
&+\lambda_n w(x;a,b,c,d|q)p_n(x)=0,
\end{aligned}
\tag{7.7.17}
$$

where

$$
\lambda_n=-4q(1-q^{-n})(1-abcdq^{n-1})(1-q)^{-2}. \tag{7.7.18}
$$

Exercises

7.1 If $\{p_n(x)\}$ is an orthogonal system of polynomials on $(-\infty,\infty)$ with respect to a positive measure $d\alpha(x)$ that has infinitely many points of support, prove that they satisfy a three-term recurrence relation of the form

$$
xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x),\quad n\ge 0,
$$

with $p_{-1}(x)=0$, $p_0(x)=1$, where A_n,B_n,C_n are real and $A_nC_{n+1}>0$ for $n\ge 0$.

7.2 Let $p_0(x),p_1(x),\ldots,p_N(x)$ be a system of polynomials that satisfies a three-term recurrence relation

$$
xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x),
$$

$n = 0, 1, \ldots, N$, where $p_{-1}(x) = 0, p_0(x) = 1$. Prove the Christoffel-Darboux formula

$$(x-y)\sum_{j=0}^{n} p_j(x)p_j(y)v_j = A_n v_n \left[p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\right],$$

$0 \le n \le N$, where $v_0 = 1$ and $v_n C_n = v_{n-1}A_{n-1}, 1 \le n \le N$. Deduce that

$$\sum_{j=0}^{n} p_j^2(x)v_j = A_n v_n \left[p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right]$$

and hence

$$\sum_{j=0}^{N} p_j^2(x_k)v_j = A_N v_N p_N(x_k)p'_{N+1}(x_k)$$

if x_k is a zero of $p_{N+1}(x)$.

7.3 Show that when $n = N$, the recurrence relation (7.2.1) reduces to

$$\left(1-q^{-j}\right)\left(1-cdq^{j+1}\right)p_N(x_j) = C_N\left[p_N(x_j) - p_{N-1}(x_j)\right],$$

where $p_n(x_j)$ is given by (7.2.11), x_j by (7.2.9), and A_n and C_n by (7.2.5) and (7.2.6). Hence show that (7.2.1) holds with $x = x_j, j = 0, 1, \ldots, N$. (Askey and Wilson [1979])

7.4 If $p_n(x_j)$, v_n and w_j are defined by (7.2.11), (7.2.3) and (7.2.15), respectively, prove directly (i.e. without the use of Favard's theorem) that

$$\sum_{j=0}^{N} p_m(x_j)p_n(x_j)w_j = v_n^{-1}\sum_{j=0}^{N} w_j \; \delta_{m,n}$$

and

$$\sum_{n=0}^{N} p_n(x_j)p_n(x_k)v_n = w_j^{-1}\sum_{n=0}^{N} w_n \; \delta_{j,k}.$$

[Hint: First transform one of the polynomials, say $p_n(x)$, to be a multiple of the ${}_4\phi_3$ series on the left side of (7.2.14)].

7.5 Let one of a, b, c, d be a nonnegative integer power of q^{-1} and let

$$\phi(a,b) = {}_4\phi_3\left[\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q, q\right],$$

where $efg = abcdq$. Prove the Askey and Wilson [1979] contiguous relation

$$A\phi(aq^{-1}, bq) + B\phi(a,b) + C\phi(aq, bq^{-1}) = 0,$$

where

$$\begin{aligned} A &= b(1-b)(aq-b)(a-e)(a-f)(a-g), \\ B &= ab(a-bq)(b-a)(aq-b)(1-c)(1-d) \\ &\quad - b(1-b)(aq-b)(a-e)(a-f)(a-g) \\ &\quad + a(1-a)(a-bq)(e-b)(f-b)(g-b), \\ C &= -a(1-a)(a-bq)(e-b)(f-b)(g-b), \end{aligned}$$

and derive the following limit case which was found independently by Lassalle [1999]:

$$\left((c-b)+\frac{ab(1-c)}{y}-\frac{ac(1-b)}{x}\right) {}_3\phi_2\left(\begin{matrix} a,b,c \\ x,y \end{matrix};q,\frac{xy}{abc}\right)$$
$$=\frac{(1-c)(y-a)(y-b)}{y(1-y)}\ {}_3\phi_2\left(\begin{matrix} a,b,qc \\ x,qy \end{matrix};q,\frac{xy}{abc}\right)$$
$$-\frac{(1-b)(x-a)(x-c)}{x(1-x)}\ {}_3\phi_2\left(\begin{matrix} a,qb,c \\ qx,y \end{matrix};q,\frac{xy}{abc}\right).$$

7.6 Determine the conditions that a,b,c,d must satisfy so that $A_nC_{n+1}>0$ for $0\le n\le N-1$, where A_n and C_n are as defined in (7.2.5) and (7.2.6) and one of aq,cq,bdq is q^{-N}, N a nonnegative integer.

7.7 Prove (7.2.22) directly by using the appropriate transformation and summation formulas derived in Chapters 1-3. Verify that

$$\sum_n h_n(q)W_n(x;q)W_n(y;q)=\delta_{x,y}/\rho(x;q)$$

for $x,y=0,1,\ldots,N$, which is the dual of (7.2.18).

7.8 (i) Prove that the *q-Krawtchouk polynomials*

$$K_n(x;a,N;q)=\ {}_3\phi_2\left[\begin{matrix} q^{-n},x,-a^{-1}q^n \\ q^{-N},0 \end{matrix};q,q\right]$$

satisfy the orthogonality relation

$$\sum_{x=0}^{N}K_m(q^{-x};a,N;q)K_n(q^{-x};a,N;q)\frac{(q^{-N};q)_x}{(q;q)_x}(-a)^x$$
$$=\left(-qa^{-1};q\right)_N a^N q^{-\binom{N+1}{2}}\frac{(q;q)_n(1+a^{-1})(-a^{-1}q^{N+1};q)_n}{(-a^{-1};q)_n(1+a^{-1}q^{2n})(q^{-N};q)_n}$$
$$\times(-aq^{N+1})^{-n}q^{n(n+1)}\delta_{m,n},$$

and find their three-term recurrence relation. (Stanton [1980b])

(ii) Let

$$K_n(x;a,N|q)=\ {}_2\phi_1(q^{-n},x;q^{-N};q,aq^{n+1})$$

be another family of *q-Krawtchouk polynomials*. Prove that they satisfy the orthogonality relation

$$\sum_{x=0}^{N}K_m(q^{-x};a,N|q)\ K_n(q^{-x};a,N|q)\frac{(aq;q)_{N-x}(-1)^{N-x}q^{\binom{x}{2}}}{(q;q)_x(q;q)_{N-x}}$$
$$=\frac{(q,aq;q)_n(q;q)_{N-n}}{(q,q;q)_N}(-1)^n a^N q^{\binom{N+n+1}{2}-n(n+1)}\delta_{m,n}.$$

7.9 Prove that

$$xp_n(x)=A_n\left[p_{n+1}(x)-p_n(x)\right]-C_n\left[p_n(x)-p_{n-1}(x)\right],\quad n\ge 0,$$

where $p_n(x) = p_n(x; a, b; q)$ are the little q-Jacobi polynomials and

$$A_n = \frac{-q^n\left(1-aq^{n+1)}\right)\left(1-abq^{n+1}\right)}{\left(1-abq^{2n+1}\right)\left(1-abq^{2n+2}\right)},\ C_n = \frac{(1-q^n)(1-bq^n)(-aq^n)}{(1-abq^{2n})\left(1-abq^{2n+1}\right)}.$$

7.10 Prove that

$$(x-1)P_n(x) = A_n\left[P_{n+1}(x) - P_n(x)\right] + C_n\left[P_n(x) - P_{n-1}(x)\right], \quad n \geq 0,$$

where $P_n(x) = P_n(x; a, b, c; q)$ are the big q-Jacobi polynomials and

$$A_n = \frac{\left(1-aq^{n+1}\right)\left(1-cq^{n+1}\right)\left(1-abq^{n+1}\right)}{\left(1-abq^{2n+1}\right)\left(1-abq^{2n+2}\right)},$$

$$C_n = \frac{(1-q^n)(1-bq^n)\left(1-abc^{-1}q^n\right)}{(1-abq^{2n})\left(1-abq^{2n+1}\right)}acq^{n+1}.$$

7.11 The *affine q-Krawtchouk polynomials* are defined by

$$K_n^{\text{Aff}}(x; a, N; q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, & x, & 0 \\ & aq, & q^{-N} \end{matrix}; q, q\right], \ 0 < aq < 1.$$

Prove that they satisfy the orthogonality relation

$$\sum_{x=0}^{N} K_m^{\text{Aff}}\left(q^{-x}; a, N; q\right) K_n^{\text{Aff}}\left(q^{-x}; a, N; q\right) \frac{(aq, q^{-N}; q)_x}{(q;q)_x}$$
$$\times \left(-\frac{q^{N-1}}{a}\right)^x q^{-\binom{x}{2}} = \frac{\delta_{m,n}}{h_n}, \quad m, n = 0, 1, \ldots, N,$$

where

$$h_n = \frac{\left(aq, q^{-N}; q\right)_n}{(q;q)_n}(-1)^n (aq)^{N-n} q^{Nn-\binom{n}{2}}.$$

(Delsarte [1976a,b], Dunkl [1977])

7.12 The *q-Meixner polynomials* are defined by

$$M_n(x; a, c; q) = {}_2\phi_1\left(q^{-n}, x; aq; q, -q^{n+1}/c\right),$$

with $0 < aq < 1$ and $c > 0$. Show that they satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} M_m(q^{-x}; a, c; q) M_n(q^{-x}; a, c; q) \frac{(aq;q)_x}{(q, -acq; q)_x} c^x q^{\binom{x}{2}} = \frac{\delta_{m,n}}{h_n},$$

where

$$h_n = \frac{(-acq;q)_\infty (aq;q)_n}{(-c;q)_\infty (q, -qc^{-1}; q)_n} q^n.$$

(When $a = q^{-r-1}$, the q-Meixner polynomials reduce to the q-Krawtchouk polynomials considered in Koornwinder [1989].)

7.13 The *q-Charlier polynomials* are defined by

$$c_n(x; a; q) = {}_2\phi_1\left(q^{-n}, x; 0; q, -q^{n+1}/a\right).$$

Show that

$$\sum_{x=0}^{\infty} c_m(q^{-x}; a; q) c_n(q^{-x}; a; q) \frac{a^x}{(q;q)_x} q^{\binom{x}{2}}$$
$$= (-a; q)_\infty (q, -qa^{-1}; q)_n q^{-n} \delta_{m,n}.$$

7.14 Show that, for $x = \cos\theta$,

$$C_n(x; q|q) = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(x), \quad n \geq 0$$

and

$$\lim_{\beta \to 1} \frac{1-q^n}{2(1-\beta)} C_n(x; \beta|q) = \cos n\theta = T_n(x), \quad n \geq 1,$$

where $T_n(x)$ and $U_n(x)$ are the Tchebichef polynomials of the first and second kind, respectively.

7.15 Verify that formulas (7.6.2) and (7.6.14) follow from the q-analogue of the Fields-Wimp formula (3.7.9).

7.16 Let $x = \cos\theta$, $|t| < 1$, and $|q| < 1$. Show that

$$\sum_{n=0}^{\infty} C_n(x; \beta|q) \frac{(\lambda; q)_n}{(\beta^2; q)_n} t^n = \frac{2i \sin\theta}{(1-q) W_\beta(x|q)} \frac{(\beta, \beta; q)_\infty}{(q, \beta^2; q)_\infty}$$
$$\times \int_{e^{i\theta}}^{e^{-i\theta}} \frac{\left(que^{i\theta}, que^{-i\theta}, \lambda tu; q\right)_\infty}{\left(\beta ue^{i\theta}, \beta ue^{-i\theta}, tu; q\right)_\infty} d_q u$$

and deduce that

(i)
$$\sum_{n=0}^{\infty} C_n(x; \beta|q) \frac{t^n}{(\beta^2; q)_n}$$
$$= (te^{-i\theta}; q)_\infty^{-1} \, {}_2\phi_1\left(\beta, \beta e^{-2i\theta}; \beta^2; q, te^{i\theta}\right).$$

(ii)
$$\sum_{n=0}^{\infty} C_n(x; \beta|q) q^{\binom{n}{2}} \frac{(\beta t)^n}{(\beta^2; q)_n}$$
$$= (-te^{-i\theta}; q)_\infty \, {}_2\phi_1\left(\beta, \beta e^{2i\theta}; \beta^2; q, -te^{-i\theta}\right).$$

7.17 Using (1.8.1), or otherwise, prove that

$$C_n(0; \beta|q) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^{n/2} \dfrac{\left(\beta^2; q^2\right)_{n/2}}{\left(q^2; q^2\right)_{n/2}}, & \text{if } n \text{ is even.} \end{cases}$$

7.18 If $-1 < q, \beta < 1$, show that

$$|C_n(x; \beta|q)| \leq C_n(1; \beta|q).$$

7.19 Derive the recurrence relations

$$2xC_n(x; \beta|q) = \frac{1-q^{n+1}}{1-\beta q^n} C_{n+1}(x; \beta|q) + \frac{1-\beta^2 q^{n-1}}{1-\beta q^n} C_{n-1}(x; \beta|q),$$

and

$$C_n(x;\beta|q) = \frac{1-\beta}{1-\beta q^n} C_n(x;\beta q|q) - \frac{\beta(1-\beta)}{1-\beta q^n} C_{n-2}(x;\beta q|q),$$

with $C_{-2}(x;\beta|q) = C_{-1}(x;\beta|q) = 0$, $C_0(x;\beta|q) = 1$, and $n \geq 0$.
(Ismail and Zhang [1994])

7.20 Prove that

$$\int_0^\pi C_n(\cos\theta;\beta|q)\cos(n+2k)\theta\; W_\beta(\cos\theta|q)\; d\theta = \frac{\pi(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty}\beta^k$$
$$\times \frac{(\beta^{-1};q)_k(\beta^2;q)_n(q;q)_{n+k}}{(q;q)_k(q;q)_n(\beta q;q)_{n+k}}\frac{1-q^{n+2k}}{1-q^{n+k}}, \qquad n,k \geq 0,$$

where $W_\beta(x|q)$ is defined in (7.4.6).

7.21 Using (7.4.15) and (7.6.14) prove that

$$\frac{h(x;\gamma)}{h(x;\beta)} C_n(x;\beta|q) = \frac{(\gamma^2,\beta,\beta q;q)_\infty}{(\gamma,\gamma q,\beta^2;q)_\infty}\sum_{k=0}^\infty d_{k,n} C_{n+2k}(x;\gamma|q),$$

where $h(x;a)$ is as defined in (6.1.2) and

$$d_{k,n} = \frac{\beta^k(\gamma/\beta;q)_k(q;q)_{n+2k}(\beta^2;q)_n(\gamma;q)_{n+k}(1-\gamma q^{n+2k})}{(q;q)_k(\gamma^2;q)_{n+2k}(q;q)_n(\beta q;q)_{n+k}(1-\gamma)}.$$

(Askey and Ismail [1983])

7.22 Prove that the continuous q-Hermite polynomials defined in Ex. 1.28 satisfy the orthogonality relation

$$\int_0^\pi H_m(\cos\theta|q)H_n(\cos\theta|q)\left|\left(e^{2i\theta};q\right)_\infty\right|^2 d\theta = \frac{2\pi(q;q)_n}{(q;q)_\infty}\delta_{m,n}.$$

7.23 Setting

$$C_n(x;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n} c_n(x;\beta|q)$$

in Ex. 7.19, show that

$$2x(1-\beta q^n)c_n(x;\beta|q) = (1-\beta^2 q^n)c_{n+1}(x;\beta|q) + (1-q^n)c_{n-1}(x;\beta|q),$$

for $n \geq 0$, with $c_{-1}(x;\beta|q) = 0$, $c_0(x;\beta|q) = 1$. Now set $\beta = s^{\lambda k}$ and $q = s\omega_k$, where $\omega_k = \exp(2\pi i/k)$ is a k-th root of unity, divide the above recurrence relation by $1 - s\omega_k^n$ and take the limit as $s \to 1$ to show that the limiting polynomials, $c_n^\lambda(x;k)$, called the *sieved ultraspherical polynomials of the first kind*, satisfy the recurrence relation

$$2xc_n^\lambda(x;k) = c_{n+1}^\lambda(x;k) + c_{n-1}^\lambda(x;k), \quad n \neq mk,$$
$$2x(m+\lambda)c_{mk}^\lambda(x;k) = (m+2\lambda)c_{mk+1}^\lambda(x;k) + mc_{mk-1}^\lambda(x;k)$$

where $k,m,n = 0,1,\ldots$, $c_0^\lambda(x;k) = 1$ and $c_1^\lambda(x;k) = x$.
(Al-Salam, Allaway and Askey [1984b])

7.24 Rewrite the orthogonality relation (7.4.15) in terms of the sieved orthogonal polynomial $c_n(x;\beta|q)$ defined in Ex. 7.23 and set $\beta = s^{\lambda k}$ and $q = s\omega_k$. By carefully taking the limits of the q-shifted factorials prove that

$$\int_{-1}^{1} c_m^\lambda(x;k)c_n^\lambda(x;k)w(x)\,dx = \frac{\delta_{m,n}}{h_n},$$

where

$$w(x) = 2^{2\lambda(k-1)}(1-x^2)^{-\frac{1}{2}}\prod_{j=0}^{k-1}|x^2-\cos^2(\pi j/k)|^\lambda$$

and

$$h_n = \frac{\Gamma(\lambda+1)}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}\frac{(\lambda+1)_{\lfloor n/k\rfloor}(2\lambda)_{\lceil n/k\rceil}}{(1)_{\lfloor n/k\rfloor}(\lambda)_{\lceil n/k\rceil}},$$

where the *roof* and *floor* functions are defined by

$$\begin{aligned}\lceil a\rceil &= \text{ smallest integer greater than or equal to } a,\\ \lfloor a\rfloor &= \text{ largest integer less than or equal to } a.\end{aligned}$$

(Al-Salam, Allaway and Askey [1984b])

7.25 The *sieved ultraspherical polynomials of the second kind* are defined by

$$B_n^\lambda(x;k) = \lim_{s\to 1} C_n(x;s^{\lambda k+1}\omega_k|s\omega_k),\ \omega_k = \exp(2\pi i/k).$$

Show that $B_n^\lambda(x;k)$ satisfies the recurrence relation

$$\begin{aligned}&2xB_n^\lambda(x;k) = B_{n+1}^\lambda(x;k) + B_{n-1}^\lambda(x;k), \quad n+1\neq mk,\\ &2x(m+\lambda)B_{mk-1}^\lambda(x;k) = mB_{mk}^\lambda(x;k) + (m+2\lambda)B_{mk-2}^\lambda(x;k),\end{aligned}$$

where $B_0^\lambda(x;k) = 1;\ B_1^\lambda(x;k) = 2x$ if $k\geq 2$; $B_1^\lambda(x;1) = 2(\lambda+1)x$. Show also that $B_n^\lambda(x;k)$ satisfies the orthogonality relation

$$\int_{-1}^{1} B_m^\lambda(x;k)B_n^\lambda(x;k)w(x)\,dx = \frac{\delta_{m,n}}{h_n},$$

where

$$w(x) = 2^{2\lambda(k-1)}(1-x^2)^{\frac{1}{2}}\prod_{j=0}^{k-1}\left|x^2-\cos^2(\pi j/k)\right|^\lambda$$

and

$$h_n = \frac{2\Gamma(\lambda+1)}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}\frac{(1)_{\lfloor n/k\rfloor}(\lambda+1)_{\lfloor\frac{n+1}{k}\rfloor}}{(\lambda+1)_{\lfloor n/k\rfloor}(2\lambda+1)_{\lfloor\frac{n+1}{k}\rfloor}}.$$

(Al-Salam, Allaway and Askey [1984b])

7.26 Using (2.5.1) show that

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,bc,q;q)_n}{(abcdq^{-1};q)_n}$$
$$\times \sum_{k=0}^{n} \frac{\left(abcq^{-1}e^{i\theta};q\right)_k \left(1-abce^{i\theta}q^{2k-1}\right)\left(ae^{i\theta},be^{i\theta},ce^{i\theta};q\right)_k}{(q;q)_k(1-abcq^{-1}e^{i\theta})(bc,ac,ab;q)_k}$$
$$\times \frac{\left(abcdq^{-1};q\right)_{n+k}\left(de^{-i\theta};q\right)_{n-k}}{\left(abce^{i\theta};q\right)_{n+k}(q;q)_{n-k}} e^{i(n-2k)\theta}.$$

Deduce that the polynomials

$$p_n(x) = \lim_{q\to 0} p_n(x;a,b,c,d|q)$$

are given by

$$\begin{aligned} p_0(x) &= 1 = U_0(x),\\ p_1(x) &= (1-s_4)U_1(x) + (s_3-s_1)U_0(x),\\ p_2(x) &= U_2(x) - s_1U_1(x) + (s_2-s_4)U_0(x),\\ p_n(x) &= \sum_{k=0}^{4}(-1)^k s_k U_{n-k}(x), \quad n\ge 3, \end{aligned}$$

where

$$s_0 = 1,\ s_1 = a+b+c+d,\ s_2 = ab+ac+ad+bc+bd+cd,$$
$$s_3 = abc+abd+acd+bcd,\ s_4 = abcd,$$

and $U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta$, $U_{-1}(x) = 0$. When $\max(|a|,|b|,|c|,|d|) < 1$ show that these polynomials satisfy the orthogonality relation

$$\frac{2}{\pi}\int_{-1}^{1} \frac{p_m(x)p_n(x)(1-x^2)^{\frac12}\,dx}{(1-2ax+a^2)(1-2bx+b^2)(1-2cx+c^2)(1-2dx+d^2)}$$
$$= \begin{cases} 0, & m\ne n,\\ \dfrac{1-abcd}{(1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)}, & m=n=0,\\ 1-abcd, & m=n=1,\\ 1, & m=n\ge 2. \end{cases}$$

(Askey and Wilson [1985])

7.27 Prove that

(i) $$p_n(\cos\theta;q,-q,q^{\frac12},-q^{\frac12}|q) = \left(q^{n+2};q\right)_n \frac{\sin(n+1)\theta}{\sin\theta},$$

(ii) $$p_n(\cos\theta;1,-1,q^{\frac12},-q^{\frac12}|q) = 2(q^n;q)_n\cos n\theta, \quad n\ge 1,$$

(iii) $$p_n(\cos\theta;q,-1,q^{\frac12},-q^{\frac12}|q) = \left(q^{n+1};q\right)_n \frac{\sin(n+\frac12)\theta}{\sin(\theta/2)},$$

(iv) $$p_n(\cos\theta;1,-q,q^{\frac12},-q^{\frac12}|q) = \left(q^{n+1};q\right)_n \frac{\cos(n+\frac12)\theta}{\cos(\theta/2)}.$$

7.28 Use the orthogonality relations (7.5.28) and (7.5.29) to prove the quadratic transformation formula (7.5.26).

7.29 Verify the orthogonality relations (7.5.28) and (7.5.29).

7.30 Verify formula (7.5.36).

7.31 Suppose that a, b, c, d are complex parameters with $\max(|b|, |c|, |d|, |q|) < 1 < |a|$ such that $|aq^{N+1}| < 1 < |aq^N|$, where N is a nonnegative integer. Use (6.6.12) to prove that

$$\int_{-1}^{1} p_m(x)p_n(x)w(x; a, b, c, d|q)\, dx + 2\pi \sum_{k=0}^{N} p_m(x_k)p_n(x_k)w_k$$
$$= \frac{\delta_{m,n}}{h_n(a, b, c, d|q)},$$

where $p_n(x) = p_n(x; a, b, c, d|q)$, $x_k = \frac{1}{2}\left(aq^k + a^{-1}q^{-k}\right)$ and w_k is given by (7.5.22).

7.32 Prove that

(i) $$P_n^{(\alpha,\beta)}(-x; q) = (-1)^n P_n^{(\beta,\alpha)}(x; q),$$

(ii) $$P_n^{(\alpha,\beta)}(-x|q) = (-1)^n q^{(\alpha-\beta)n/2} P_n^{(\beta,\alpha)}(x|q).$$

7.33 Using (7.4.1), (7.4.7) and (2.11.2) prove that

(i) $$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} C_n(x;\beta|q)C_n(y;\beta|q)t^n$$
$$= \frac{(t^2, \beta; q)_\infty}{(\beta t^2, \beta^2; q)_\infty} \left| \frac{\left(\beta te^{i\theta+i\phi}, \beta te^{i\theta-i\phi}; q\right)_\infty}{(te^{i\theta+i\phi}, te^{i\theta-i\phi}; q)_\infty} \right|^2$$
$$\times {}_8W_7\left(\beta t^2 q^{-1}; \beta, te^{i\theta+i\phi}, te^{-i\theta-i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}; q, \beta\right),$$

(ii) $$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} \frac{1-\beta q^n}{1-\beta} C_n(x;\beta|q)C_n(y;\beta|q)t^n$$
$$= \frac{\left(t^2, \beta; q\right)_\infty}{\left(q\beta t^2, \beta^2; q\right)_\infty} \left| \frac{\left(\beta te^{i\theta+i\phi}, q\beta te^{i\theta-i\phi}; q\right)_\infty}{\left(te^{i\theta+i\phi}, te^{i\theta-i\phi}; q\right)_\infty} \right|^2$$
$$\times {}_8W_7\left(\beta t^2; \beta, qte^{i\theta+i\phi}, qte^{-i\theta-i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}; q, \beta\right),$$

where $-1 < q, \beta, t < 1$ and $x = \cos\theta$, $y = \cos\phi$.
(Gasper and Rahman [1983a], Rahman and Verma [1986a])

7.34 Show that

$$p_n(\cos\theta; a, b, c, d|q) = D^{-1}(\theta)(ab, ac, bc; q)_n$$
$$\times \int_{qe^{i\theta}/d}^{qe^{-i\theta}/d} \frac{\left(due^{i\theta}, due^{-i\theta}, abcdu/q; q\right)_\infty}{(dau/q, dbu/q, dcu/q; q)_\infty} \frac{(q/u;q)_n}{(abcdu/q;q)_n} \left(\frac{du}{q}\right)^n d_q u,$$

where

$$D(\theta) = \frac{-iq(1-q)}{2d}(q, ab, ac, bc; q)_\infty h(\cos\theta; d)w(\cos\theta; a, b, c, d|q).$$

Hence show that

$$\sum_{n=0}^{\infty} \frac{(a^2c^2;q)_n p_n(\cos\theta;a,aq^{\frac{1}{2}},c,cq^{\frac{1}{2}}|q)}{(q;q)_n(a^2q^{\frac{1}{2}},ac,acq^{\frac{1}{2}};q)_n} t^n$$
$$= \frac{(at,ac^2t,-acte^{i\theta},-acte^{-i\theta};q)_\infty}{(-ct,-a^2ct,te^{i\theta},te^{-i\theta};q)_\infty}$$
$$\times {}_8W_7\left(-a^2ctq^{-1};-ac,-a/c,-ctq^{-\frac{1}{2}},ae^{i\theta},ae^{-i\theta};q,-ctq^{\frac{1}{2}}\right).$$

(Gasper and Rahman [1986])

7.35 Show that Ex. 6.9 is equivalent to

$$\int_{-1}^{1} w(y;a,b,\mu e^{i\theta},\mu e^{-i\theta}|q)p_n(y;a,b,c,d|q)\,dy$$
$$= \frac{2\pi(ab\mu^2;q)_\infty}{(q,ab,\mu^2,a\mu e^{i\theta},a\mu e^{-i\theta},b\mu e^{i\theta},b\mu e^{-i\theta};q)_\infty}\frac{(ab;q)_n}{(ab\mu^2;q)_n}\mu^n$$
$$\times p_n(\cos\theta;a\mu,b\mu,c\mu^{-1},d\mu^{-1}|q),$$

where $\max(|a|,|b|,|\mu|) < 1$.

7.36 Show that if for $|q| < 1$ we define

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu;q)_\infty},$$

where ν is a complex number and the principal value of q^ν is taken, then (7.7.6) extends to

$$D_q\left(ae^{i\theta},ae^{-i\theta};q\right)_\nu = -\frac{2a(1-q^\nu)}{1-q}\left(aq^{\frac{1}{2}}e^{i\theta},aq^{\frac{1}{2}}e^{-i\theta};q\right)_{\nu-1}.$$

7.37 Let $n = 1,2,\ldots,r$, $x = \cos\theta$, and

$$U_n(x) = A_{n,r}\left(q^{\nu+1}e^{i\theta},q^{\nu+1}e^{-i\theta};q\right)_n$$
$$\times {}_6\phi_5\left[\begin{matrix} q^{n-r},q^{n+r+2\nu+2\lambda+1},q^{n+\nu+\frac{1}{2}},q^{n+2\nu},q^{n+\nu+1}e^{i\theta},q^{n+\nu+1}e^{-i\theta} \\ q^{2n+2\nu+1},q^{n+\nu+\lambda+1},q^{n+2\nu+1},-q^{n+\nu+\lambda+1},-q^{n+\nu+\frac{1}{2}} \end{matrix};q,q\right]$$

with

$$A_{n,r} = \frac{(q;q)_n(q^{2\nu+2\lambda+1};q)_{n+r}q^{\frac{3}{2}n^2+n(2\nu+\frac{1}{2}-r)}}{(q;q)_{r-n}(q^{\nu+1},q^{2\nu+1},q^{\nu+\lambda+1},-q^{\nu+1},-q^{\nu+\frac{1}{2}},-q^{\nu+\frac{1}{2}},-q^{\nu+\lambda+1};q)_n}.$$

Show that $U_n(x)$ satisfies the q-differential equation

$$D_q[(q^{n+\nu+1}e^{i\theta},q^{n+\nu+1}e^{-i\theta};q)_{-n}U_n(x)]$$
$$= -\frac{1-q^{n+2\nu}}{1-q^{n+1}}\left(q^{n+\nu+\frac{3}{2}}e^{i\theta},q^{n+\nu+\frac{3}{2}}e^{-i\theta};q\right)_{-2n-2\nu-1}$$
$$\times D_q[(q^{-n-\nu}e^{i\theta},q^{-n-\nu}e^{-i\theta};q)_{n+2\nu+1}U_{n+1}(x)].$$

(Gasper [1989b])

7.38 Show that the *discrete q-Hermite polynomials*

$$H_n(x;q) = \sum_{k=0}^{[n/2]} \frac{(q;q)_n}{(q^2;q^2)_k (q;q)_{n-2k}} (-1)^k q^{k(k-1)} x^{n-2k}$$

satisfy the recurrence relation

$$H_{n+1}(x;q) = xH_n(x;q) - q^{n-1}(1-q^n)H_{n-1}(x;q), \quad n \geq 1,$$

and the orthogonality relation

$$\int_{-1}^{1} H_m(x;q)H_n(x;q)\ d\psi(x) = q^{\binom{n}{2}}(q;q)_n \delta_{m,n},$$

where $\psi(x)$ is a step function with jumps

$$\frac{|x|}{2} \frac{(x^2q^2, q; q^2)_\infty}{(q^2;q^2)_\infty}$$

at the points $x = \pm q^j,\ j = 0, 1, 2, \ldots$.
(Al-Salam and Carlitz [1965], Al-Salam and Ismail [1988])

7.39 Let $a < 0$ and $0 < q < 1$. Show that

$$\int_{-\infty}^{\infty} U_m^{(a)}(x;q)U_n^{(a)}(x;q)\ d\alpha^{(a)}(x)$$
$$= (1-a)(-a)^n (q;q)_n q^{\binom{n}{2}} \delta_{m,n},$$

where

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}}\ {}_2\phi_1(q^{-n}, x^{-1}; 0; q, qx/a)$$

and $\alpha^{(a)}(x)$ is a step function with jumps

$$\frac{q^k}{(aq;q)_\infty (q, q/a; q)_k}$$

at the points $x = q^k,\ k = 0, 1, \ldots,$ and jumps

$$\frac{-aq^k}{(q/a;q)_\infty (q, aq; q)_k}$$

at the points $x = aq^k,\ k = 0, 1, \ldots$. Verify that when $a = -1$ this orthogonality relation reduces to the orthogonality relation for the discrete q-Hermite polynomials in Ex. 7.38.
(See Al-Salam and Carlitz [1957, 1965], Chihara [1978, (10.7)], and Ismail [1985b, p. 590])

7.40 Show that if

$$h_n(x;q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} x^k,$$

then

$$\begin{aligned} &h_n^2(x;q) - h_{n+1}(x;q)h_{n-1}(x;q) \\ &= (1-q)(q;q)_{n-1}\sum_{k=0}^{n} h_{n-k}^2(x;q)\frac{q^{n-k}x^k}{(q;q)_{n-k}}, \quad n \geq 1. \end{aligned}$$

Deduce that the polynomials $h_n(x;q)$, which are called the *Rogers-Szegő polynomials*, satisfy the Turán-type inequality

$$h_n^2(x;q) - h_{n+1}(x;q)h_{n-1}(x;q) \geq 0$$

for $x \geq 0$ when $0 < q < 1$ and $n = 1, 2, \ldots$.
(Carlitz [1957b])

7.41 Derive the addition formula

$$\begin{aligned} &p_n(q^z;1,1;q)\ p_y(q^z;q^x,0;q) \\ &= p_n(q^{x+y};1,1;q)\ p_n(q^y;1,1;q)\ p_y(q^z;q^x,0;q) \\ &\quad + \sum_{k=1}^{n} \frac{(q;q)_{x+y+k}(q;q)_{n+k}q^{k(k+y-n)}}{(q;q)_{x+y}(q;q)_{n-k}(q;q)_k^2} \\ &\quad \times p_{n-k}(q^{x+y};q^k,q^k;q)\ p_{n-k}(q^y;q^k,q^k;q) \\ &\quad \times p_{y+k}(q^z;q^x,0;q) + \sum_{k=1}^{n} \frac{(q;q)_y(q;q)_{n+k}q^{k(x+y-n+1)}}{(q;q)_{y-k}(q;q)_{n-k}(q;q)_k^2} \\ &\quad \times p_{n-k}(q^{x+y-k};q^k,q^k;q)\ p_{n-k}(q^{y-k};q^k,q^k;q)\ p_{y-k}(q^z;q^x,0;q) \end{aligned}$$

where $x, y, z, n = 0, 1, \ldots,$ and $p_n(t;a,b;q)$ is the little q-Jacobi polynomial defined in Ex. 1.32.
(Koornwinder [1991a])

7.42 Derive the product formula

$$p_n(q^x;1,1;q)\ p_n(q^y;1,1;q) = (1-q)\sum_{z=0}^{\infty} p_n(q^z;1,1;q)\,K(q^x,q^y,q^z;q)q^z,$$

where $x, y, z, n = 0, 1, \ldots,$ $p_n(t;a,b;q)$ is the little q-Jacobi polynomial, and

$$\begin{aligned} K(q^x,q^y,q^z;q) &= \frac{(q^{x+1},q^{y+1},q^{z+1};q)_\infty}{(1-q)(q,q;q)_\infty} q^{xy+xz+yz} \\ &\quad \times \left\{ {}_3\phi_2(q^{-x},q^{-y},q^{-z};0,0;q,q) \right\}^2. \end{aligned}$$

(Koornwinder [1991a])

7.43 The *q-Laguerre polynomials* are defined by

$$L_n^{\alpha}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, {}_1\phi_1(q^{-n};q^{\alpha+1};q,-x(1-q)q^{n+\alpha+1}).$$

Show that if $\alpha > -1$ then these polynomials satisfy the orthogonality relation

$$\text{(i)} \qquad \int_0^\infty L_m^\alpha(x;q) L_n^\alpha(x;q) \frac{x^\alpha dx}{(-(1-q)x;q)_\infty} = \frac{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)(q;q)_n q^n} \delta_{m,n}$$

and the discrete orthogonality relation

$$\text{(ii)} \quad \sum_{k=-\infty}^{\infty} L_m^\alpha(cq^k;q) L_n^\alpha(cq^k;q) \frac{q^{k(\alpha+1)}}{(-c(1-q)q^k;q)_\infty} = A \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \delta_{m,n},$$

where

$$A = \frac{(q, -c(1-q)q^{\alpha+1}, -1/cq^\alpha(1-q);q)_\infty}{(q^{\alpha+1}, -c(1-q), -q/c(1-q);q)_\infty}.$$

(Moak [1981])

7.44 Let

$$v(y; a_1, a_2, a_3, a_4, a_5|q) = \frac{h(y; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_1 a_2 a_3 a_4 a_5)}{h(y; a_1, a_2, a_3, a_4, a_5)} (1-y^2)^{-\frac{1}{2}}, \quad y = \cos\phi.$$

and

$$g(a_1, a_2, a_3, a_4, a_5|q) = \int_{-1}^{1} v(y; a_1, a_2, a_3, a_4, a_5|q)\, dy.$$

Show that

$$\int_{-1}^{1} v(y; a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(abc\mu e^{i\theta}, abc\mu e^{-i\theta};q)_n}{(abc\mu^2 e^{i\phi}, abc\mu^2 e^{-i\phi};q)_n} \times p_n(y; a, b, c, d|q)\, dy$$
$$= g(a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(ab, ac, bc;q)_n}{(ab\mu^2, ac\mu^2, bc\mu^2;q)_n} \mu^n \times p_n(x; a\mu, b\mu, c\mu, d\mu^{-1}|q), \quad x = \cos\theta,$$

where $p_n(x; a, b, c, d|q)$ are the Askey-Wilson polynomials defined in (7.5.2) and $\max(|a|, |b|, |c|, |\mu|, |q|) < 1$.
(Rahman [1988a])

7.45 Defining the *q-ultraspherical function of the second kind* by

$$D_n(x;\beta|q) = 4 \frac{\sin\theta\ h(\cos 2\theta;\beta)}{h(\cos 2\theta;1)} \sum_{k=0}^{\infty} b(k,n;\beta) \cos(n+2k+1)\theta$$

with $b(k,n;\beta)$ as in (7.4.9), prove that

$$\begin{aligned}
&C_n^2(\cos\theta;\beta|q)+D_n^2(\cos\theta;\beta|q)\\
&=\left[4\sin\theta\frac{(\beta,\beta q^{n+1};q)_\infty h(\cos 2\theta;\beta)}{(q,\beta^2q^n;q)_\infty h(\cos 2\theta;1)}\right]^2\\
&\quad\times\Bigg(\frac{(q^{n+1},\beta^2q^n,qe^{2i\theta},qe^{-2i\theta};q)_\infty}{(\beta q^n,\beta q^{n+1},\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty}\\
&\quad\times{}_5\phi_4\left[\begin{matrix}\beta,q/\beta,-\sqrt{q},-q\\ q^{1-n}/\beta,\beta q^{n+1},qe^{2i\theta},eq^{-2i\theta}\end{matrix};q,q\right]\\
&\quad+\frac{(\beta,q/\beta,\beta e^{2i\theta}q^{n+1},\beta e^{-2i\theta}q^{n+1};q)_\infty}{(\beta q^{n+1},q^{-n}/\beta,\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty}\\
&\quad\times{}_5\phi_4\left[\begin{matrix}q^{n+1},\beta^2q^n,\beta q^{n+1/2},-\beta q^{n+1/2},-\beta q^{n+1}\\ \beta q^{n+1},\beta^2q^{2n+1},\beta e^{2i\theta}q^{n+1},\beta e^{-2i\theta}q^{n+1}\end{matrix};q,q\right]\Bigg).
\end{aligned}$$

(Rahman [1992a])

7.46 Let $f(x)$ be continuous on $[-1,1]$. Show that

$$D_q\left[\int_{-1}^{1}K(x,y)\frac{f(y)}{\sqrt{1-y^2}}dy\right]=f(x),$$

where

$$\begin{aligned}
K(x,y)&=\frac{(1-q)(q,q,e^{2i\phi},e^{-2i\phi};q)_\infty h(x;-q^{1/4},-q^{3/4})}{4\pi q^{1/4}h(y;-q^{1/4},-q^{3/4},q^{1/2}e^{i\theta},q^{1/2}e^{-i\theta})}\\
&\quad-\frac{(q,q,e^{2i\phi},e^{-2i\phi};q)_\infty}{4\pi q^{1/4}(q^{1/2},q^{3/2};q)_\infty h(y;q^{3/4},q^{5/4},-q^{1/4},-q^{3/4})},
\end{aligned}$$

with $x=\cos\theta$, $y=\cos\phi$. Note that this defines a formal inverse of the Askey-Wilson operator D_q.
(Ismail and Rahman [2002a,b])

7.47 Bustoz and Suslov's q-trigonometric functions are defined by

$$\begin{aligned}
C_q(x;\omega)&=\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\,{}_2\phi_1(-qe^{2i\theta},-qe^{-2i\theta};q;q^2,-\omega^2),\\
S_q(x;\omega)&=\frac{2\omega q^{1/4}(-\omega^2;q^2)_\infty}{(1-q)(-q\omega^2;q^2)_\infty}\,{}_2\phi_1(-q^2e^{2i\theta},-q^2e^{-2i\theta};q^3;q^2,-\omega^2),
\end{aligned}$$

where $x=\cos\theta$.
(i) Show that

$$\lim_{q\to 1^-}C_q(x;(1-q)\omega/2)=\cos\omega x\ \text{ and }\ \lim_{q\to 1^-}S_q(x;(1-q)\omega/2)=\sin\omega x.$$

(ii) Prove the orthogonality relations

$$\begin{aligned}&\int_{-1}^{1} W_{1/2}(x|q)C_q(x;\omega_m)C_q(x;\omega_n)(1-x^2)^{-1/2}dx\\ =&\int_{-1}^{1} W_{1/2}(x|q)S_q(x;\omega_m)S_q(x;\omega_n)(1-x^2)^{-1/2}dx\\ =&\pi\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}C_q(\eta;\omega_n)\Big[\frac{\partial}{\partial\omega}S_q(\eta;\omega)\Big]_{\omega=\omega_n}\delta_{m,n},\end{aligned}$$

where $\eta=(q^{1/4}+q^{-1/4})/2$, and ω_m, ω_n are two distinct roots of the equation $S_q(\eta;\omega)=0$.

(Bustoz and Suslov [1998], Suslov [2003])

Notes

§7.1 See also Atakishiyev and Suslov [1988a,b], Atakishiyev, Rahman and Suslov [1995], Nikiforov and Uvarov [1988], Nikiforov, Suslov and Uvarov [1991], and Szegő [1968, 1982]. For a classical polynomial system with complex weight function see Ismail, Masson and Rahman [1991]. The familiar connection between continued fractions and orthogonality was extended by Ismail and Masson [1995] to what they call R-fractions of type I and II, which lead to biorthogonal rational functions. See further work on continued fractions related to elliptic functions in Ismail and Masson [1999], Ismail, Valent and Yoon [2001], and Milne [2002]. In T.S. Chihara and Ismail [1993] extremal measures for a system of orthogonal polynomials in an indeterminate moment problem are examined. For orthogonal polynomials on the unit circle see Ismail and Ruedemann [1992]. Some classical orthogonal polynomials that can be represented by moments are discussed in Ismail and Stanton [1997, 1998].

§7.2 Andrews and Bressoud [1984] used the concept of a crossing number to provide a combinatorial interpretation of the q-Hahn polynomials. Koelink and Koornwinder [1989] showed that the q-Hahn and dual q-Hahn polynomials admit a quantum group theoretic interpretation, analogous to an interpretation of (dual) Hahn polynomials in terms of Clebsch-Gordan coefficients for $SU(2)$. For how Clebsch-Gordan coefficients arise in quantum mechanics, see Biedenharn and Louck [1981a,b]. L. Chihara [1987] considered the locations of zeros of q-Racah polynomials and employed her results to prove non-existence of perfect codes and tight designs in the classical association schemes. The correspondence between q-Racah polynomials and Leonard pairs is outlined in Terwilliger [2003]. For the relationship between orthogonal polynomials and association schemes, see Bannai and Ito [1984], L. Chihara and Stanton [1986], Delsarte [1976b], and Leonard [1982]. A multivariable extension of the q-Racah polynomials is considered in Gasper and Rahman [2003c], while a system of multivariable biorthogonal polynomials is given in Gasper and Rahman [2003a], which are q-analogues of those found in Tratnik [1991b] and [1989], respectively.

§7.3 Al-Salam and Ismail [1977] constructed a family of reproducing kernels (bilinear formulas) for the little q-Jacobi polynomials. In Al-Salam and Ismail [1983] they considered a related family of orthogonal polynomials associated with the Rogers–Ramanujan continued fraction. A biorthogonal extension of the little q-Jacobi polynomials is studied in Al-Salam and Verma [1983a]. When $b = 0$ the little q-Jacobi polynomials reduce (after changing variables and renormalizing) to the *Wall polynomials*

$$W_n(x; b, q) = (-1)^n (b; q)_n q^{\binom{n+1}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{q^{\binom{j}{2}}(-q^{-n}x)^j}{(b; q)_j}$$

and to the *generalized Stieltjes–Wigert polynomials*

$$S_n(x; p, q) = (-1)^n q^{-n(2n+1)/2} (p; q)_n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{q^{j^2}(-q^{\frac{1}{2}}x)^j}{(p; q)_j},$$

which are q-analogues of the Laguerre polynomials that are different from those considered in Ex. 7.43. Since the Hamburger and Stieltjes moment problems corresponding to these polynomials are both indeterminate, there are infinitely many nonequivalent measures on $[0, \infty)$ for which these polynomials are orthogonal. See Chihara [1978, Chapter VI], [1968b, 1971, 1979, 1982, 1985], Al-Salam and Verma [1982b], L. Chihara and T.S. Chihara [1987], and Shohat and Tamarkin [1950].

§7.4 An integral of the product of two continuous q-ultraspherical polynomials and a q-ultraspherical function of the second kind is evaluated in Askey, Koornwinder and Rahman [1986]. Al-Salam, Allaway and Askey [1984a] gave a characterization of the continuous q-ultraspherical polynomials as orthogonal polynomial solutions of certain integral equations. Askey [1989b] showed that the polynomials $C_n(ix; \beta|q)$, $0 \le n \le N$, are orthogonal on the real line with respect to a positive measure when $0 < q < 1$ and $\beta > q^{-N}$. Ismail and Rahman [1991] showed that the associated Askey-Wilson polynomials $r_n^\alpha(x; a, b, c, d|q)$ defined in Ex. 8.26 have an orthogonality property on $[-1, 1]$. A survey of classical associated orthogonal polynomials is in Rahman [2001], and an integral representation is given in Rahman [1996b]. A projection formula and a reproducing kernel for $r_n^\alpha(x; a, b, c, d|q)$ is given in Rahman and Tariq Qazi [1997b]. Berg and Ismail [1996] showed how to generate one to four parameter orthogonal polynomials in the Askey-Wilson family by starting from the continuous q-Hermite polynomials. Also see Koelink [1995b] and Rahman and Verma [1987].

§7.5 Asymptotic formulas and generating functions for the Askey-Wilson polynomials and their special cases are derived in Ismail and Wilson [1982] and Ismail [1986c]. Kalnins and Miller [1989] employed symmetry techniques to give an elementary proof of the orthogonality relation for the Askey-Wilson polynomials. Following Hahn's approach to the classification of classical orthogonal polynomials N.M. Atakishiyev and Suslov [1992b] gave a generalized moment representation for the Askey-Wilson polynomials. Brown, Evans and Ismail [1996] showed that the Askey-Wilson polynomials are solutions of a

q-Sturm-Liouville problem and gave an operator theoretic description of the Askey-Wilson operator $\mathcal{D}_q$. L. Chihara [1993] extended her work on q-Racah polynomials [1987] to the Askey-Wilson polynomials. Floreanini, LeTourneux and Vinet [1999] also employed symmetry techniques to study systems of continuous q-orthogonal polynomials. A multivariable extension of Askey-Wilson polynomials is given in Gasper and Rahman [2003b] as a q-analogue of Tratnik [1991a], see Ex. 8.29. Also see Gasper, Ismail, Koornwinder, Nevai and Stanton [2000], Spiridonov and Zhedanov [1995–1997], Stokman [1997a–2003a], Stokman and Koornwinder [1998], Vinet and Zhedanov [2001], and Wilson [1991].

§7.6 For additional results on connection coefficients (and the corresponding projection formulas), see Andrews [1979a], Gasper [1974, 1975a].

Ex. 7.5 A contiguous relation satisfied by a WP-balanced, generally nonterminating ${}_8\phi_7$ series is given in Ismail and Rahman [1991]. For more results on contiguous relations and orthogonal polynomials, see Gupta, Ismail and Masson [1992, 1996], Gupta and Masson [1998] and Ismail and Libis [1989].

Ex. 7.7 Ismail [1995] gave a simple proof of (1.7.2) by making use of iterations of (7.7.6) and evaluating them at $x_j = \frac{1}{2}(aq^j + q^{-j}/a)$. An operator calculus for D_q is developed in Ismail [2001a].

Ex. 7.8 Stanton [1981b] showed that the q-Krawtchouk polynomials $K_n(x; a, N; q)$ are spherical functions for three different Chevalley groups over finite fields and derived three addition theorems for these polynomials by decomposing the irreducible representations with respect to maximal parabolic subgroups. In Koornwinder [1989] it is shown that the orthogonality relation for the q-Krawtchouk polynomials $K_n(x; a, N|q)$ expresses the fact that the matrix representations of the quantum group $S_\mu U(2)$ are unitary.

Ex. 7.11 The affine q-Krawtchouk polynomials are the eigenvalues of the association schemes of bilinear, alternating, symmetric and hermitian forms over a finite field (see Carlitz and Hodges [1955], Delsarte [1978], Delsarte and Goethals [1975], and Stanton [1981a,b, 1984]). L. Chihara and Stanton [1987] showed that the zeros of the affine q-Krawtchouk polynomials are never zero at integral values of x, and they gave some interlacing theorems for the zeros of q-Krawtchouk polynomials.

Ex. 7.12 and 7.43 N.M. Atakishiyev, M.N. Atakishiyev and Klimyk [2003] found the connection between big q-Laguerre and q-Meixner polynomials and representations of the group $U_q(su_{1,1})$. Ciccoli, Koelink and Koornwinder [1999] extended Moak's q-Laguerre polynomials to an orthogonal system for a doubly infinite Jacobi matrix originating from analysis on $SU_q(1,1)$, and found the orthogonality and dual orthogonality relations for q-Bessel functions originating in $E_q(2)$.

Ex. 7.22 Askey [1989b] proved that the polynomials $H_n(ix|q)$ are orthogonal on the real line with respect to a positive measure when $q > 1$.

Exercises 7.23–7.25 Other sieved orthogonal polynomials are considered in Al-Salam and Chihara [1987], Askey [1984b], Charris and Ismail [1986, 1987, 1993], Charris, Ismail and Monslave [1994], Ismail [1985a, 1986a,b], and Ismail and Li [1992].

Exercises 7.38–7.40 For additional material on q-analogues of Hermite polynomials, see Allaway [1980], Al-Salam and Chihara [1976], Al-Salam and Ismail [1988], Carlitz [1963b, 1972], Chihara [1968a, 1982, 1985], Dehesa [1979], Désarménien [1982], Hou, Lascoux and Mu [2003], Ismail [1985b], Ismail, Stanton and Viennot [1987], Lubinsky and Saff [1987], and Szegő [1926].

Ex. 7.41 Rahman [1989a] gave a simple proof for this addition formula. For derivations of the addition formula for Jacobi polynomials, see Koornwinder [1974a,b] and Laine [1982].

Ex. 7.43 See also Cigler [1981] and Pastro [1985]. In view of the two different orthogonality relations for the q-Laguerre polynomials, it follows that there are infinitely many measures for which these polynomials are orthogonal. The *Stieltjes–Wigert polynomials* (see Chihara [1978, pp. 172–174], Szegő [1975, p. 33] and the above Notes for §7.3)

$$s_n(x) = (-1)^n q^{(2n+1)/4} (q;q)_n^{-\frac{1}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j^2} (-q^{\frac{1}{2}} x)^j,$$

which are orthogonal with respect to the log normal weight function

$$w(x) = k\pi^{-\frac{1}{2}} \exp(-k^2 \log^2 x), \qquad 0 < x < \infty,$$

where $q = \exp[-(2k^2)^{-1}]$ and $k > 0$, are a limit case of the q-Laguerre polynomials. Askey [1986] gave the orthogonality relation for these polynomials (with a slightly different definition) that follows as a limit case of the first orthogonality relation in this exercise. Al-Salam and Verma [1983b,c] studied a pair of biorthogonal sets of polynomials, called the q-Konhauser polynomials, which were suggested by the q-Laguerre polynomials. Ismail and Rahman [1998] studied two indeterminate Hamburger moment problems associated with $L_n^\alpha(x;q)$, thus completing earlier work of Moak [1981]. For asymptotics of basic Bessel functions and q-Laguerre polynomials, see Chen, Ismail and Muttalib [1994].

Ex. 7.46 A right inverse of the Askey-Wilson operator is derived in Brown and Ismail [1995].

Ex. 7.47 The question of completeness of the q-trigonometric functions for use in a q-Fourier type analysis is dealt with in Ismail [2001b] and Suslov [2001c, 2003].

8

FURTHER APPLICATIONS

8.1 Introduction

In this chapter we derive some formulas that are related to products of the q-orthogonal polynomials introduced in the previous chapter and use these formulas to obtain q-analogues of various product formulas, Poisson kernels and linearization formulas for ultraspherical and Jacobi polynomials. The method in Gasper and Rahman [1984] originates with the observation that since

$$\left(q^{-x}, aq^{x}; q\right)_j = \prod_{k=0}^{j-1} \left(1 - q^k(q^{-x} + aq^x) + aq^{2k}\right)$$

is a polynomial of degree j in powers of $q^{-x}+aq^x$, there must exist an expansion of the form

$$\left(q^{-x}, aq^{x}; q\right)_j \left(q^{-x}, aq^{x}; q\right)_k = \sum_{m=0}^{j+k} A_m(j,k,a;q) \left(q^{-x}, aq^{x}; q\right)_m . \tag{8.1.1}$$

Since, for $k \geq j$,

$${}_3\phi_2\left(q^{-j}, q^{k-x}, aq^{k+x}; aq^k, q^{1+k-j}; q, q\right) = \frac{\left(q^{-x}, aq^x; q\right)_j}{\left(q^{-k}, aq^k; q\right)_j}$$

by the q-Saalschütz formula (1.7.2), it is easy to verify that

$$\begin{aligned}\left(q^{-x}, aq^{x}; q\right)_j \left(q^{-x}, aq^{x}; q\right)_k &= (q;q)_j (q;q)_k (a;q)_{j+k} \\ &\times \sum_{m=\max(j,k)}^{j+k} \frac{\left(q^{-x}, aq^x; q\right)_m q^{\binom{j}{2}+\binom{k}{2}+\binom{m+1}{2}-m(j+k)}(-1)^{j+k+m}}{(a;q)_m (q;q)_{m-j}(q;q)_{m-k}(q;q)_{j+k-m}}. \end{aligned} \tag{8.1.2}$$

This linearizes the product on the left side and forms the basis for the product formulas derived in the following section.

Suppose $\{B_j\}_{j=0}^{\infty}$ and $\{C_j\}_{j=0}^{\infty}$ are arbitrary complex sequences and b, c are complex numbers such that $(b;q)_k, (c;q)_k$ do not vanish for $k = 1, 2, \ldots$. Then, setting

$$F_n = \sum_{j=0}^{n} \frac{\left(q^{-n}, aq^n; q\right)_j}{(q, b; q)_j} B_j \sum_{k=0}^{n} \frac{\left(q^{-n}, aq^n; q\right)_k}{(q, c; q)_k} C_k, \tag{8.1.3}$$

we find by using (8.1.2) that

$$F_n = \sum_{m=0}^{n} \left(q^{-n}, aq^n; q\right)_m \sum_{k=0}^{m} \frac{C_k q^{k^2-mk}}{(q, c; q)_k (q, b; q)_{m-k}}$$

$$\times \sum_{j=0}^{k} \frac{\left(q^{-k}, aq^m; q\right)_j}{\left(q, bq^{m-k}; q\right)_j} B_{m-k+j} \tag{8.1.4}$$

for $n = 0, 1, 2, \ldots$. This formula does not extend to noninteger values of n because, in general, the triple sum on the right side does not converge.

8.2 A product formula for balanced ${}_4\phi_3$ polynomials

Since we are mainly interested in q-orthogonal polynomials which are expressible as balanced ${}_4\phi_3$ series or their limit cases, we shall now specialize (8.1.4) to such cases. Set

$$B_j = \frac{(b_1, b_2; q)_j}{(b_3, qab_1b_2/bb_3; q)_j} q^j, \ \ C_j = \frac{(c_1, c_2; q)_j}{(c_3, qac_1c_2/cc_3; q)_j} q^j, \tag{8.2.1}$$

where it is assumed that the parameters are such that no zero factors appear in the denominators. Then formula (8.1.4) gives

$$\begin{aligned} f_n = & \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(q^{-n}, aq^n; q)_m \, (c_1, c_2; q)_{m-k}}{(q, b; q)_k (q, c, c_3, qac_1c_2/cc_3; q)_{m-k}} q^{k^2 - mk + m} \\ & \times \frac{(b_1, b_2; q)_k}{(b_3, qab_1b_2/bb_3; q)_k} \, {}_4\phi_3 \left[\begin{matrix} q^{k-m}, aq^m, b_1q^k, b_2q^k \\ bq^k, b_3q^k, ab_1b_2q^{k+1}/bb_3 \end{matrix} ; q, q \right], \end{aligned} \tag{8.2.2}$$

where f_n is the right side of (8.1.3) with B_j and C_j as defined in (8.2.1). The crucial step in the next round of calculations is to convert the ${}_4\phi_3$ series in (8.2.2) into a very-well-poised ${}_8\phi_7$ series by Watson's formula (2.5.1), i.e.,

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{k-m}, aq^m, b_1q^k, b_2q^k \\ bq^k, b_3q^k, ab_1b_2q^{k+1}/bb_3 \end{matrix} ; q, q \right] \\ & = \frac{\left(bb_3q^{k-m}/ab_1, bb_3q^{k-m}/ab_2; q\right)_{m-k}}{\left(bb_3q^{2k-m}/a, bb_3q^{-m}/ab_1b_2; q\right)_{m-k}} \\ & \quad \times {}_8W_7 \left(\frac{bb_3q^{2k-m-1}}{a}; \frac{b_3q^{k-m}}{a}, \frac{bq^{k-m}}{a}, b_1q^k, b_2q^k, q^{k-m}; q, \frac{bb_3q^{m-k}}{b_1b_2} \right). \end{aligned} \tag{8.2.3}$$

Substituting this into (8.2.2) gives

$$\begin{aligned} f_n = & \sum_{m=0}^{n} \frac{(q^{-n}, aq^n, qab_1/bb_3, qab_2/bb_3; q)_m}{(q, c, qa/bb_3, qab_1b_2/bb_3; q)_m} q^m \\ & \times \sum_{k=0}^{m} \sum_{j=0}^{m-k} \frac{\left(bb_3q^{-m-1}/a; q\right)_{2k+j} \left(1 - bb_3q^{2j+2k-m-1}/a\right) (c_1, c_2; q)_{m-k}}{(q; q)_j \left(1 - bb_3q^{-m-1}/a\right) (c_3, qac_1c_2/cc_3; q)_{m-k}} \\ & \times \frac{\left(q^{1-m}/c; q\right)_k (b_1, b_2, bq^{-m}/a, b_3q^{-m}/a, q^{-m}; q)_{j+k}}{(q, bq^{-m}/a, b_3q^{-m}/a; q)_k \, (b, b_3, bb_3/a, bb_3q^{-m}/ab_1, bb_3q^{-m}/ab_2; q)_{j+k}} \\ & \times (-1)^k q^{mk - \binom{k}{2}} \left(\frac{bb_3c}{qab_1b_2} \right)^k \left(\frac{bb_3q^{m-k}}{b_1b_2} \right)^j. \end{aligned} \tag{8.2.4}$$

Then, replacing j by $j-k$ in the sum on the right side of (8.2.4), we obtain

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ b_1,\ b_2 \\ b,\ \ b_3,\ qab_1b_2/bb_3 \end{matrix};q,q\right]\ {}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ c_1,\ c_2 \\ c,\ \ c_3,\ qac_1c_2/cc_3 \end{matrix};q,q\right] \\
&= \sum_{m=0}^{n} \frac{(q^{-n},aq^n,c_1,c_2,qab_1/bb_3,qab_2/bb_3;q)_m}{(q,c,c_3,qa/bb_3,qac_1c_2/cc_3,qab_1b_2/bb_3;q)_m} q^m \\
&\times \sum_{j=0}^{m} \frac{(bb_3q^{-m-1}/a;q)_j(1-bb_3q^{2j-m-1}/a)(b_1,b_2,bq^{-m}/a,b_3q^{-m}/a,q^{-m};q)_j}{(q;q)_j(1-bb_3q^{-m-1}/a)(bb_3q^{-m}/ab_1,bb_3q^{-m}/ab_2,b_3,b,bb_3/a;q)_j} \\
&\times \left(\frac{bb_3q^m}{b_1b_2}\right)^j {}_5\phi_4\left[\begin{matrix} q^{-j},q^{1-m}/c,q^{1-m}/c_3,bb_3q^{j-m-1}/a,cc_3q^{-m}/ac_1c_2 \\ q^{1-m}/c_1,q^{1-m}/c_2,bq^{-m}/a,b_3q^{-m}/a \end{matrix};q,q\right].
\end{aligned}
\tag{8.2.5}
$$

Note that the ${}_5\phi_4$ series in (8.2.5) is balanced and, in the special case $c=aq/b$ and $c_3=aq/b_3$, becomes a ${}_3\phi_2$ which is summable by (1.7.2). Thus, we obtain the formula

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ b_1,b_2 \\ b,\ \ b_3,\ qab_1b_2/bb_3 \end{matrix};q,q\right]\ {}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ c_1,\ c_2 \\ aq/b,\ aq/b_3,\ bb_3c_1c_2/aq \end{matrix};q,q\right] \\
&= \sum_{m=0}^{n} \frac{(q^{-n},aq^n,c_1,c_2,qab_1/bb_3,qab_2/bb_3;q)_m}{(q,aq/b,aq/b_3,aq/bb_3,qab_1b_2/bb_3,bb_3c_1c_2/aq;q)_m} q^m \\
&\times {}_{10}\phi_9\left[\begin{matrix} \lambda,\ q\lambda^{\frac12},\ -q\lambda^{\frac12},\ b_1,\ b_2,\ bb_3c_1/aq, \\ \lambda^{\frac12},\ -\lambda^{\frac12},\ bb_3q^{-m}/ab_1,\ bb_3q^{-m}/ab_2,\ q^{1-m}/c_1, \end{matrix}\right. \\
&\qquad \left.\begin{matrix} bb_3c_2/aq,\ bq^{-m}/a,\ b_3q^{-m}/a,\ q^{-m} \\ q^{1-m}/c_2,\ b_3,\ b,\ bb_3/a, \end{matrix};q,\frac{aq^2}{b_1b_2c_1c_2}\right],
\end{aligned}
\tag{8.2.6}
$$

where $\lambda = bb_3q^{-m-1}/a$. This formula is a q-analogue of Bailey's [1933] product formula

$$
\begin{aligned}
&{}_2F_1(-n,a+n;b;x)\ {}_2F_1(-n,a+n;1+a-b;y) \\
&\quad = F_4(-n,a+n;b,1+a-b;x(1-y),y(1-x)),
\end{aligned}
\tag{8.2.7}
$$

where

$$
F_4(a,b;c,d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(d)_n} x^m y^n. \tag{8.2.8}
$$

However, even though (8.2.6) is valid only when the series on both sides terminate, (8.2.7) holds whether or not n is a nonnegative integer, subject to the absolute convergence of the two ${}_2F_1$ series on the left and the F_4 series on the right.

Application of Sears' transformation formula (2.10.4) enables us to transform one or both of the ${}_4\phi_3$ series on the left side of (8.2.6) and derive a number of equivalent forms. Two particularly interesting ones are

$$
{}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ b_1,\ b_2 \\ b,\ b_3,\ qab_1b_2/bb_3 \end{matrix};q,q\right]\ {}_4\phi_3\left[\begin{matrix} q^{-n},\ aq^n,\ bb_3c_1/aq,\ bb_3c_2/aq \\ b,\ b_3,\ bb_3c_1c_2/aq \end{matrix};q,q\right]
$$

$$= \frac{(aq/b, aq/b_3; q)_n}{(b, b_3; q)_n} \left(\frac{bb_3}{aq}\right)^n$$

$$\times \sum_{m=0}^{n} \frac{(q^{-n}, aq^n, c_1, c_2, qab_1/bb_3, qab_2/bb_3; q)_m}{(q, aq/b, aq/b_3, aq/bb_3, qab_1b_2/bb_3, bb_3c_1c_2/aq; q)_m} q^m$$

$$\times {}_{10}\phi_9 \left[\begin{matrix} \lambda, & q\lambda^{\frac{1}{2}}, & -q\lambda^{\frac{1}{2}}, & b_1, & b_2, & bb_3c_1/aq, & bb_3c_2/aq, \\ & \lambda^{\frac{1}{2}}, & -\lambda^{\frac{1}{2}}, & bb_3q^{-m}/ab_1, & bb_3q^{-m}/ab_2, & q^{1-m}/c_1, & q^{1-m}/c_2, \end{matrix}\right.$$

$$\left.\begin{matrix} bq^{-m}/a, & b_3q^{-m}/a, & q^{-m} \\ b_3, & b, & bb_3/a \end{matrix}; q, \frac{aq^2}{b_1b_2c_1c_2}\right], \tag{8.2.9}$$

and

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, & aq^n, & qab_1/bb_3, & qab_2/bb_3 \\ & aq/b, & aq/b_3, & qab_1b_2/bb_3 \end{matrix}; q, q\right] {}_4\phi_3 \left[\begin{matrix} q^{-n}, & aq^n, & c_1, & c_2 \\ aq/b, & aq/b_3, & bb_3c_1c_2/aq \end{matrix}; q, q\right]$$

$$= \frac{(b, b_3; q)_n}{(aq/b, aq/b_3; q)_n} \left(\frac{aq}{bb_3}\right)^n$$

$$\times \sum_{m=0}^{n} \frac{(q^{-n}, aq^n, c_1, c_2, qab_1/bb_3, qab_2/bb_3; q)_m}{(q, aq/b, aq/b_3, aq/bb_3, qab_1b_2/bb_3, bb_3c_1c_2/aq; q)_m} q^m$$

$$\times {}_{10}\phi_9 \left[\begin{matrix} \lambda, & q\lambda^{\frac{1}{2}}, & -q\lambda^{\frac{1}{2}}, & b_1, & b_2, & bb_3c_1/aq, & bb_3c_2/aq, \\ & \lambda^{\frac{1}{2}}, & -\lambda^{\frac{1}{2}}, & bb_3q^{-m}/ab_1, & bb_3q^{-m}/ab_2, & q^{1-m}/c_1, & q^{1-m}/c_2, \end{matrix}\right.$$

$$\left.\begin{matrix} bq^{-m}/a, & b_3q^{-m}/a, & q^{-m} \\ b_3, & b, & bb_3/a \end{matrix}; q, \frac{aq^2}{b_1b_2c_1c_2}\right], \tag{8.2.10}$$

where $\lambda = bb_3q^{-m-1}/a$.

Either of the formulas (8.2.9) and (8.2.10) may be regarded as a q-analogue of Watson's [1922] product formula for the Jacobi polynomials

$$\begin{aligned} &{}_2F_1(-n, a+n; b; x)\ {}_2F_1(-n, a+n; b; y) \\ &= (-1)^n \frac{(1+a-b)_n}{(b)_n}\ F_4\left(-n, a+n; b, 1+a-b; xy, (1-x)(1-y)\right), \end{aligned} \tag{8.2.11}$$

where $n = 0, 1, \ldots$.

The special case in which the ${}_{10}\phi_9$ series in (8.2.6), (8.2.9) or (8.2.10) become balanced is also of interest in some applications. Thus, if we set $c_2 = aq/b_1b_2c_1$, then by using Bailey's transformation formula (2.10.8) we may express (8.2.9) in the form

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, & aq^n, & b_1, & b_2 \\ b, & b_3, & qab_1b_2/bb_3 \end{matrix}; q, q\right] {}_4\phi_3 \left[\begin{matrix} q^{-n}, & aq^n, & bb_3c_1/aq, & bb_3/b_1b_2c_1 \\ & b, & b_3, & bb_3/b_1b_2 \end{matrix}; q, q\right]$$

$$= \frac{(aq/b, aq/b_3; q)_n}{(b, b_3; q)_n} \left(\frac{bb_3}{aq}\right)^n \sum_{m=0}^{n} \frac{(q^{-n}, aq^n, b_1c_1, b_2c_1, aq/b_1b_2c_1; q)_m}{(q, aq/b, aq/b_3, b_1b_2c_1, bb_3/b_1b_2; q)_m} q^m$$

$$\times {}_{10}\phi_9 \left[\begin{matrix} \mu, & q\mu^{\frac{1}{2}}, & -q\mu^{\frac{1}{2}}, & b_1, & b_2 & b_1b_2c_1/b, & b_1b_2c_1/b_3, \\ & \mu^{\frac{1}{2}}, & -\mu^{\frac{1}{2}}, & b_2c_1, & b_1c_1, & b, & b_3, \end{matrix}\right.$$

$$\left.\begin{matrix} bb_3c_1/aq, & aq^m, & q^{-m} \\ qab_1b_2/bb_3, & b_1b_2c_1q^{-m}/a, & b_1b_2c_1q^m \end{matrix} ;q,q\right], \tag{8.2.12}$$

where $\mu = b_1b_2c_1q^{-1}$. This provides a q-analogue of Bateman's [1932, p. 392] product formula

$$\begin{aligned} &{}_2F_1(-n,a+n;b;x)\ {}_2F_1(-n,a+n;b;y) \\ &= (-1)^n\frac{(1+a-b)_n}{(b)_n}\sum_{k=0}^{n}\frac{(-n)_k(a+n)_k}{k!(1+a-b)_k}(1-x-y)^k \\ &\quad\times {}_2F_1\left(-k,a+k;b;-xy/(1-x-y)\right). \end{aligned} \tag{8.2.13}$$

8.3 Product formulas for q-Racah and Askey-Wilson polynomials

Let us replace the parameters $a, b, b_1, b_2, b_3, c_1, c_2$ in (8.2.9) by abq, aq, q^{-x}, $cq^{x-N}, bcq, c^{-1}q^{-y}, q^{y-N}$, respectively, to obtain the following product formula for the q-Racah polynomials introduced in §7.2:

$$\begin{aligned} &W_n(x;a,b,c,N;q)\ W_n(y;a,b,c,N;q) \\ &= \frac{(bq,qac^{-1};q)_n}{(aq,bcq;q)_n}c^n\sum_{m=0}^{n}\frac{\left(q^{-n},abq^{n+1},q^{x-N},q^{y-N},c^{-1}q^{-x},c^{-1}q^{-y};q\right)_m}{(q,bq,qac^{-1},c^{-1},q^{-N},q^{-N};q)_m}q^m \\ &\quad\times {}_{10}\phi_9\left[\begin{matrix} cq^{-m}, q(cq^{-m})^{\frac12}, -q(cq^{-m})^{\frac12}, ca^{-1}q^{-m}, b^{-1}q^{-m}, q^{-m}, q^{-x}, \\ (cq^{-m})^{\frac12}, -(cq^{-m})^{\frac12}, aq, bcq, cq, cq^{x+1-m}, \end{matrix}\right. \\ &\qquad\left.\begin{matrix} q^{-y}, cq^{x-N}, cq^{y-N} \\ cq^{y+1-m}, q^{N-x+1-m}, q^{N-y+1-m} \end{matrix};q,abq^{2N+3}\right], \end{aligned} \tag{8.3.1}$$

where

$$\begin{aligned} &W_n(x;a,b,c,N;q) \\ &= {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N} \\ aq, q^{-N}, bcq \end{matrix};q,q\right] \end{aligned} \tag{8.3.2}$$

is the q-Racah polynomial defined in (7.2.17). This is a Watson-type formula. Two additional Watson-type formulas are given in Ex. 8.1.

Letting $c \to 0$ in (8.3.1) gives a product formula for the q-Hahn polynomials defined in (7.2.21):

$$\begin{aligned} &Q_n(x;a,b,N;q)\ Q_n(y;a,b,N;q) \\ &= (-aq)^n q^{\binom{n}{2}}\frac{(bq;q)_n}{(aq;q)_n}\sum_{m=0}^{n}\frac{\left(q^{-n},abq^{n+1},q^{x-N},q^{y-N};q\right)_m}{(q,bq,q^{-N},q^{-N};q)_m}\left(aq^{x+y}\right)^{-m} \\ &\quad\times {}_4\phi_3\left[\begin{matrix} q^{-x}, q^{-y}, b^{-1}q^{-m}, q^{-m} \\ aq, q^{N-x+1-m}, q^{N-y+1-m} \end{matrix};q,abq^{2N+3}\right]. \end{aligned} \tag{8.3.3}$$

To obtain a Watson-type product formula for the Askey-Wilson polynomials defined in (7.5.2) we replace $a, b, b_1, b_2, b_3, c_1, c_2$ in (8.2.9) by $abcdq^{-1}, ab$, $ae^{i\theta}, ae^{-i\theta}, ac, de^{i\phi}, de^{-i\phi}$, respectively, where $x = \cos\theta,\ y = \cos\phi$. This gives

$$\begin{aligned}
&p_n(x; a, b, c, d|q)\ p_n(y; a, b, c, d|q)\\
&= (ab, ac, ad, ad, bd, cd; q)_n (ad)^{-n}\\
&\quad\times \sum_{m=0}^{n} \frac{\left(q^{-n}, abcdq^{n-1}, de^{i\theta}, de^{-i\theta}, de^{i\phi}, de^{-i\phi}; q\right)_m}{(q, ad, ad, bd, cd, da^{-1}; q)_m} q^m\\
&\quad\times {}_{10}\phi_9 \left[\begin{matrix} aq^{-m}/d, q(aq^{-m}/d)^{\frac12}, -q(aq^{-m}/d)^{\frac12}, q^{1-m}/bd, q^{1-m}/cd, q^{-m},\\ (aq^{-m}/d)^{\frac12}, -(aq^{-m}/d)^{\frac12}, ab, ac, aq/d,\end{matrix}\right.\\
&\qquad\left.\begin{matrix} ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}\\ q^{1-m}e^{-i\theta}/d, q^{1-m}e^{i\theta}/d, q^{1-m}e^{-i\phi}/d, q^{1-m}e^{i\phi}/d\end{matrix}; q, \frac{bcq}{ad}\right]. \qquad (8.3.4)
\end{aligned}$$

When $b = aq^{\frac12}$ and $d = cq^{\frac12}$, the ${}_{10}\phi_9$ series in (8.3.4) becomes balanced and hence can be transformed to another balanced ${}_{10}\phi_9$ via (2.9.1). This leads to a Bateman-type product formula

$$\begin{aligned}
&p_n(x; a, aq^{\frac12}, c, cq^{\frac12}|q)\ p_n(y; a, aq^{\frac12}, c, cq^{\frac12}|q)\\
&= \left(a^2q^{\frac12}, ac, acq^{\frac12}, acq^{\frac12}, acq, c^2q^{\frac12}; q\right)_n \left(acq^{\frac12}\right)^{-n}\\
&\quad\times \sum_{m=0}^{n} \frac{\left(q^{-n}, a^2c^2q^n, acq^{\frac12}e^{i\theta+i\phi}, acq^{\frac12}e^{i\phi-i\theta}, cq^{\frac12}e^{-i\phi}; q\right)_m}{\left(q, c^2q^{\frac12}, acq^{\frac12}, acq, a^2cq^{\frac12}e^{i\phi}; q\right)_m} q^m\\
&\quad\times {}_{10}\phi_9 \left[\begin{matrix} \nu, & q\nu^{\frac12}, & -q\nu^{\frac12}, & ae^{i\phi}, & aq^{\frac12}e^{i\phi}, & ce^{i\phi}, & ae^{i\theta},\\ & \nu^{\frac12}, & -\nu^{\frac12}, & acq^{\frac12}, & ac, & a^2q^{\frac12}, & acq^{\frac12}e^{i\phi-i\theta},\end{matrix}\right.\\
&\qquad\left.\begin{matrix} ae^{-i\theta}, & a^2c^2q^m, & q^{-m}\\ acq^{\frac12}e^{i\phi+i\theta}, & q^{\frac12-m}e^{i\phi}/c, & a^2ce^{i\phi}q^{m+\frac12}\end{matrix}; q, q\right], \qquad (8.3.5)
\end{aligned}$$

where $\nu = a^2ce^{i\phi}q^{-\frac12}$. In fact, if we replace a and c by $q^{(2\alpha+1)/4}$ and $-q^{(2\beta+1)/4}$, respectively, then this gives a Bateman-type product formula for the continuous q-Jacobi polynomials (7.5.24) which, on letting $q \to 1$, gives Bateman's [1932] product formula for the Jacobi polynomials:

$$\begin{aligned}
\frac{P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)P_n^{(\alpha,\beta)}(1)} &= (-1)^n \frac{(\beta+1)_n}{(\alpha+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\beta+1)_k} \left(\frac{x+y}{2}\right)^k\\
&\quad\times P_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) / P_k^{(\alpha,\beta)}(1), \qquad (8.3.6)
\end{aligned}$$

which is equivalent to (8.2.13).

For terminating series there is really no difference between the Watson formula (8.2.11) and the Bailey formula (8.2.7) since one can be transformed into the other in a trivial way. However, for the continuous q-ultraspherical polynomials given in (7.4.14), there is an interesting Bailey-type product formula that can be obtained from (8.2.6) by replacing $a, b, b_1, b_2, b_3, c_1, c_2$ by

$a^4, a^2q^{\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}, -a^2q^{\frac{1}{2}}, ae^{i\phi}$ and $ae^{-i\phi}$, respectively:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4q^n, & ae^{i\theta}, & ae^{-i\theta} \\ a^2q^{\frac{1}{2}}, & -a^2q^{\frac{1}{2}}, & -a^2 & \end{matrix};q,q\right] {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^4q^n, & ae^{i\phi}, & ae^{-i\phi} \\ a^2q^{\frac{1}{2}}, & -a^2q^{\frac{1}{2}}, & -a^2 & \end{matrix};q,q\right]
$$

$$
= \sum_{m=0}^{n} \frac{\left(q^{-n}, a^4q^n, ae^{i\phi}, ae^{-i\phi}, -ae^{i\theta}, -ae^{-i\theta};q\right)_m}{\left(q, a^2q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, -1, -a^2, -a^2;q\right)_m} q^m
$$

$$
\times {}_{10}\phi_9\left[\begin{matrix} -q^{-m}, & q(-q^{-m})^{\frac{1}{2}}, & -q(-q^{-m})^{\frac{1}{2}}, & q^{\frac{1}{2}-m}/a^2, & -q^{\frac{1}{2}-m}/a^2, & q^{-m}, \\ & (-q^{-m})^{\frac{1}{2}}, & -(-q^{-m})^{\frac{1}{2}}, & -a^2q^{\frac{1}{2}}, & a^2q^{\frac{1}{2}}, & -q, \end{matrix}\right.
$$

$$
\left.\begin{matrix} ae^{i\theta}, & ae^{-i\theta}, & -ae^{i\phi}, & -ae^{-i\phi} \\ -q^{1-m}e^{-i\theta}/a, & -q^{1-m}e^{i\theta}/a, & q^{1-m}e^{-i\phi}/a, & q^{1-m}e^{i\phi}/a \end{matrix};q,q^2\right]. \tag{8.3.7}
$$

For further information about product formulas see Rahman [1982] and Gasper and Rahman [1984].

8.4 A product formula in integral form for the continuous q-ultraspherical polynomials

As an application of the Bateman-type product formula (8.3.5) for the Askey-Wilson polynomials we shall now derive a product formula for the continuous q-ultraspherical polynomials in the integral form

$$
\begin{aligned}
&C_n(x;\beta|q)C_n(y;\beta|q) \\
&= \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-n/2}\int_{-1}^{1} K(x,y,z;\beta|q)C_n(z;\beta|q)\,dz,
\end{aligned} \tag{8.4.1}
$$

where

$$
\begin{aligned}
K(x,y,z;\beta|q) &= \frac{(q,\beta,\beta;q)_\infty \left|\left(\beta e^{2i\theta}, \beta e^{2i\phi};q\right)_\infty\right|^2}{2\pi(\beta^2;q)_\infty} \\
&\times w\left(z;\beta^{\frac{1}{2}}e^{i\theta+i\phi}, \beta^{\frac{1}{2}}e^{-i\theta-i\phi}, \beta^{\frac{1}{2}}e^{i\theta-i\phi}, \beta^{\frac{1}{2}}e^{i\phi-i\theta}\right)
\end{aligned} \tag{8.4.2}
$$

with $w(z;a,b,c,d)$ defined as in (6.3.1) and $x=\cos\theta, y=\cos\phi$.

First, we set $c=-a$ in (8.3.5) and rewrite it in the form

$$
\begin{aligned}
r_n(x;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}}|q)r_n(y;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}}|q) &= \frac{1+a^2q^n}{1+a^2}\left(-q^{-\frac{1}{2}}\right)^n \\
\times \sum_{m=0}^{n} \frac{\left(q^{-n}, a^4q^n, -a^2q^{\frac{1}{2}}e^{i\theta+i\phi}, -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -aq^{\frac{1}{2}}e^{-i\phi};q\right)_m}{\left(q, a^2q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, -a^2q, -a^3q^{\frac{1}{2}}e^{i\phi};q\right)_m} q^m \\
\times {}_{10}W_9\left(-a^3q^{-\frac{1}{2}}e^{i\phi}; ae^{i\phi}, -ae^{i\phi}, aq^{\frac{1}{2}}e^{i\phi}, ae^{i\theta}, ae^{-i\theta}, a^4q^m, q^{-m};q,q\right),
\end{aligned} \tag{8.4.3}
$$

where

$$r_n(x;a,b,c,d|q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right]. \tag{8.4.4}$$

The key step now is to use the $d = -(aq)^{\frac{1}{2}}$ case of Bailey's transformation formula (2.8.3) to transform the balanced ${}_{10}\phi_9$ series in (8.4.3) to a balanced ${}_4\phi_3$ series:

$$\begin{aligned}&{}_{10}W_9\left(-a^3q^{-\frac{1}{2}}e^{i\phi}; ae^{i\phi}, -ae^{i\phi}, aq^{\frac{1}{2}}e^{i\phi}, ae^{i\theta}, ae^{-i\theta}, a^4q^m, q^{-m}; q, q\right)\\ &= \frac{\left(a^2e^{-2i\phi}, -a^3q^{\frac{1}{2}}e^{i\phi}; q\right)_m}{\left(a^4, -aq^{\frac{1}{2}}e^{-i\phi}; q\right)_m}\\ &\times {}_4\phi_3\left[\begin{matrix} a^2e^{2i\phi}, -q^{\frac{1}{2}}e^{i\theta+i\phi}, -q^{\frac{1}{2}}e^{i\phi-i\theta}, q^{-m} \\ -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -a^2q^{\frac{1}{2}}e^{i\theta+i\phi}, q^{1-m}e^{2i\phi}/a^2 \end{matrix}; q, q\right].\end{aligned} \tag{8.4.5}$$

So (8.4.3) reduces to

$$\begin{aligned}&r_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)\ r_n(y; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q) = \frac{1+a^2q^n}{1+a^2}\left(-q^{-\frac{1}{2}}\right)^n\\ &\times \sum_{m=0}^{n} \frac{\left(q^{-n}, a^4q^n, a^2e^{-2i\phi}, -a^2q^{\frac{1}{2}}e^{i\theta+i\phi}, -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}; q\right)_m}{\left(q, a^4, a^2q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, -a^2q; q\right)_m} q^m\\ &\times {}_4\phi_3\left[\begin{matrix} a^2e^{2i\phi}, -q^{\frac{1}{2}}e^{i\theta+i\phi}, -q^{\frac{1}{2}}e^{i\phi-i\theta}, q^{-m} \\ -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -a^2q^{\frac{1}{2}}e^{i\theta+i\phi}, q^{1-m}e^{2i\phi}/a^2 \end{matrix}; q, q\right].\end{aligned} \tag{8.4.6}$$

Transforming this ${}_4\phi_3$ series by Sears' transformation formula (2.10.4), we obtain a further reduction

$$\begin{aligned}&r_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)\ r_n(y; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)\\ &= \frac{1+a^2q^n}{1+a^2}\left(-q^{-\frac{1}{2}}\right)^n \sum_{m=0}^{n} \frac{\left(q^{-n}, a^4q^n, -q^{\frac{1}{2}}e^{i\theta-i\phi}, -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}; q\right)_m}{\left(q, a^2q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, -a^2q; q\right)_m} q^m\\ &\times {}_4\phi_3\left[\begin{matrix} q^{-m}, a^2, a^2e^{2i\phi}, a^2e^{-2i\theta} \\ a^4, -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -q^{\frac{1}{2}-m}e^{i\phi-i\theta} \end{matrix}; q, q\right].\end{aligned} \tag{8.4.7}$$

Now observe that, by (6.1.1),

$$\begin{aligned}&\int_{-1}^{1} w\left(z; ae^{i\phi-i\theta}, ae^{i\theta-i\phi}, ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}\right)\left(ae^{i\phi-i\theta+i\psi}, ae^{i\phi-i\theta-i\psi}; q\right)_j dz\\ &= \int_{-1}^{1} w\left(z; aq^je^{i\phi-i\theta}, ae^{i\theta-i\phi}, ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}\right) dz\\ &= \frac{2\pi(a^4;q)_\infty}{(q, a^2, a^2; q)_\infty |(a^2e^{2i\phi}, a^2e^{-2i\theta}; q)_\infty|^2} \frac{\left(a^2, a^2e^{2i\phi}, a^2e^{-2i\theta}; q\right)_j}{(a^4;q)_j},\end{aligned} \tag{8.4.8}$$

where $|a| < 1$ and $z = \cos\psi$. Hence

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-m}, a^2, a^2e^{2i\phi}, a^2e^{-2i\theta} \\ a^4, -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -q^{\frac{1}{2}-m}e^{i\phi-i\theta} \end{matrix}; q, q\right] \\
&= \frac{\left(q, a^2, a^2; q\right)_\infty \left|\left(a^2e^{2i\phi}, a^2e^{-2i\theta}; q\right)_\infty\right|^2}{2\pi(a^4; q)_\infty} \\
&\quad \times \int_{-1}^{1} w\left(z; ae^{i\phi-i\theta}, ae^{i\theta-i\phi}, ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}\right) \\
&\quad \times {}_3\phi_2\left[\begin{matrix} q^{-m}, ae^{i\phi-i\theta+i\psi}, ae^{i\phi-i\theta-i\psi} \\ -a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -q^{\frac{1}{2}-m}e^{i\phi-i\theta} \end{matrix}; q, q\right] dz \\
&= \frac{\left(q, a^2, a^2; q\right)_\infty \left|\left(a^2e^{2i\phi}, a^2e^{-2i\theta}; q\right)_\infty\right|^2}{2\pi(a^4; q)_\infty} \\
&\quad \times \int_{-1}^{1} w\left(z; ae^{i\phi-i\theta}, ae^{i\theta-i\phi}, ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}\right) \\
&\quad \times \frac{\left(-aq^{\frac{1}{2}}e^{i\psi}, -aq^{\frac{1}{2}}e^{-i\psi}; q\right)_m}{\left(-a^2q^{\frac{1}{2}}e^{i\phi-i\theta}, -q^{\frac{1}{2}}e^{i\theta-i\phi}; q\right)_m} dz. \qquad (8.4.9)
\end{aligned}
$$

Substituting (8.4.9) into (8.4.7) and using (2.10.4) we finally obtain

$$
\begin{aligned}
&r_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)\; r_n(y; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q) \\
&= \int_{-1}^{1} K(x, y, z; a^2|q)\; r_n(z; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)\, dz. \qquad (8.4.10)
\end{aligned}
$$

This yields (8.4.1) if we replace a by $\beta^{\frac{1}{2}}$ and use (7.4.14). By setting $\beta = q^\lambda$ in (8.4.1) and taking the limit $q \to 1$, Rahman and Verma [1986b] showed that (8.4.1) tends to Gegenbauer's [1874] product formula

$$
\frac{C_n^\lambda(x)C_n^\lambda(y)}{C_n^\lambda(1)C_n^\lambda(1)} = \int_{-1}^{1} K(x, y, z)\frac{C_n^\lambda(z)}{C_n^\lambda(1)}dz, \qquad (8.4.11)
$$

where

$$
K(x, y, z) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\frac{\left(1-x^2-y^2-z^2+2xyz\right)^{\lambda-1}}{\left[(1-x^2)(1-y^2)\right]^{\lambda-\frac{1}{2}}} \text{ or } 0, \qquad (8.4.12)
$$

according as $1 - x^2 - y^2 - z^2 + 2xyz$ is positive or negative.

Rahman and Verma [1986b] were also able to derive an addition formula for the continuous q-ultraspherical polynomials corresponding to the product formula (8.4.1). This is left as an exercise (Ex. 8.11).

8.5 Rogers' linearization formula for the continuous q-ultraspherical polynomials

Rogers [1895] used an induction argument to prove the linearization formula

$$\begin{aligned} &C_m(x;\beta|q)C_n(x;\beta|q) \\ &= \sum_{k=0}^{\min(m,n)} \frac{(q;q)_{m+n-2k}(\beta;q)_{m-k}(\beta;q)_{n-k}(\beta;q)_k(\beta^2;q)_{m+n-k}}{(\beta^2;q)_{m+n-2k}(q;q)_{m-k}(q;q)_{n-k}(q;q)_k(\beta q;q)_{m+n-k}} \\ &\times \frac{\left(1-\beta q^{m+n-2k}\right)}{(1-\beta)} C_{m+n-2k}(x;\beta|q). \end{aligned} \tag{8.5.1}$$

Different proofs of (8.5.1) have been given by Bressoud [1981d], Rahman [1981] and Gasper [1985]. We shall give Gasper's proof since it appears to be the simplest.

We use (7.4.2) for $C_n(x;\beta|q)$ and, via Heine's transformation formula (1.4.3),

$$\begin{aligned} C_m(x;\beta|q) = &\frac{\left(\beta e^{-2i\theta};q\right)_\infty}{\left(q\beta^{-1}e^{-2i\theta};q\right)_\infty} \frac{(\beta;q)_m}{(q;q)_m} e^{im\theta} \\ &\times {}_2\phi_1\left(q\beta^{-1},\beta^{-2}q^{1-m};\beta^{-1}q^{1-m};q,\beta e^{-2i\theta}\right), \end{aligned} \tag{8.5.2}$$

where $x=\cos\theta$. Then, temporarily assuming that $|q|<|\beta|<1$, we have

$$\begin{aligned} &C_m(x;\beta|q)C_n(x;\beta|q) = A_{m,n} \sum_{r=0}^{n} \frac{\left(q^{-n},\beta;q\right)_r}{\left(q,\beta^{-1}q^{1-n};q\right)_r} \left(q\beta^{-1}e^{-2i\theta}\right)^r \\ &\times \sum_{s=0}^{\infty} \frac{\left(q\beta^{-1},\beta^{-2}q^{1-m};q\right)_s}{\left(q,\beta^{-1}q^{1-m};q\right)_s} \left(\beta e^{-2i\theta}\right)^s \\ &= A_{m,n} \sum_{k=0}^{\infty} \frac{\left(q\beta^{-1},\beta^{-2}q^{1-m};q\right)_k}{\left(q,\beta^{-1}q^{1-m};q\right)_k} \left(\beta e^{-2i\theta}\right)^k \\ &\times {}_4\phi_3\left[\begin{matrix} q^{-k},q^{-n},\beta,\beta q^{m-k} \\ \beta^2 q^{m-k},\beta q^{-k},\beta^{-1}q^{1-n} \end{matrix};q,q\right], \end{aligned} \tag{8.5.3}$$

where

$$A_{m,n} = \frac{\left(\beta e^{-2i\theta};q\right)_\infty}{\left(q\beta^{-1}e^{-2i\theta};q\right)_\infty} \frac{(\beta;q)_m(\beta;q)_n}{(q;q)_m(q;q)_n} e^{i(m+n)\theta}. \tag{8.5.4}$$

The crucial point here is that the ${}_4\phi_3$ series in (8.5.3) is balanced and so, by (2.5.1),

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-k},q^{-n},\beta,\beta q^{m-k} \\ \beta^2 q^{m-k},\beta q^{-k},\beta^{-1}q^{1-n} \end{matrix};q,q\right] = \frac{\left(\beta^{-2}q^{1-m-n},\beta^{-1}q^{1-m};q\right)_k}{\left(\beta^{-1}q^{1-m-n},\beta^{-2}q^{1-m};q\right)_k} \\ &\times {}_8W_7\left(\beta^{-1}q^{-m-n};\beta,\beta^{-2}q^{1+k-m-n},q^{-m},q^{-n},q^{-k};q,q\beta^{-1}\right). \end{aligned} \tag{8.5.5}$$

Substituting this into (8.5.3) and interchanging the order of summation, we obtain

$$\begin{aligned}
&C_m(x;\beta|q)C_n(x;\beta|q) = A_{m,n} \\
&\times \sum_{k=0}^{\min(m,n)} \frac{\left(\beta^{-1}q^{-m-n};q\right)_k\left(1-\beta^{-1}q^{2k-m-n}\right)\left(\beta,q^{-m},q^{-n};q\right)_k}{(q;q)_k\left(1-\beta^{-1}q^{-m-n}\right)\left(\beta^{-2}q^{1-m-n},\beta^{-1}q^{1-n},\beta^{-1}q^{1-m};q\right)_k} \\
&\times \frac{\left(\beta^{-2}q^{1-m-n};q\right)_{2k}}{\left(\beta^{-1}q^{1-m-n};q\right)_{2k}}\left(q\beta^{-1}e^{-2i\theta}\right)^k \\
&\times {}_2\phi_1\left(q\beta^{-1},\beta^{-2}q^{1+2k-m-n};\beta^{-1}q^{1+2k-m-n};q,\beta e^{-2i\theta}\right),
\end{aligned} \tag{8.5.6}$$

which gives (8.5.1) by using (8.5.2) and observing that both sides of (8.5.1) are polynomials in x. Notice that the linearization coefficients in (8.5.1) are nonnegative when $-1<\beta<1$ and $-1<q<1$.

For an extension of the linearization formula to the continuous q-Jacobi polynomials, see Ex. 8.24.

8.6 The Poisson kernel for $C_n(x;\beta|q)$

For a system of orthogonal polynomials $\{p_n(x)\}$ which satisfies an orthogonality relation of the form (7.1.6), the bilinear generating function

$$K_t(x,y) = \sum_{n=0}^{\infty} h_n p_n(x)p_n(y)t^n \tag{8.6.1}$$

is called a Poisson kernel for these polynomials provided that $h_n = cv_n$ for some constant $c>0$. The Poisson kernel for the continuous q-ultraspherical polynomials is defined by

$$K_t(x,y;\beta|q) = \sum_{n=0}^{\infty} \frac{(q;q)_n(1-\beta q^n)}{(\beta^2;q)_n(1-\beta)} C_n(x;\beta|q)C_n(y;\beta|q)t^n, \tag{8.6.2}$$

where $|t|<1$.

Gasper and Rahman [1983a] used (8.5.1) to show that

$$\begin{aligned}
K_t(x,y;\beta|q) &= \frac{(\beta,t^2;q)_\infty}{(\beta^2,q\beta t^2;q)_\infty}\left|\frac{\left(\beta te^{i\theta+i\phi},q\beta te^{i\theta-i\phi};q\right)_\infty}{\left(te^{i\theta+i\phi},te^{i\theta-i\phi};q\right)_\infty}\right|^2 \\
&\times {}_8\phi_7\left[\begin{matrix}\beta t^2, & qt\beta^{\frac12}, & -qt\beta^{\frac12}, & qte^{i\theta+i\phi}, & qte^{-i\theta-i\phi}, \\ & t\beta^{\frac12}, & -t\beta^{\frac12}, & \beta te^{-i\theta-i\phi}, & \beta te^{i\theta+i\phi},\end{matrix}\right. \\
&\qquad \left.\begin{matrix} te^{i\theta-i\phi}, & te^{i\phi-i\theta}, & \beta \\ q\beta te^{i\phi-i\theta}, & q\beta te^{i\theta-i\phi}, & qt^2\end{matrix};q,\beta\right],
\end{aligned} \tag{8.6.3}$$

where $x=\cos\theta, y=\cos\phi$ and $\max(|q|,|t|,|\beta|)<1$. They also computed a closely related kernel

$$L_t(x,y;\beta|q) = \sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} C_n(x;\beta|q)C_n(y;\beta|q)t^n$$

$$= \frac{(\beta, t^2; q)_\infty}{(\beta^2, \beta t^2; q)_\infty} \left| \frac{\left(\beta te^{i\theta+i\phi}, \beta te^{i\theta-i\phi}; q\right)_\infty}{\left(te^{i\theta+i\phi}, te^{i\theta-i\phi}; q\right)_\infty} \right|^2$$

$$\times {}_8\phi_7 \left[\begin{matrix} \beta t^2 q^{-1}, & t(q\beta)^{\frac{1}{2}}, & -t(q\beta)^{\frac{1}{2}}, & te^{i\theta+i\phi}, & te^{-i\theta-i\phi}, \\ & t(\beta q^{-1})^{\frac{1}{2}}, & -t(\beta q^{-1})^{\frac{1}{2}}, & \beta te^{-i\theta-i\phi}, & \beta te^{i\theta+i\phi}, \end{matrix} \right.$$

$$\left. \begin{matrix} te^{i\theta-i\phi}, & te^{i\phi-i\theta}, & \beta \\ \beta te^{i\phi-i\theta}, & \beta te^{i\theta-i\phi}, & t^2 \end{matrix} ; q, \beta \right]. \tag{8.6.4}$$

Alternative derivations of (8.6.3) and (8.6.4) were given by Rahman and Verma [1986a]. In view of the product formula (8.4.1), however, one can now give a simpler proof. Let us assume, for the moment, that $|t\beta^{-\frac{1}{2}}| < 1$ and $|\beta| < 1$. Then, by (7.4.1) and (8.4.1), we find that, with $z = \cos\psi$,

$$\begin{aligned} L_t(x, y; \beta|q) &= \int_{-1}^{1} K(x, y, z; \beta|q) \frac{\left(t\beta^{\frac{1}{2}} e^{i\psi}, t\beta^{\frac{1}{2}} e^{-i\psi}; q\right)_\infty}{\left(t\beta^{-\frac{1}{2}} e^{i\psi}, t\beta^{-\frac{1}{2}} e^{-i\psi}; q\right)_\infty} dz \\ &= \frac{(q, \beta, \beta; q)_\infty | \left(\beta e^{2i\theta}, \beta e^{2i\phi}; q\right)_\infty |^2}{2\pi(\beta^2; q)_\infty} \\ &\quad \times \int_0^\pi \frac{h\left(\cos\psi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, t\beta^{\frac{1}{2}}\right)}{h\left(\cos\psi; \beta^{\frac{1}{2}} e^{i\theta+i\phi}, \beta^{\frac{1}{2}} e^{-i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\phi-i\theta}, t\beta^{-\frac{1}{2}}\right)} d\psi. \end{aligned} \tag{8.6.5}$$

By (6.3.9),

$$\begin{aligned} &\int_0^\pi \frac{h\left(\cos\psi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, t\beta^{\frac{1}{2}}\right)}{h\left(\cos\psi; \beta^{\frac{1}{2}} e^{i\theta+i\phi}, \beta^{\frac{1}{2}} e^{-i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\phi-i\theta}, t\beta^{-\frac{1}{2}}\right)} d\psi \\ &= \frac{2\pi \left(\beta, t^2; q\right)_\infty \left|\left(\beta te^{i\theta+i\phi}, \beta te^{i\theta-i\phi}; q\right)_\infty\right|^2}{(q, \beta, \beta, \beta t^2; q)_\infty \left|\left(\beta e^{2i\theta}, \beta e^{2i\phi}, te^{i\theta+i\phi}, te^{i\theta-i\phi}; q\right)_\infty\right|^2} \\ &\quad \times {}_8W_7\left(\beta t^2 q^{-1}; te^{i\theta+i\phi}, te^{-i\theta-i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, \beta; q, \beta\right). \end{aligned} \tag{8.6.6}$$

Formula (8.6.4) follows immediately from (8.6.5) and (8.6.6).

It is slightly more complicated to compute (8.6.1). Consider the generating function

$$\begin{aligned} G_t(z) &= \sum_{n=0}^{\infty} \frac{1-\beta q^n}{1-\beta} C_n(z; \beta|q) t^n \\ &= \frac{\left(\beta te^{i\psi}, \beta te^{-i\psi}; q\right)_\infty}{(1-\beta)(te^{i\psi}, te^{-i\psi}; q)_\infty} - \frac{\beta}{1-\beta} \frac{\left(\beta qte^{i\psi}, \beta qte^{-i\psi}; q\right)_\infty}{\left(qte^{i\psi}, qte^{-i\psi}; q\right)_\infty} \\ &= (1-\beta t^2) \frac{\left(\beta qte^{i\psi}, \beta qte^{-i\psi}; q\right)_\infty}{\left(te^{i\psi}, te^{-i\psi}; q\right)_\infty}. \end{aligned} \tag{8.6.7}$$

Then

$$\begin{aligned} K_t(x, y; \beta|q) &= \int_{-1}^{1} K(x, y, z; \beta|q)\, G_{t\beta^{-\frac{1}{2}}}(z)\, dz \\ &= \frac{(1-t^2)(q, \beta, \beta; q)_\infty \left|(\beta e^{2i\theta}, \beta e^{2i\phi}; q)_\infty\right|^2}{2\pi(\beta^2; q)_\infty} \\ &\times \int_0^\pi \frac{h\left(\cos\psi; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, qt\beta^{\frac{1}{2}}\right)}{h\left(\cos\psi; \beta^{\frac{1}{2}} e^{i\theta+i\phi}, \beta^{\frac{1}{2}} e^{-i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\theta-i\phi}, \beta^{\frac{1}{2}} e^{i\phi-i\theta}, t\beta^{-\frac{1}{2}}\right)} d\psi. \end{aligned} \tag{8.6.8}$$

This gives (8.6.3) via (7.4.1) and an application of (2.10.1).

By analytic continuation, formulas (8.6.3) and (8.6.4) hold when $\max(|q|, |t|, |\beta|) < 1$.

Even though it is clear from (8.6.3) and (8.6.4) that these kernels are positive when $-1 < q, t < 1$ and $-1 \le x, y \le 1$ if $0 \le \beta < 1$, it is not clear what happens when $-1 < \beta < 0$ since both ${}_8\phi_7$ series in (8.6.3) and (8.6.4) become alternating series. It is shown in Gasper and Rahman [1983a] that the Poisson kernel $K_t(x, y; \beta|q)$ is also positive for $-1 < t < 1$ when $-1 < q < \beta < 0$ and when $2^{3/2} - 3 \le \beta < 0,\ -1 < q \le 0$.

For the nonnegativity of the Poisson kernel for the continuous q-Jacobi polynomials, see Gasper and Rahman [1986].

8.7 Poisson kernels for the q-Racah polynomials

For the q-Racah polynomials

$$W_n(x; q) \equiv W_n(x; a, b, c, N; q)$$

we shall give conditions under which the Poisson kernel

$$\sum_{n=0}^{N} h_n(q) W_n(x; q) W_n(y; q) t^n, \quad 0 \le t < 1, \tag{8.7.1}$$

and the so-called discrete Poisson kernel

$$\sum_{n=0}^{z} \frac{(q^{-z}; q)_n}{(q^{-N}; q)_n} h_n(q) W_n(x; q) W_n(y; q), \tag{8.7.2}$$

$z = 0, 1, \ldots, N$, are nonnegative for $x, y = 0, 1, \ldots, N$.

Let us first consider a more general bilinear sum

$$\begin{aligned} P_z(x, y) &\equiv P_z(x, y; a, b, c, \alpha, \gamma, K, M, N; q) \\ &= \sum_{n=0}^{z} \frac{(q^{-z}; q)_N}{(q^{-K}; q)_n} h_n(a, b, c, N; q) W_n(x; a, b, c, N; q) \\ &\quad \times W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q), \end{aligned} \tag{8.7.3}$$

where $z = 0, 1, \ldots, \min(K, N)$ and $N \le M$. If $\alpha = a, \gamma = c$ and $M = N$, then (8.7.3) reduces to (8.7.2) when $K = N$, and it has the Poisson kernel (8.7.1) as a limit case.

From the product formula (8.2.5) it follows that

$$\begin{aligned} &W_n(x; a, b, c, N; q) W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q) \\ &= \frac{(bq, aq/c; q)_n}{(aq, bc; q)_n} c^n \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{\left(q^{-n}, abq^{n+1}; q\right)_{r+s} \left(q^{-x}, cq^{x-N}, q^{-y}, \gamma q^{y-N}; q\right)_r}{(q^{-N}; q)_{r+s} \left(q, \alpha q, q^{-M}, ab\gamma q\alpha^{-1}, cq^{1-s}; q\right)_r} \\ &\quad \times \frac{\left(q^{x-N}, c^{-1}q^{1-x}; q\right)_s}{(q, bq, ac^{-1}q; q)_s} \frac{1 - cq^{r-s}}{1 - cq^{-s}} q^{r+s-rs} A_{r,s}, \end{aligned} \tag{8.7.4}$$

where

$$A_{r,s} = {}_5\phi_4 \left[\begin{matrix} q^{-s}, q^{r-y}, \gamma q^{r+y-M}, aq^{r+1}, bcq^{r+1} \\ \alpha q^{r+1}, q^{r-M}, ab\gamma\alpha^{-1}q^{r+1}, cq^{1+r-s} \end{matrix} ; q, q \right]. \tag{8.7.5}$$

Using (8.7.5) in (8.7.4) and changing the order of summation, we find that

$$\begin{aligned} P_z(x, y) &= \frac{\left(bq, ac^{-1}q; q\right)_N}{(abq^2, c^{-1}; q)_N} \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(q^{-z}; q)_{r+s} (abq^2; q)_{2r+2s}}{(q^{-K}, abq^{N+2}; q)_{r+s}} \\ &\quad \times \frac{\left(q^{-x}, cq^{x-N}, q^{-y}, \gamma q^{y-M}; q\right)_r \left(q^{x-N}, c^{-1}q^{-x}; q\right)_s}{\left(q, \alpha q, q^{-M}, ab\gamma q\alpha^{-1}, cq^{1-s}; q\right)_r \left(q, bq, ac^{-1}q; q\right)_s} \frac{1 - cq^{r-s}}{1 - cq^{-s}} \\ &\quad \times (-1)^{r+s} q^{(r+s)(2N-r-s+1)/2 - rs} A_{r,s} B_{r,s}, \end{aligned} \tag{8.7.6}$$

where

$$B_{r,s} = {}_5\phi_4 \left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, q^{r+s-N}, q^{r+s-z} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, abq^{N+r+s+2}, q^{r+s-K} \end{matrix} ; q, q^{N-r-s} \right], \tag{8.7.7}$$

with $\lambda = abq^{2r+2s+1}$. We shall now show that when $K = N$,

$$B_{r,s} = \frac{\left(q^{N-z}; q\right)_{z-r-s}}{\left(abq^{N+r+s+2}; q\right)_{z-r-s}} \, {}_2\phi_1 \left(q^{r+s-z}, q^{1+r+s-z}; q^{1+N-z}; q, abq^{N+z+2}\right). \tag{8.7.8}$$

To prove (8.7.8) it suffices to show that

$$\begin{aligned} &{}_4\phi_3 \left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & w \end{matrix} ; q, \frac{w}{aq} \right] \\ &= \frac{(wb/aq, w/b; q)_\infty}{(w/aq, w; q)_\infty} \, {}_2\phi_1(b, bq; wb/a; q, w/b) \end{aligned} \tag{8.7.9}$$

whether or not b is a negative integer power of q, provided the series on both sides converge. Since $1 - aq^{2k} = 1 - q^k + q^k(1 - aq^k)$, the left side of (8.7.9) equals

$$\begin{aligned} &{}_2\phi_1(aq, b; w; q, w/a) + \frac{w(1-b)}{aq(1-w)} \, {}_2\phi_1(aq, bq; wq; q, w/aq) \\ &= \frac{(wb/a, w/b; q)_\infty}{(w/a, w; q)_\infty} \, {}_2\phi_1(b, bq; wb/a; q, w/b) \end{aligned}$$

$$+\frac{w(1-b)(wb/a, w/b;q)_\infty}{aq(w/aq, w;q)_\infty}\ {}_2\phi_1(b, bq; wb/a; q, w/b)$$

$$=\frac{(wb/aq, w/b;q)_\infty}{(w/aq, w;q)_\infty}\ {}_2\phi_1(b, bq; wb/a; q, w/b) \tag{8.7.10}$$

by (1.4.5). Also, for $\alpha = a$ the ${}_5\phi_4$ series in (8.7.5) reduces to a ${}_4\phi_3$ series which, by (2.10.4), equals

$$\frac{\left(q^{M+1-y-s}, b^{-1}\gamma^{-1}q^{-y-s};q\right)_s}{(q^{r-M}, b\gamma q^{r+1};q)_s}\left(b\gamma q^{r+s+y-M}\right)^s$$

$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-s}, q^{r-y}, b^{-1}q^{-s}, c\gamma^{-1}q^{M+1-y-s} \\ cq^{1+r-s}, q^{M+1-y-s}, b^{-1}\gamma^{-1}q^{-y-s}\end{matrix}; q, q\right]. \tag{8.7.11}$$

From (8.7.6), (8.7.8) and (8.7.11) it follows that

$$P_z(x, y; a, b, c, a, \gamma, N, M, N; q) \ge 0 \tag{8.7.12}$$

for $x = 0, 1, \ldots, N,\ y = 0, 1, \ldots, M,\ z = 0, 1, \ldots, N$ when $0 < q < 1$, $0 < aq < 1$, $0 \le bq < 1$, $0 < c < aq^N$ and $cq \le \gamma < q^{M-1} \le q^{N-1}$. Hence the discrete Poisson kernel (8.7.2) is nonnegative for $x, y, z = 0, 1, \ldots, N$ when $0 < q < 1$, $0 < aq < 1$, $0 \le bq < 1$ and $0 < c < aq^N$.

If in (8.7.3) we write the sum with N as the upper limit of summation, replace $(q^{-z};q)_n$ by $(tq^{-K};q)_n$ and let $K \to \infty$, it follows from (8.7.6) that

$$\begin{aligned} &L_t(x, y; a, b, c, \alpha, \gamma, M, N; q) \\ &= \sum_{n=0}^{N} t^n h_n(a, b, c, N; q) W_n(x; a, b, c, N; q) W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q) \\ &= \frac{(bq, aq/c;q)_N}{(abq^2, c^{-1};q)_N}\sum_{r=0}^{x}\sum_{s=0}^{N-x}\frac{(abq^2;q)_{2r+2s}\left(q^{-x}, cq^{x-N}, q^{-y}, \gamma q^{y-M};q\right)_r}{(abq^{N+2};q)_{r+s}\,(q, \alpha q, q^{-M}, ab\gamma q\alpha^{-1}, cq^{1-s};q)_r} \\ &\quad\times\frac{\left(q^{x-N}, c^{-1}q^{-x};q\right)_s (1-cq^{r-s})}{(q, bq, ac^{-1}q;q)_s(1-cq^{-s})}A_{r,s}C_{r,s}(-t)^{r+s}q^{(r+s)(2N-r-s+1)/2-rs}, \end{aligned} \tag{8.7.13}$$

for $x = 0, 1, \ldots, N, y = 0, 1, \ldots, M$ with $A_{r,s}$ defined in (8.7.5) and

$$C_{r,s} = {}_4\phi_3\left[\begin{matrix} \lambda, q\lambda^{\frac12}, -q\lambda^{\frac12}, q^{r+s-N} \\ \lambda^{\frac12}, -\lambda^{\frac12}, abq^{N+r+s+2}\end{matrix}; q, tq^{N-r-s}\right], \tag{8.7.14}$$

where $\lambda = abq^{2r+2s+1}$. However, by Ex. 2.2,

$${}_4\phi_3\left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & b^{-1} \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & \lambda bq\end{matrix}; q, tb\right]$$

$$=\frac{(t, \lambda q;q)_\infty}{(tb, \lambda bq;q)_\infty}\ {}_2\phi_1(b, tb; tq; q, \lambda q) \tag{8.7.15}$$

for $\max(|tb|, |\lambda q|) < 1$. Use of this in (8.7.14) yields

$$C_{r,s} = (t, abq^{2r+2s+2};q)_{N-r-s}\ {}_2\phi_1\left(q^{N-r-s}, tq^{N-r-s}; tq; q, abq^{2r+2s+2}\right), \tag{8.7.16}$$

from which it is obvious that $C_{r,s} \geq 0$ for $0 \leq t < 1$, $r + s \leq N$ when $0 \leq abq^2 < 1$. Combining this with our previous observation that $A_{r,s}$ equals the expression in (8.7.11) when $\alpha = a$, it follows from (8.7.13) that

$$L_t(x, y; a, b, c, a, \gamma, M, N; q) > 0 \qquad (8.7.17)$$

for $x = 0, 1, \ldots, N$, $y = 0, 1, \ldots, M$, $0 \leq t < 1$ when $0 < q < 1$, $0 < aq < 1$, $0 \leq bq < 1$, $0 < c < aq^N$ and $cq \leq \gamma < q^{M-1} \leq q^{N-1}$. In particular, the Poisson kernel (8.7.1) is positive for $x, y = 0, 1, \ldots, N$, $0 \leq t < 1$ when $0 < q < 1$, $0 < aq < 1$, $0 \leq bq < 1$ and $0 < c < aq^N$.

For further details on the nonnegative bilinear sums of discrete orthogonal polynomials, see Gasper and Rahman [1984] and Rahman [1982].

8.8 q-analogues of Clausen's formula

Clausen's [1828] formula

$$\left\{ {}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; z\right]\right\}^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+\frac{1}{2} \end{matrix}; z\right], \qquad (8.8.1)$$

where $|z| < 1$, provides a rare example of the square of a hypergeometric series that is expressible as a hypergeometric series. Ramanujan's [1927, pp. 23–39] rapidly convergent series representations of $1/\pi$, which have been used to compute π to millions of decimal digits, are based on special cases of (8.8.1); see the Chudnovskys' [1988] survey paper. Clausen's formula was used in Askey and Gasper [1976] to prove that

$${}_3F_2\left[\begin{matrix} -n, n+\alpha+2, \frac{1}{2}(\alpha+1) \\ \alpha+1, \frac{1}{2}(\alpha+3) \end{matrix}; \frac{1-x}{2}\right] \geq 0 \qquad (8.8.2)$$

when $\alpha > -2$, $-1 \leq x \leq 1$, $n = 0, 1, \ldots$, which was then used to prove the positivity of certain important kernels involving sums of Jacobi polynomials; see Askey [1975] and the extensions in Gasper [1975a, 1977]. The special cases $\alpha = 2, 4, 6, \ldots$ of (8.8.2) turned out to be the inequalities de Branges [1985] needed to complete the last step in his celebrated proof of the Bieberbach conjecture. In this section we consider q-analogues of (8.8.1).

Jackson [1940, 1941] derived the product formula given in Ex. 3.11 and additional proofs of it have been given by Singh [1959], Nassrallah [1982], and Jain and Srivastava [1986]. But, unfortunately, the left side of it is not a square and so Jackson's formula cannot be used to write certain basic hypergeometric series as sums of squares as was done with Clausen's formula in Askey and Gasper [1976] to prove (8.8.2).

In order to obtain a q-analogue of Clausen's formula which expressed the square of a basic hypergeometric series as a basic hypergeometric series, the authors derived the formula

$$\left\{ {}_4\phi_3\left[\begin{matrix} a, b, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, abz, ab/z \\ a^2b^2, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right], \qquad (8.8.3)$$

which holds when the series terminate. See Gasper [1989b], where it was pointed out that there are several ways of proving (8.8.3), such as using the Rogers' linearization formula (8.5.1), the product formula in §8.2, or the Rahman and Verma integral (8.4.10).

In this section we derive a nonterminating q-analogue of Clausen's formula which reduces to (8.8.3) when it terminates. The key to the discovery of this formula is the observation that the proof of Rogers' linearization formula given in §8.5 is independent of the fact that the parameter n in the ${}_2\phi_1$ series in (7.4.2) is a nonnegative integer. In view of (7.4.2) let

$$f(z) = {}_2\phi_1(\alpha, \beta; \alpha q/\beta; q, zq/\beta), \tag{8.8.4}$$

which reduces to the ${}_2\phi_1$ series in (7.4.2) when $\alpha = q^{-n}$ and $z = e^{-2i\theta}$. Temporarily assume that $|q| < |\beta| < 1$ and $|z| \le 1$. From Heine's transformation (1.4.3),

$$f(z) = \frac{(\beta z; q)_\infty}{(zq/\beta; q)_\infty}\, {}_2\phi_1(\alpha q/\beta^2, q/\beta; \alpha q/\beta; q, \beta z). \tag{8.8.5}$$

Hence, if we multiply the two ${}_2\phi_1$ series in (8.8.4) and (8.8.5) and collect the coefficients of z^j, we get

$$f^2(z) = \frac{(\beta z; q)_\infty}{(zq/\beta; q)_\infty} \sum_{k=0}^{\infty} A_k \frac{(\alpha q/\beta^2, q/\beta; q)_k}{(q, \alpha q/\beta; q)_k} (\beta z)^k, \tag{8.8.6}$$

where

$$A_k = {}_4\phi_3\left[\begin{matrix} q^{-k}, \beta, \beta q^{-k}/\alpha, \alpha \\ \beta q^{-k}, \beta^2 q^{-k}/\alpha, \alpha q/\beta \end{matrix}; q, q\right] \tag{8.8.7}$$

is a terminating balanced series. As in (8.5.5) we now apply (2.5.1) to the ${}_4\phi_3$ series in (8.8.7) to obtain that

$$A_k = \frac{(\alpha q/\beta, \alpha^2 q/\beta^2; q)_\infty}{(\alpha^2 q/\beta, \alpha q/\beta^2; q)_\infty}\, {}_8W_7(\alpha^2/\beta; \alpha, \alpha, \beta, \alpha^2 q^{k+1}/\beta^2, q^{-k}; q, q/\beta). \tag{8.8.8}$$

Using (8.8.8) in (8.8.6) and changing the order of summation we get the formula

$$\begin{aligned} f^2(z) = \frac{(\beta z; q)_\infty}{(zq/\beta; q)_\infty} &\sum_{k=0}^{\infty} \frac{(1-\alpha^2 q^{2k}/\beta)(\alpha^2/\beta, \alpha, \alpha, \beta; q)_k}{(1-\alpha^2/\beta)(q, \alpha q/\beta, \alpha q/\beta, \alpha^2 q/\beta^2; q)_k} \\ &\times \frac{(\alpha^2 q/\beta^2; q)_{2k}}{(\alpha^2 q/\beta; q)_{2k}} \left(\frac{zq}{\beta}\right)^k {}_2\phi_1\left[\begin{matrix} \alpha^2 q^{2k+1}/\beta^2, q/\beta \\ \alpha^2 q^{2k+1}/\beta \end{matrix}; q, \beta z\right]. \end{aligned} \tag{8.8.9}$$

Observe that since the ${}_2\phi_1$ series in (8.8.9) is well-poised we may transform it by applying the quadratic transformation formula (3.4.7) to express it as an ${}_8\phi_7$ series and then apply (2.10.10) to get the transformation formula

$$\begin{aligned} {}_2\phi_1\left[\begin{matrix} \alpha^2 q^{2k+1}/\beta^2, q/\beta \\ \alpha^2 q^{2k+1}/\beta \end{matrix}; q, \beta z\right] &= \frac{(\alpha z q/\beta; q)_\infty (-\alpha/\beta z)^k q^{\binom{k+1}{2}}}{(\beta z/\alpha; q)_\infty (\alpha q/\beta z, \alpha z q/\beta; q)_k} \\ \times\ {}_4\phi_3&\left[\begin{matrix} \alpha q^k, \alpha q^{k+\frac{1}{2}}/\beta, -\alpha q^{k+\frac{1}{2}}/\beta, -\alpha q^{k+1}/\beta \\ \alpha^2 q^{2k+1}/\beta, \alpha z q^{k+1}/\beta, \alpha q^{k+1}/\beta z \end{matrix}; q, q\right] \end{aligned}$$

$$+\frac{(zq,\alpha,\alpha zq,\alpha^2q/\beta^2;q)_\infty}{(\beta z,\alpha q/\beta,\alpha/\beta z,\alpha^2q/\beta;q)_\infty}\frac{(\alpha q/\beta,\alpha/\beta z;q)_k(\alpha^2q/\beta;q)_{2k}}{(\alpha,\alpha zq;q)_k(\alpha^2q/\beta^2;q)_{2k}}$$
$$\times\ {}_4\phi_3\left[\begin{matrix}\beta z, zq^{\frac12},-zq^{\frac12},-zq\\ qz^2,\alpha zq^{k+1},\beta zq^{1-k}/\alpha\end{matrix};q,q\right]. \tag{8.8.10}$$

We can now substitute (8.8.10) into (8.8.9) and change the orders of summation to find that

$$f^2(z)=\frac{(\beta z,\alpha zq/\beta;q)_\infty}{(zq/\beta,\beta z/\alpha;q)_\infty}\sum_{m=0}^\infty\frac{(\alpha;q)_m(\alpha^2q/\beta^2;q)_{2m}}{(q,\alpha zq/\beta,\alpha q/\beta z,\alpha^2q/\beta,\alpha q/\beta;q)_m}$$
$$\times q^m\ {}_6W_5(\alpha^2/\beta;\alpha,\beta,q^{-m};q,\alpha q^{m+1}/\beta^2)$$
$$+\frac{(\alpha,\alpha^2q/\beta,zq,\alpha zq;q)_\infty}{(\alpha q/\beta,\alpha^2q/\beta,zq/\beta,\alpha/\beta z;q)_\infty}\sum_{m=0}^\infty\frac{(\beta z,zq^{\frac12},-zq^{\frac12},-zq;q)_m}{(q,qz^2,\alpha zq,\beta zq/\alpha;q)_m}q^m$$
$$\times\ {}_6W_5(\alpha^2/\beta;\alpha,\beta,\alpha q^{-m}/\beta z;q,zq^{m+1}/\beta). \tag{8.8.11}$$

Summing the above ${}_6W_5$ series by means of (2.7.1), we obtain the formula

$$\{{}_2\phi_1(\alpha,\beta;\alpha q/\beta;q,zq/\beta)\}^2=\frac{(\beta z,\alpha zq/\beta;q)_\infty}{(zq/\beta,\beta z/\alpha;q)_\infty}$$
$$\times\ {}_5\phi_4\left[\begin{matrix}\alpha,\alpha q/\beta^2,\alpha q^{\frac12}/\beta,-\alpha q^{\frac12}/\beta,-\alpha q/\beta\\ \alpha q/\beta,\alpha^2q/\beta^2,\alpha q/\beta z,\alpha zq/\beta\end{matrix};q,q\right]$$
$$+\frac{(\alpha,\alpha q/\beta^2,zq,zq,\alpha zq/\beta;q)_\infty}{(\alpha q/\beta,\alpha q/\beta,zq/\beta,zq/\beta,\alpha/\beta z;q)_\infty}$$
$$\times\ {}_5\phi_4\left[\begin{matrix}\beta z,zq/\beta,zq^{\frac12},-zq^{\frac12},-zq\\ zq,z^2q,\beta zq/\alpha,\alpha zq/\beta\end{matrix};q,q\right], \tag{8.8.12}$$

which gives the square of a well-poised ${}_2\phi_1$ series as the sum of the two balanced ${}_5\phi_4$ series. By analytic continuation, (8.8.12) holds whenever $|q|<1$ and $|zq/\beta|<1$.

To derive (8.8.3) from (8.8.12), observe that if $\alpha=q^{-n}$, $n=0,1,\ldots$, then $(\alpha;q)_\infty=0$ and (8.8.12) gives

$$f^2(z)$$
$$=\frac{(\beta z,zq^{1-n}/\beta;q)_\infty}{(\beta zq^n,zq/\beta;q)_\infty}\ {}_5\phi_4\left[\begin{matrix}q^{-n},q^{\frac12-n}/\beta,q^{1-n}/\beta^2,-q^{\frac12-n}/\beta,-q^{1-n}/\beta\\ q^{1-n}/\beta,q^{1-2n}/\beta^2,zq^{1-n}/\beta,q^{1-n}/\beta z\end{matrix};q,q\right]$$
$$=\frac{(\beta^2,\beta^2;q)_n}{(\beta,\beta;q)_n}\left(\frac{z}{\beta}\right)^n\ {}_5\phi_4\left[\begin{matrix}q^{-n},\beta^2q^n,\beta,\beta z,\beta/z\\ \beta^2,\beta q^{\frac12},-\beta q^{\frac12},-\beta\end{matrix};q,q\right] \tag{8.8.13}$$

by reversing the order of summation. Since

$$f(z)=\frac{(\beta^2;q)_n}{(\beta;q)_n}\left(\frac{z}{\beta}\right)^n\ {}_4\phi_3\left[\begin{matrix}q^{-n},\beta^2q^n,(\beta z)^{\frac12},(\beta/z)^{\frac12}\\ \beta q^{\frac12},-\beta q^{\frac12},-\beta\end{matrix};q,q\right] \tag{8.8.14}$$

by (7.4.14), it follows from (8.8.13) that

$$\left\{{}_4\phi_3\left[\begin{matrix}q^{-n},\beta^2q^n,(\beta z)^{\frac12},(\beta/z)^{\frac12}\\ \beta q^{\frac12},-\beta q^{\frac12},-\beta\end{matrix};q,q\right]\right\}^2$$

$$= {}_5\phi_4\left[\begin{matrix} q^{-n}, \beta^2 q^n, \beta, \beta z, \beta/z \\ \beta^2, \beta q^{\frac12}, -\beta q^{\frac12}, -\beta \end{matrix}; q, q\right] \tag{8.8.15}$$

for $n = 0, 1, \ldots$, which is formula (8.8.3) written in an equivalent form.

Now note that

$$\begin{aligned} &{}_2\phi_1(\alpha, \beta; \alpha q/\beta; q, zq/\beta) \\ &= \frac{(z(\alpha q)^{\frac12}, -z(\alpha q)^{\frac12}, zq\alpha^{\frac12}/\beta, -zq\alpha^{\frac12}/\beta; q)_\infty}{(zq^{\frac12}, -zq^{\frac12}, zq/\beta, -\alpha zq/\beta; q)_\infty} \\ &\quad \times {}_8W_7(-\alpha z/\beta; \alpha^{\frac12}, -\alpha^{\frac12}, (\alpha q)^{\frac12}/\beta, -(\alpha q)^{\frac12}/\beta, -z; q, -zq), \end{aligned} \tag{8.8.16}$$

by (3.4.7) and (2.10.1), and set $a = \alpha^{\frac12}, b = (\alpha q)^{\frac12}/\beta$ to obtain from (8.8.12) the following q-analogue of Clausen's formula:

$$\begin{aligned} &\left\{\frac{(a^2z^2q, b^2z^2q; q^2)_\infty}{(z^2q, a^2b^2z^2q; q^2)_\infty}\, {}_8W_7(-abzq^{-\frac12}; a, -a, b, -b, -z; q, -zq)\right\}^2 \\ &= \frac{(azq^{\frac12}/b, bzq^{\frac12}/a; q)_\infty}{(zq^{\frac12}/ab, abzq^{\frac12}; q)_\infty}\, {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, -ab, -abq^{\frac12} \\ a^2b^2, abq^{\frac12}, abzq^{\frac12}, abq^{\frac12}/z \end{matrix}; q, q\right] \\ &\quad + \frac{(zq, zq, a^2, b^2; q)_\infty}{(abq^{\frac12}, abq^{\frac12}, ab/zq^{\frac12}, abzq^{\frac12}; q)_\infty} \\ &\quad \times {}_5\phi_4\left[\begin{matrix} azq^{\frac12}/b, bzq^{\frac12}/a, zq^{\frac12}, -zq^{\frac12}, -zq \\ zq, z^2q, abzq^{\frac12}, zq^{\frac32}/ab \end{matrix}; q, q\right], \end{aligned} \tag{8.8.17}$$

where $|q| < 1$ and $|zq| < 1$.

To see that (8.8.17) is a nonterminating q-analogue of Clausen's formula, it suffices to replace a by q^a, b by q^b and let $q \to 1^-$; then the left side and the first term on the right side of (8.8.17) tend to the left and right sides of (8.8.1) with z replaced by $-4z(1-z)^{-2}$ and so, by (8.8.1), the second term on the right side of (8.8.17) must tend to zero.

It is shown in Gasper and Rahman [1989] that the nonterminating extension (3.4.1) of the Sears-Carlitz quadratic transformation can be used in place of (8.8.10) to derive the product formula

$$\begin{aligned} &{}_2\phi_1(a, b; c; q, z)\ {}_2\phi_1(a, aq/c; aq/b; q, z) = \frac{(az, abz/c; q)_\infty}{(z, bz/c; q)_\infty} \\ &\times {}_6\phi_5\left[\begin{matrix} a,\ c/b,\ (ac/b)^{\frac12},\ -(ac/b)^{\frac12},\ (acq/b)^{\frac12},\ -(acq/b)^{\frac12} \\ aq/b,\ c,\ ac/b,\ az,\ cq/bz \end{matrix}; q, q\right] \\ &+ \frac{(a, c/b, az, bz, azq/c; q)_\infty}{(c, aq/b, z, z, c/bz; q)_\infty} \\ &\times {}_6\phi_5\left[\begin{matrix} z,\ abz/c,\ z(ab/c)^{\frac12},\ -z(ab/c)^{\frac12},\ z(abq/c)^{\frac12},\ -z(abq/c)^{\frac12} \\ az,\ bz,\ azq/c,\ bzq/c,\ abz^2/c \end{matrix}; q, q\right], \end{aligned} \tag{8.8.18}$$

where $|z| < 1$ and $|q| < 1$. This formula reduces to (8.8.12) when $a = \alpha,\ b = \beta,\ c = \alpha q/\beta$ and z is replaced by zq/β.

By applying various transformation formulas to the ${}_2\phi_1$ series in (8.8.12) and (8.8.18), these formulas can be written in many equivalent forms. For instance, by replacing b in (8.8.18) by c/b and applying (1.5.4) we obtain

$$\begin{aligned}
&{}_2\phi_2(a,b;c,az;q,cz/b)\ {}_2\phi_2(a,b;abq/c,az;q,azq/c)\\
&= \frac{(z,az/b;q)_\infty}{(az,z/b;q)_\infty}\ {}_6\phi_5\left[\begin{matrix} a,b,(ab)^{\frac12},-(ab)^{\frac12},(abq)^{\frac12},-(abq)^{\frac12}\\ abq/c,c,ab,az,bq/z\end{matrix};q,q\right]\\
&\quad + \frac{(a,b,cz/b,azq/c;q)_\infty}{(c,abq/c,az,b/z;q)_\infty}\\
&\quad \times {}_6\phi_5\left[\begin{matrix} z,abz,z(a/b)^{\frac12},-z(a/b)^{\frac12},z(aq/b)^{\frac12},-z(aq/b)^{\frac12}\\ az,cz/b,aqz/c,zq/b,az^2/b\end{matrix};q,q\right], \qquad (8.8.19)
\end{aligned}$$

where $\max(|q|,|azq/c|,|cz/b|) < 1$. If we replace a,b,z in (8.8.19) by $q^a, b^b, z/(z-1)$, respectively, and let $q \to 1^-$, we obtain the Ramanujan [1957, Vol. 2] and Bailey [1933, 1935a] product formula

$$\begin{aligned}
&{}_2F_1(a,b;c;z)\ {}_2F_1(a,b;a+b-c+1;z)\\
&= {}_4F_3(a,b,(a+b)/2,(a+b+1)/2;c,a+b,a+b-c+1;4z(1-z)),
\end{aligned} \qquad (8.8.20)$$

where $|z| < 1$ and $|4z(1-z)| < 1$. This is an extension of Clausen's formula in the sense that by replacing $a,b,c,4z(1-z)$ in (8.8.20) by $2a, 2b, a+b+\frac12, z$, respectively, and using the quadratic transformation (Erdélyi [1953, 2.11 (2)])

$$ {}_2F_1(2a,2b;a+b+\frac12;z) = {}_2F_1(a,b;a+b+\frac12;4z(1-z)), \qquad (8.8.21)$$

we get (8.8.1). See Askey [1989d].

8.9 Nonnegative basic hypergeometric series

Our main aim in this section is to show how the terminating q-Clausen formula (8.8.3) can be used to derive q-analogues of the Askey-Gasper inequalities (8.8.2) and of the nonnegative hypergeometric series in Gasper [1975a, Equations (8.19), (8.20), (8.22)].

As in Gasper [1989b], let us set

$$\begin{gathered}
\gamma = q^{2b}, a_1 = q^b, a_2 = q^b e^{i\theta}, a_3 = q^b e^{-i\theta}, b_1 = q^{2b}, b_2 = -q^b,\\
c_1 = q^{-n}, c_2 = q^{n+a}, d_1 = q^{\frac12(a+1)} = -d_2, e_1 = q^{b+\frac12} = -e_3, x = q, w = 1,
\end{gathered}$$

in the $r=3, s=t=u=k=2$ case of (3.7.9) to obtain the expansion

$$\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} q^{-n},q^{n+a},q^b,q^be^{i\theta},q^be^{-i\theta}\\ q^{2b},q^{\frac12(a+1)},-q^{\frac12(a+1)},-q^b\end{matrix};q,q\right]\\
&= \sum_{j=0}^{n} \frac{(q^{-n},q^{n+a},q^{b+\frac12},-q^{b+\frac12};q)_j}{(q,q^{\frac12(a+1)},-q^{\frac12(a+1)},q^{j+2b};q)_j}(-1)^j q^{j+\binom{j}{2}}
\end{aligned}$$

$$\times {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^{j+n+a}, q^{j+b+\frac{1}{2}}, -q^{j+b+\frac{1}{2}} \\ q^{2j+2b+1}, q^{j+\frac{1}{2}(a+1)}, -q^{j+\frac{1}{2}(a+1)} \end{matrix}; q, q\right]$$

$$\times {}_5\phi_4\left[\begin{matrix} q^{-j}, q^{j+2b}, q^b, q^b e^{i\theta}, q^b e^{-i\theta} \\ q^{2b}, q^{b+\frac{1}{2}}, -q^{b+\frac{1}{2}}, -q^b \end{matrix}; q, q\right], \tag{8.9.1}$$

where, as throughout this section, $n = 0, 1, \ldots$. By Ex. 2.8 the ${}_4\phi_3$ series in (8.9.1) equals zero when $n - j$ is odd and equals

$$\frac{(q, q^{a-2b}; q^2)_k}{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}; q^2)_k} q^{2k(n-2k+b+1/2)}$$

when $n - j = 2k$ and $k = 0, 1, \ldots$. Hence, using (8.8.3) to write the ${}_5\phi_4$ as the square of a ${}_4\phi_3$ series, we have

$$\begin{aligned} &{}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^b e^{i\theta}, q^b e^{-i\theta} \\ q^{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^b \end{matrix}; q, q\right] \\ &= \sum_{k=0}^{[n/2]} \frac{(-1)^n (q^{-n}, q^{n+a}, q^{b+\frac{1}{2}}, -q^{b+\frac{1}{2}}; q)_{n-2k}}{(q, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, q^{n-2k+2b}; q)_{n-2k}} \\ &\times \frac{(q, q^{a-2b}; q^2)_k}{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}; q^2)_k} \; q^{2k(n-2k+b+\frac{1}{2})+\frac{1}{2}(n-2k)(n-2k+1)} \\ &\times \left\{ {}_4\phi_3\left[\begin{matrix} q^{2k-n}, q^{n-2k+2b}, q^{\frac{1}{2}b} e^{\frac{1}{2}i\theta}, q^{\frac{1}{2}b} e^{-\frac{1}{2}i\theta} \\ q^{b+\frac{1}{2}}, -q^{b+\frac{1}{2}}, -q^b \end{matrix}; q, q\right]\right\}^2 . \end{aligned} \tag{8.9.2}$$

Since $(-1)^n (q^{-n}; q)_{n-2k} \geq 0$, it is clear from (8.9.2) that

$$ {}_5\phi_4\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, q^b e^{i\theta}, q^b e^{-i\theta} \\ q^{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^b \end{matrix}; q, q\right] \geq 0 \tag{8.9.3}$$

when $a \geq 2b > -1$ and $0 < q < 1$. By setting $x = \cos\theta$ and letting $q \to 1^-$, it follows from (8.9.3) that

$$ {}_3F_2\left[\begin{matrix} -n, n+a, b \\ 2b, \frac{1}{2}(a+1) \end{matrix}; \frac{1-x}{2}\right] \geq 0, \quad -1 \leq x \leq 1, \tag{8.9.4}$$

when $a \geq 2b > -1$ which shows that (8.9.3) is a q-analogue of (8.9.4). When $a = \alpha + 2$ and $b = \frac{1}{2}(\alpha + 1)$, (8.9.4) reduces to (8.8.2). Special cases of (8.9.4) were used by de Branges [1986] in his work on coefficient estimates for Riemann mapping functions.

Another q-analogue of (8.9.4) can be derived by using (8.9.1), (8.8.3) and Ex. 2.8 to obtain

$$\begin{aligned} &{}_6\phi_5\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, -q^b, q^{\frac{1}{2}a} e^{i\theta}, q^{\frac{1}{2}a} e^{-i\theta} \\ q^{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a}, -q^{\frac{1}{2}a} \end{matrix}; q, q\right] \\ &= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+a}, q^{\frac{1}{2}a}, q^{\frac{1}{2}a} e^{i\theta}, q^{\frac{1}{2}a} e^{-i\theta}; q)_j}{(q, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a}, q^{j+a-1}; q)_j} (-1)^j q^{j+\binom{j}{2}} \\ &\times {}_5\phi_4\left[\begin{matrix} q^{j-n}, q^{n+j+a}, q^{j+\frac{1}{2}a}, q^{j+\frac{1}{2}a} e^{i\theta}, q^{j+\frac{1}{2}a} e^{-i\theta} \\ q^{2j+a}, q^{j+\frac{1}{2}(a+1)}, -q^{j+\frac{1}{2}(a+1)}, -q^{j+\frac{1}{2}a} \end{matrix}; q, q\right] \end{aligned}$$

$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-j}, q^{j+a-1}, q^b, -q^b \\ q^{2b}, q^{\frac{1}{2}a}, -q^{\frac{1}{2}a} \end{matrix}; q, q\right]$$

$$= \sum_{k=0}^{[n/2]} \frac{(q^{-n}, q^{n+a}, q^{\frac{1}{2}a}, q^{\frac{1}{2}a}e^{i\theta}, q^{\frac{1}{2}a}e^{-i\theta}; q)_{2k}}{(q, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a}, q^{2k+a-1}; q)_{2k}}$$

$$\times \frac{(q, q^{a-2b}; q^2)_k}{(q^a, q^{2b+1}; q^2)_k}\, q^{2k^2+k+2kb}$$

$$\times \left\{ {}_4\phi_3\left[\begin{matrix} q^{2k-n}, q^{n+2k+a}, q^{k+\frac{1}{4}a}e^{\frac{1}{2}i\theta}, q^{k+\frac{1}{4}a}e^{-\frac{1}{2}i\theta} \\ q^{2k+\frac{1}{2}(a+1)}, -q^{2k+\frac{1}{2}(a+1)}, -q^{2k+\frac{1}{2}a} \end{matrix}; q, q\right]\right\}^2, \qquad (8.9.5)$$

which shows that

$$ {}_6\phi_5\left[\begin{matrix} q^{-n}, q^{n+a}, q^b, -q^b, q^{\frac{1}{2}a}e^{i\theta}, q^{\frac{1}{2}a}e^{-i\theta} \\ q^{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a}, -q^{\frac{1}{2}a} \end{matrix}; q, q\right] \geq 0 \qquad (8.9.6)$$

when $a \geq 2b > -1$ and $0 < q < 1$.

The expansions (8.12) and (8.17) in Gasper [1975a] are special cases of the $q \to 1^-$ limit cases of (8.9.2) and (8.9.5), respectively, when (7.4.14) and Gasper [1975a, (8.10)] are used. A q-analogue of the expansion (Gasper [1975a, 1989b])

$$ {}_3F_2\left[\begin{matrix} -n, n+\alpha+2, \frac{1}{2}(\alpha+1) \\ \alpha+1, \frac{1}{2}(\alpha+3) \end{matrix}; (1-x^2)(1-y^2)\right]$$

$$= \sum_{j=0}^{n} \frac{n!(n+\alpha+2)_j \left(\frac{\alpha+2}{2}\right)_j}{j!(n-j)! \left(\frac{\alpha+3}{2}\right)_j (j+\alpha+1)_j} (1-y^2)^j$$

$$\times \left\{ \frac{j!(n-j)!}{(\alpha+1)_j(2j+\alpha+2)_{n-j}} C_j^{\frac{1}{2}(\alpha+1)}(x) C_{n-j}^{j+\frac{1}{2}(\alpha+2)}(y)\right\}^2 \qquad (8.9.7)$$

is easily derived by employing (3.7.9), (8.8.3) and (7.4.14) to obtain

$$ {}_7\phi_6\left[\begin{matrix} q^{-n}, q^{n+\alpha+2}, q^{\frac{1}{2}(\alpha+1)}, q^{\frac{1}{2}(\alpha+1)}e^{2i\theta}, q^{\frac{1}{2}(\alpha+1)}e^{-2i\theta}, \\ q^{\alpha+1}, q^{\frac{1}{2}(\alpha+3)}, -q^{\frac{1}{2}(\alpha+3)}, -q^{\frac{1}{2}(\alpha+2)}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{\frac{1}{2}(\alpha+2)}e^{2i\tau}, q^{\frac{1}{2}(\alpha+2)}e^{-2i\tau} \\ -q^{\frac{1}{2}(\alpha+2)}, -q^{\frac{1}{2}(\alpha+1)} \end{matrix}; q, q\right]$$

$$= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+\alpha+2}, q^{\frac{1}{2}(\alpha+2)}, q^{\frac{1}{2}(\alpha+2)}e^{2i\tau}, q^{\frac{1}{2}(\alpha+2)}e^{-2i\tau}; q)_j}{(q, q^{\frac{1}{2}(\alpha+3)}, -q^{\frac{1}{2}(\alpha+3)}, -q^{\frac{1}{2}(\alpha+2)}, q^{j+\alpha+1}; q)_j} (-1)^j q^{j+\binom{j}{2}}$$

$$\times \left\{ \frac{(q;q)_j (q;q)_{n-j}}{(q^{\alpha+1};q)_j (q^{2j+\alpha+2};q)_{n-j}} q^{\frac{1}{2}(j+\alpha+\frac{3}{2})}\right.$$

$$\left.\times\, C_j(\cos\theta; q^{\frac{1}{2}(\alpha+1)}|q) C_{n-j}(\cos\tau; q^{j+\frac{1}{2}(\alpha+2)}|q)\right\}^2, \qquad (8.9.8)$$

which is obviously nonnegative for real θ and τ when $\alpha > -2$.

If we proceed as in (8.9.5), but use the q-Saalschütz summation formula (1.7.2) instead of Ex. 2.8, we find that

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} q^{-n}, q^{n+a}, q^{\frac{1}{2}a}, q^b, q^{\frac{1}{2}a}e^{i\theta}, q^{\frac{1}{2}a}e^{-i\theta} \\ q^{\frac{1}{2}(a+1)}, q^c, q^{a+b-c}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a} \end{matrix}; q, q\right] \\
&= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+a}, q^{\frac{1}{2}a}, q^{a-c}, q^{c-b}, q^{\frac{1}{2}a}e^{i\theta}, q^{\frac{1}{2}a}e^{-i\theta}; q)_j}{(q, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a}, q^{j+a-1}, q^c, q^{a+b-c}; q)_j} \\
&\quad \times (-1)^j q^{j(b+1)+\binom{j}{2}} \left\{ {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^{j+n+a}, q^{\frac{1}{2}j+\frac{1}{4}a}e^{\frac{1}{2}i\theta}, q^{\frac{1}{2}j+\frac{1}{4}a}e^{-\frac{1}{2}i\theta} \\ q^{j+\frac{1}{2}(a+1)}, -q^{j+\frac{1}{2}(a+1)}, -q^{j+\frac{1}{2}a} \end{matrix}; q, q\right]\right\}^2
\end{aligned} \tag{8.9.9}
$$

and hence

$$
{}_6\phi_5\left[\begin{matrix} q^{-n}, q^{n+a}, q^{\frac{1}{2}a}, q^b, q^{\frac{1}{2}a}e^{i\theta}, q^{\frac{1}{2}a}e^{-i\theta} \\ q^{\frac{1}{2}(a+1)}, q^c, q^{a+b-c}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}a} \end{matrix}; q; q\right] \geq 0 \tag{8.9.10}
$$

for real θ when $a \geq c \geq b$, $a+b > c > 0$ and $0 < q < 1$. By letting $q \to 1^-$ in (8.9.10) it follows that

$$
{}_4F_3\left[\begin{matrix} -n, n+a, \frac{1}{2}a, b \\ \frac{1}{2}(a+1), c, a+b-c \end{matrix}; \frac{1-x}{2}\right] \geq 0 \tag{8.9.11}
$$

for $-1 \leq x \leq 1$ when $a \geq c \geq b$ and $a+b > c > 0$, which gives the nonnegativity of the ${}_3F_2$ series in Gasper [1975a, (8.22)] when $a = \alpha + \frac{3}{2}, b = \frac{1}{2}\alpha + \frac{5}{4}$ and $c = \alpha + 1$. Also see Gasper [1989d].

8.10 Applications in the theory of partitions of positive integers

In the applications given so far we have dealt almost exclusively with orthogonal polynomials which are representable as basic hypergeometric series. These are important results and most of them have appeared in print during the last thirty years. They constitute the main bulk of applications as far as this book is concerned. However, our task would remain incomplete if we failed to mention some of the earliest examples where basic hypergeometric series played crucial roles. The simplest among these examples is Euler's [1748] enumeration $p(n)$ of the partitions of a positive integer n, where a *partition* of a positive integer n is a finite monotone decreasing sequence of positive integers (called the parts of the partition) whose sum is n. To illustrate how $p(n)$ arises in a q-series let us consider a formal series expansion of the infinite product $(q;q)_\infty^{-1}$ in powers of q:

$$
\begin{aligned}
(q;q)_\infty^{-1} &= \prod_{k=0}^{\infty}(1-q^{k+1})^{-1} \\
&= \sum_{k_1\geq 0}\sum_{k_2\geq 0}\cdots q^{k_1\cdot 1+k_2\cdot 2+\cdots} \\
&= \sum_{n=0}^{\infty} p(n)q^n.
\end{aligned} \tag{8.10.1}
$$

where

$$n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n, \tag{8.10.2}$$

$p(0) = 1$ and, for a positive integer n, $p(n)$ is the number of partitions of n into parts $\leq n$. In the partition (8.10.2) of n there are k_m m's and hence $0 \leq k_m \leq n/m, 1 \leq m \leq n$. For small values of n, $p(n)$ can be calculated quite easily, but the number increases very rapidly. For example, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, but $p(243) = 133978259344888$. Hardy and Ramanujan [1918] found the following asymptotic formula for large n:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left[\pi\left(\frac{2n}{3}\right)^{\frac{1}{2}}\right]. \tag{8.10.3}$$

Also of interest are the enumerations of partitions of a positive integer n into parts restricted in certain ways such as:

(i) $p_N(n)$, the number of partitions of n into parts $\leq N$, which is given by the generating function

$$(q;q)_N^{-1} = \sum_{n=0}^{\infty} p_N(n) q^n, \tag{8.10.4}$$

where $n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_N \cdot N$ has k_m m's ;

(ii) $p_e(n)$, the number of partitions of an even integer n into even parts, generated by

$$\begin{aligned}(q^2;q^2)_\infty^{-1} &= \prod_{k=0}^{\infty}(1-q^{2k+2})^{-1} \\ &= \sum_{n=0}^{\infty} p_e(n) q^n; \end{aligned} \tag{8.10.5}$$

(iii) $p_{\text{dist}}(n)$, the number of partitions of n into distinct positive integers, generated by

$$\begin{aligned}(-q;q)_\infty &= \sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots q^{k_1\cdot 1+k_2\cdot 2+\cdots} \\ &= \sum_{n=0}^{\infty} p_{\text{dist}}(n) q^n, \end{aligned} \tag{8.10.6}$$

where $n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n$, $0 \leq k_i \leq 1$, $1 \leq i \leq n$, and

(iv) $p_0(n)$, the number of partitions of n into odd parts, generated by

$$\begin{aligned}(q;q^2)_\infty^{-1} &= \prod_{k=0}^{\infty}(1-q^{2k+1})^{-1} \\ &= \sum_{n=0}^{\infty} p_0(n) q^n. \end{aligned} \tag{8.10.7}$$

Euler's partition identity

$$p_{\text{dist}}(n) = p_0(n) \tag{8.10.8}$$

follows from (8.10.6), (8.10.7) and the fact that

$$(-q;q)_\infty = \left(q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q\right)_\infty^{-1} = (q;q^2)_\infty^{-1}. \tag{8.10.9}$$

Other combinatorial identities of this type can be discovered from q-series identities similar to, but perhaps somewhat more complicated than, (8.10.9). For example, let us consider Euler's [1748] identity involving the *pentagonal numbers* $n(3n \pm 1)/2$:

$$\begin{aligned}(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n+1)/2} + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2},\end{aligned} \tag{8.10.10}$$

which is given in Ex. 2.18. A formal power series expansion gives

$$\begin{aligned}(q;q)_\infty &= \prod_{k=0}^{\infty}(1-q^{k+1}) \\ &= \sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots(-1)^{k_1+k_2+\cdots}q^{k_1\cdot 1+k_2\cdot 2+\cdots},\end{aligned} \tag{8.10.11}$$

which differs from the multiple series in (8.10.6) only in the factor $(-1)^{k_1+k_2+\cdots}$. This factor is ± 1 according as the partition has an even or odd number of parts. Denoting these numbers by $p_{\text{even}}(n)$ and $p_{\text{odd}}(n)$, respectively, we find that

$$(q;q)_\infty = 1 + \sum_{n=1}^{\infty}\left[p_{\text{even}}(n) - p_{\text{odd}}(n)\right]q^n. \tag{8.10.12}$$

From (8.10.10) and (8.10.12) it follows that

$$p_{\text{even}}(n) - p_{\text{odd}}(n) = \begin{cases} (-1)^k & \text{for } n = k(3k\pm 1)/2, \\ 0 & \text{otherwise.}\end{cases} \tag{8.10.13}$$

Thus Euler's identity (8.10.10) expresses the important property that a positive integer n which is not a pentagonal number of the form $k(3k \pm 1)/2$ can be partitioned as often into an even number of parts as into an odd number of parts. However, if $n = k(3k \pm 1)/2$, $k = 1, 2, \ldots$, then $p_{\text{even}}(n)$ exceeds $p_{\text{odd}}(n)$ by $(-1)^k$. See Hardy and Wright [1979], Rademacher [1973] and Andrews [1976, 1983] for related results.

Two of the most celebrated identities in combinatorial analysis are the so-called Rogers–Ramanujan identities (2.7.3) and (2.7.4) which, for the purposes of the present discussion, we rewrite in the following form:

$$1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})}, \tag{8.10.14}$$

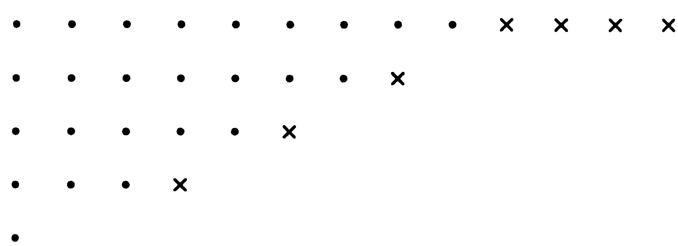
$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{\prod\limits_{k=0}^{\infty}\left(1-q^{5k+2}\right)\left(1-q^{5k+3}\right)}. \tag{8.10.15}$$

It is clear that the infinite product on the right side of (8.10.14) enumerates the partitions of a positive integer n into parts of the form $5k+1$ and $5k+4$, while that on the right side of (8.10.15) enumerates partitions of n into parts of the form $5k+2$ and $5k+3, k=0,1,\ldots$.

Following Hardy and Wright [1979] we shall now give combinatorial interpretations of the left sides of the above identities. Since

$$k^2=1+3+5+\cdots+(2k-1),$$

we can exhibit this square in a graph of k rows of dots, each row having two more dots than the lower one. We then take any partition of $n-k^2$ into at most k parts with the parts in descending order, marked with $\times$'s and placed at the ends of the rows of dots to obtain a partition of n into parts with minimal difference 2. For example, when $k=5$ and $n=32=5^2+7$ we add 4 $\times$'s to the top row, 1 each to the 2nd, 3rd and 4th rows, counted from above. This gives the partition $32=13+8+6+4+1$ displayed in the graph below.



The identity (8.10.14) states that the number of partitions of n in which the differences between parts are at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 (mod 5).

Observing that

$$k(k+1)=2+4+6+\cdots+2k,$$

a similar interpretation can be given to the left side of (8.10.15). Since the first number in the above sum is 2, one deduces that the partitions of n into parts not less than 2 and with minimal difference 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 (mod 5).

For more applications of basic hypergeometric series to partition theory, see Andrews [1976–1988], Fine [1948, 1988], Andrews and Askey [1977], and Andrews, Dyson and Hickerson [1988]. Additional results on Rogers–Ramanujan type identities are given in Slater [1951, 1952a], Jain and Verma [1980–1982] and Andrews [1975a, 1984a,b,c,d].

8.11 Representations of positive integers as sums of squares

One of the most interesting problems in number theory is the representations of positive integers as sums of squares of integers. Fermat proved that all primes of the form $4n+1$ can be uniquely expressed as the sum of two squares. Lagrange showed in 1770 that all positive integers can be represented by sums of four squares and that this number is minimal. Earlier in the same year Waring posed the general problem of representing a positive integer as a sum of a fixed number of nonnegative k-th powers of integers (positive, negative, or zero) with order taken into account and stated without proof that every integer is the sum of 4 squares, of 9 cubes, of 19 biquadrates, 'and so on'. More than 100 years later Hilbert [1909] proved that all positive integers are representable by s kth powers where $s = s(k)$ depends only on k. For an historical account of the Waring problem, see Dickson [1920], Grosswald [1985], and Hua [1982].

To illustrate the usefulness of basic hypergeometric series in the study of such representations we shall restrict ourselves to the simplest cases: sums of two and sums of four squares, where it is understood, for example, that $n = x_1^2 + x_2^2 = y_1^2 + y_2^2$ are two different representations of n as a sum of two squares if $x_1 \neq y_1$ or $x_2 \neq y_2$.

Let $r_{2k}(n)$ be the number of different representations of n as a sum of $2k$ squares, $k = 1, 2, \ldots$. We will show by basic hypergeometric series techniques that, for $n \geq 1$,

$$r_2(n) = 4(d_1(n) - d_3(n)), \tag{8.11.1}$$

$$r_4(n) = 8 \sum_{d|n,4\nmid d} d, \tag{8.11.2}$$

where $d_i(n)$, $i = 1, 3$, is the number of (positive) divisors of n congruent to i (mod 4) and the summation in (8.11.2) indicates the sum over all divisors of n not divisible by 4. The numbers 4 and 8 in (8.11.1) and (8.11.2), respectively, reflect the fact that $r_2(1) = 4$ since $1 = 0^2 + (\pm 1)^2 = (\pm 1)^2 + 0^2$, and $r_4(1) = 8$ since $1 = 0^2 + 0^2 + 0^2 + (\pm 1)^2 = 0^2 + 0^2 + (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2 + 0^2 + 0^2 = (\pm 1)^2 + 0^2 + 0^2 + 0^2$. Both of these results were proved by Jacobi by means of the theory of elliptic functions, but the proofs below are based, as in Andrews [1974a], on the formulas stated in Ex. 5.1, 5.2, and 5.3. Combining Ex. 5.1 and 5.2 we have

$$\left[\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right]^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}. \tag{8.11.3}$$

However, the bilateral sum on the left side is clearly a generating function of $(-1)^n r_2(n)$ and so it suffices to prove that

$$\sum_{n=1}^{\infty} [d_1(n) - d_3(n)](-q)^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}. \tag{8.11.4}$$

By splitting into odd and even parts and then by formal series manipulations, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} &= \sum_{r=1}^{\infty} \frac{q^{r(2r+1)}}{1+q^{2r}} - \sum_{m=0}^{\infty} \frac{q^{(m+1)(2m+1)}}{1+q^{2m+1}} \\ &= \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} (-1)^{m+r} q^{(2m+1)r} + \sum_{m=0}^{\infty} \sum_{r=m+1}^{\infty} (-1)^{m+r} q^{(2m+1)r}\end{aligned}$$

$$= \left(\sum_{r=1}^{\infty} \sum_{m=r}^{\infty} + \sum_{r=1}^{\infty} \sum_{m=0}^{r-1} \right) (-1)^{m+r} q^{(2m+1)r}$$
$$= \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{m+r} q^{(2m+1)r}$$
$$= \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{r} \left(q^{(4m+1)r} - q^{(4m+3)r} \right)$$
$$= \sum_{n=1}^{\infty} (-q)^n (d_1(n) - d_3(n)), \tag{8.11.5}$$

which completes the proof of (8.11.4).

To prove (8.11.2) we first replace q by $-q$ in Ex. 5.1 and 5.3 and find that

$$\sum_{n=0}^{\infty} r_4(n) q^n = \left[\sum_{n=-\infty}^{\infty} q^{n^2} \right]^4$$
$$= 1 + 8 \sum_{n=1}^{\infty} \frac{q^n}{(1+(-q)^n)^2}$$
$$= 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n}, \tag{8.11.6}$$

where the last line is obtained from the previous one by expanding $(1+(-q)^n)^{-2}$ and interchanging the order of summation. Now,

$$\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n}$$
$$= \sum_{\text{odd } n \geq 1} \frac{nq^n}{1-q^n} + \sum_{\text{even } n \geq 2} \frac{nq^n}{1+q^n}$$
$$= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \sum_{\text{even } n \geq 2} nq^n \left(\frac{1}{1+q^n} - \frac{1}{1-q^n} \right)$$
$$= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{\text{even } n \geq 2} \frac{2nq^{2n}}{1-q^{2n}}$$
$$= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}}$$
$$= \sum_{4 \nmid n} \frac{nq^n}{1-q^n}. \tag{8.11.7}$$

Thus,

$$\sum_{n=0}^{\infty} r_4(n) q^n = 1 + 8 \sum_{4 \nmid n} \frac{nq^n}{1-q^n}. \tag{8.11.8}$$

Since $r_4(0) = 1$, this immediately leads to (8.11.2). For a direct proof of (8.11.8) based only on Jacobi's triple product identity (1.6.1), see Hirschhorn [1987].

Ex. 5.1 and 5.4 can be employed in a similar manner to show that

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3, \quad n \geq 1. \tag{8.11.9}$$

See Andrews [1974a]. Some remarks on other applications are given in Notes §8.11.

Exercises

8.1 Prove for the little q-Jacobi polynomials $p_n(x; a, b; q)$ defined in (7.3.1) that

(i)

$$\begin{aligned} &p_n(x; a, b; q) p_n(y; a, b; q) \\ &= (-aq)^n q^{\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, x^{-1}, y^{-1}; q\right)_m}{(q, bq; q)_m} \left(\frac{xyq^{1-m}}{a}\right)^m \\ &\qquad \times \sum_{k=0}^{m} \frac{\left(q^{-m}, b^{-1}q^{-m}; q\right)_k}{\left(q, aq, xq^{1-m}, yq^{1-m}; q\right)_k} (abxy)^k q^{k^2+2k}, \end{aligned}$$

(ii)

$$\begin{aligned} &p_n(x; a, b; q) p_n(y; a, b; q) \\ &= (-bq)^{-n} q^{-\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}; q\right)_m}{(q, bq; q)_m} (-byq^2)^m q^{\binom{m}{2}} \\ &\quad \times \sum_{k=0}^{m} \frac{\left(q^{-m}, abq^{m+1}; q\right)_k}{(q, aq; q)_k} (xq)^k \, {}_2\phi_1(q^{k-m}, bxq; 0; q, b^{-1}y^{-1}). \end{aligned}$$

8.2 Derive the following product formula for the big q-Jacobi polynomials defined in (7.3.10):

$$\begin{aligned} &P_n(x; a, b, c; q) P_n(y; a, b, c; q) \\ &= (-aq)^n q^{\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, cqx^{-1}, cqy^{-1}; q\right)_m}{(q, bq, cq, cq; q)_m} \left(\frac{xy}{a}\right)^m \\ &\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-m}, b^{-1}q^{-m}, x, y \\ aq, xc^{-1}q^{-m}, yc^{-1}q^{-m} \end{matrix} ; q, abqc^{-2} \right]. \end{aligned}$$

8.3 Prove that

$$\sum_{n=0}^{\infty} \frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(q, bq; q)_n (1 - abq)} \left(\frac{t}{aq}\right)^n p_n(q^x; a, b; q) p_n(q^y; a, b; q)$$

$$= (t, abq^2; q)_\infty \sum_{r=0}^{\infty} \sum_{s=0}^{\min(x,y)} \frac{(q^{-x}, q^{-y}; q)_s\, a^{-s}}{(q, bq; q)_s (q, aq; q)_r} q^{(x+y)(r+s)}$$

$$\times\, {}_2\phi_1\left(0, 0; qt; q, abq^{2r+2s+2}\right) q^{-2rs-s^2} t^{r+s},$$

and that this gives the positivity of the Poisson kernel for the little q-Jacobi polynomials for $x, y = 0, 1, \ldots,\ 0 \le t < 1$ when $0 < q < 1, 0 < aq < 1$ and $0 < bq < 1$.

8.4 Prove for the q-Hahn polynomials that

(i)

$$Q_n(x; a, b, N; q) Q_n(y; a, b, N; q) = \frac{(bq, abq^{N+2}; q)_n}{(aq, q^{-N}; q)_n} \left(bq^{N+1}\right)^{-n}$$

$$\times \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, bq^{N-x+1}, bq^{N-y+1}; q)_m}{(q, bq, abq^{N+2}, bq^{N+1}; q)_m} q^m$$

$$\times \sum_{k=0}^{m} \frac{(b^{-1}q^{-N-m-1}; q)_k (1 - b^{-1}q^{2k-N-m-1})(a^{-1}b^{-1}q^{-N-m-1}, b^{-1}q^{-m}; q)_k}{(q; q)_k (1 - b^{-1}q^{-N-m-1})(aq, q^{-N}; q)_k}$$

$$\times \frac{(q^{-m}, q^{-x}, q^{-y}; q)_k}{(b^{-1}q^{-N}, b^{-1}q^{x-N-m}, b^{-1}q^{y-N-m}; q)_k} (ab)^k q^{(x+y+2m-k)k},$$

(ii)

$$Q_n(x; a, b, N; q) Q_n(y; a, b, N; q) = \frac{\left(bq, abq^{N+2}; q\right)_n}{\left(aq, q^{-N}; q\right)_n} \left(bq^{N+1}\right)^{-n}$$

$$\times \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, bq^{N+1-x-y}; q\right)_m}{(q, bq, abq^{N+2}; q)_m} q^{(x+1)m}$$

$$\times \sum_{k=0}^{m} \frac{\left(q^{-m}, abq^{m+1}, q^{-x}, q^{-y}; q\right)_k}{(q, aq, q^{-N}, bq^{N+1-x-y}; q)_k} \left(bq^{N-x+2}\right)^k$$

$$\times\, {}_3\phi_2\left[\begin{matrix} q^{k-m}, q^{k-x}, bq^{N+1-x} \\ 0,\ bq^{N+1-x-y+k} \end{matrix}; q, q\right].$$

8.5 Prove for the q-Racah polynomials that

(i)

$$W_n(x; a, b, c, N; q) W_n(y; a, b, c, N; q) = \frac{\left(bq, abq^{N+2}; q\right)_n}{(aq, q^{-N}; q)_n} \left(bq^{N+1}\right)^{-n}$$

$$\times \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, bq^{N-x+1}, bq^{N-y+1}; q\right)_m}{(q, bq, abq^{N+2}, bcq; q)_m}$$

$$\times \frac{\left(bcq^{x+1}, bcq^{y+1}; q\right)_m}{(bcq, bq^{N+1}; q)_m} q^m$$

$$\times {}_{10}W_9 \left(b^{-1}q^{-N-m-1}; a^{-1}b^{-1}q^{-N-m-1}, b^{-1}q^{-m},\right.$$
$$\left. q^{-m}, q^{-x}, cq^{x-N}, q^{-y}, cq^{y-N}; q, ab^{-1}c^{-2}q\right),$$

(ii)

$$W_n(x; a, b, c, N; q)W_n(y; a, b, c, N; q) = \frac{\left(aqc^{-1}, abq^{N+2}; q\right)_n}{(bcq, q^{-N}; q)_n} \left(ac^{-1}q^{N+1}\right)^{-n}$$

$$\times \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, ac^{-1}q^{N-x+1}, ac^{-1}q^{N-y+1}; q\right)_m}{(q, abq^{N+2}, aqc^{-1}, ac^{-1}q^{N+1}; q)_m}$$

$$\times \frac{\left(aq^{x+1}, aq^{y+1}; q\right)_m}{(aq, aq; q)_m} q^m$$

$$\times {}_{10}W_9 \left(ca^{-1}q^{-N-m-1}; ca^{-1}q^{-m}, a^{-1}b^{-1}q^{-N-m-1},\right.$$
$$\left. q^{-m}, q^{-x}, cq^{x-N}, q^{-y}, cq^{y-N}; q, qba^{-1}\right).$$

8.6 Show that

(i)

$$W_n(x; a, b, c, N; q)W_n(y; a, b, c, M; q)$$
$$= \frac{\left(bq, aqc^{-1}; q\right)_n}{(aq, bcq; q)_n} c^n \sum_{m=0}^{n} \frac{\left(q^{-n}, abq^{n+1}, q^{x-N}, q^{y-M}; q\right)_m}{(q, bq, aqc^{-1}, c^{-1}; q)_m}$$

$$\times \frac{\left(c^{-1}q^{-x}, c^{-1}q^{-y}; q\right)_m}{(q^{-N}, q^{-M}; q)_m} q^m$$

$$\times {}_{10}W_9 \left(cq^{-m}; ca^{-1}q^{-m}, b^{-1}q^{-m}, q^{-m}, q^{-x}, cq^{x-N}, q^{-y}, cq^{y-M}; q, abq^{M+N+3}\right),$$

where $n = 0, 1, \ldots, \min(M, N)$, $x = 0, 1, \ldots, N$, and $y = 0, 1, \ldots, M$;

(ii)

$$W_n(x; a, a, c, N; q)W_n(y; a, a, c, N; q)$$
$$= \frac{\left(aqc^{-1}, a^2q^{N+2}; q\right)_n}{(acq, q^{-N}; q)_n} \left(ac^{-1}q^{N+1}\right)^{-n} \sum_{m=0}^{n} \frac{\left(q^{-n}, a^2q^{n+1}, aq^{x+1}; q\right)_m}{(q, a^2q^{N+2}, aq; q)_m}$$

$$\times \frac{\left(aq^{1+y-x}, ac^{-1}q^{N-x-y+1}; q\right)_m}{(aqc^{-1}, aq^{1-x}; q)_m} q^m$$

$$\times {}_{10}W_9 \left(aq^{-x}; a^2q^{m+1}, q^{-m}, aq^{N-x+1}, c^{-1}q^{-x}, q^{-x}, q^{-y}, cq^{y-N}; q, q\right).$$

8.7 For the q-Hahn polynomials prove that

(i)

$$\sum_{n=0}^{z} \frac{(abq, aq, q^{-z}; q)_n (1 - abq^{2n+1})}{(q, bq, abq^{N+2}; q)_n (1 - abq)} (-aq)^{-n} q^{Nn - \binom{n}{2}}$$

$$\times Q_n(x;a,b,N;q)Q_n(y;a,b,M;q)$$

$$= \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(q^{-z};q)_{r+s}\,(abq^2;q)_{2r+2s}(q^{-x},q^{-y};q)_r(q^{x-N},q^{y-M};q)_s}{(q^{-M},q^{-N},abq^{N+2};q)_{r+s}\,(q,aq;q)_r(q,bq;q)_s}$$

$$\times \frac{\left(q^{N-z};q\right)_{z-r-s}}{\left(abq^{N+r+s+2};q\right)_{z-r-s}}(-1)^{r+s}a^{-s}q^{(r+s)(2N-r-s+1)/2-(x+y+1)s}$$

$$\times {}_2\phi_1\left(q^{r+s-z},q^{1+r+s-z};q^{1+N-z};q,abq^{N+z+2}\right) \geq 0.$$

for $x = 0,1,\ldots,N,\ y = 0,1,\ldots,M,\ z = 0,1,\ldots,N,$ when $0 < q < 1$, $0 < aq < 1$, $0 \leq bq < 1$ and $N \leq M$;

(ii)

$$\sum_{n=0}^{N} \frac{(abq,aq,q^{-N};q)_n(1-abq^{2n+1})}{(q,bq,abq^{N+2};q)_n(1-abq)}\left(-\frac{t}{aq}\right)^n q^{Nn-\binom{n}{2}}$$

$$\times Q_n(x;a,b,N;q)Q_n(y;a,b,M;q)$$

$$= \sum_{r=0}^{x}\sum_{s=0}^{N-x} \frac{(abq^2;q)_{2r+2s}\,(q^{-x},q^{-y};q)_r\left(q^{x-N},q^{y-M};q\right)_s}{(q^{-M},abq^{N+2};q)_{r+s}\,(q,aq;q)_r(q,bq;q)_s}$$

$$\times (t,abq^{2r+2s+2};q)_{N-r-s}\ {}_2\phi_1\left(q^{N-r-s},tq^{N-r-s};qt;q,abq^{2r+2s+2}\right)$$

$$\times a^{-s}q^{(r+s)(2N-r-s+1)/2-(x+y+1)s}(-t)^{r+s} > 0,$$

for $x = 0,1,\ldots,N,\ y = 0,1,\ldots,M,\ 0 \leq t < 1$ when $0 < q < 1,\ 0 < aq < 1$, $0 \leq bq < 1$ and $N \leq M$.
(Gasper and Rahman [1984])

8.8 Prove that

$$\sum_{n=0}^{z} \frac{(aq,abq,bcq;q)_n(-c)^{-n}}{(q,bq,aqc^{-1};q)_n(abq;q)_{2n}}\lambda_n(z)q^{\binom{n}{2}}$$

$$\times W_n(x;a,b,c,N;q)W_n(y;\alpha,ab\alpha^{-1},\gamma,M;q)$$

$$= \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(q^{-z};q)_{r+s}\left(q^{-x},cq^{x-N},q^{-y},\gamma q^{y-M};q\right)_r}{(q^{-N};q)_{r+s}\,(q,\alpha q,q^{-M},ab\gamma q\alpha^{-1},cq^{1-s};q)_r}$$

$$\times \frac{\left(q^{x-N},c^{-1}q^{-x};q\right)_s(1-cq^{r-s})}{(q,bq,aqc^{-1};q)_s(1-cq^{-s})}q^{-rs}A_{r,s}\ \mu_{r+s},$$

where $z = 0,1,\ldots,N,\quad N \leq M$,

$$\lambda_n(z) = \sum_{k=0}^{z-n} \frac{(q^{-z};q)_{n+k}\,\mu_{n+k}}{(q,abq^{2n+2};q)_k},$$

and $\{\mu_r\}$ is an arbitrary complex sequence, $A_{r,s}$ being the same as in (8.7.5).

8.9 Deduce from Ex. 8.8 that

$$\sum_{n=0}^{z} \frac{(abq, aq, d, e, bcq, q^{-z}; q)_n (1-abq^{2n+1})}{(q, bq, abq^2/d, abq^2/e, aq/c, abq^{z+2}; q)_n (1-abq)} \left(\frac{abq^{z+2}}{cde}\right)^n$$
$$\times W_n(x; a, b, c, N; q) W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q)$$
$$= \frac{\left(abq^2, abq^2/de; q\right)_z}{\left(abq^2/d, abq^2/e; q\right)_z} \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(q^{-z}, d, e; q)_{r+s} \left(q^{-x}, cq^{x-N}; q\right)_r}{(q^{-N}, deq^{-z-1}/ab; q)_{r+s} \left(q, q^{-M}; q\right)_r}$$
$$\times \frac{\left(q^{-y}, \gamma q^{y-M}; q\right)_r \left(q^{x-N}, c^{-1}q^{-x}; q\right)_s (1-cq^{r-s})}{(\alpha q, ab\gamma q\alpha^{-1}, cq^{1-s}; q)_r (q, bq, aq/c; q)_s (1-cq^{-s})} q^{r+s-rs} A_{r,s}.$$

8.10 Use (7.4.14) to show that (8.8.3) is equivalent to the formula

$$\{C_n(\cos\theta; \beta|q)\}^2$$
$$= \frac{(\beta^2, \beta^2; q)_n}{(q, q; q)_n} \beta^{-n} {}_5\phi_4 \left[\begin{matrix} q^{-n}, \beta^2 q^n, \beta, \beta e^{2i\theta}, \beta e^{-2i\theta} \\ \beta^2, \beta q^{\frac{1}{2}}, -\beta q^{\frac{1}{2}}, -\beta \end{matrix}; q, q\right],$$

where $n = 0, 1, \ldots$.

8.11 In view of the product formula (8.4.10) and Gegenbauer's [1874, 1893] addition formula for ultraspherical polynomials (see also Erdélyi [1953, 10.9 (34)] and Szegő [1975, p. 98]) it is natural to look for an expansion of the form

$$p_n(z; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}|q)$$
$$= \sum_{k=0}^{n} A_{k,n}(\theta, \phi)\, p_k(z; ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}, ae^{i\theta-i\phi}, ae^{i\phi-i\theta}|q),$$

where $p_n(z; a, b, c, d|q)$ is the Askey-Wilson polynomial. By multiplying both sides by

$$w(z; ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}, ae^{i\theta-i\phi}, ae^{i\phi-i\theta})$$
$$\times p_m(z; ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}, ae^{i\theta-i\phi}, ae^{i\phi-i\theta}|q)$$

and then integrating over z from -1 to 1, show that

$$A_{m,n}(\theta, \phi) = \frac{(q;q)_n \left(a^4 q^n, a^4 q^{-1}, a^2 q^{\frac{1}{2}}, -a^2 q^{\frac{1}{2}}, -a^2; q\right)_m a^{n-m}}{(q;q)_m (q;q)_{n-m} (a^4 q^{-1}; q)_{2m} \left(a^2 q^{\frac{1}{2}}, -a^2 q^{\frac{1}{2}}, -a^2; q\right)_n}$$
$$\times p_{n-m}\left(x; aq^{m/2}, aq^{(m+1)/2}, -aq^{m/2}, -aq^{(m+1)/2}|q\right)$$
$$\times p_{n-m}\left(y; aq^{m/2}, aq^{(m+1)/2}, -aq^{m/2}, -aq^{(m+1)/2}|q\right).$$

(Rahman and Verma [1986b])

8.12 Prove the inverse of the linearization formula (8.5.1), namely,

$$C_{m+n}(x; \beta|q) = \sum_{k=0}^{\min(m,n)} b(k, m, n) C_{m-k}(x; \beta|q) C_{n-k}(x; \beta|q),$$

where

$$b(k,m,n) = \frac{(q;q)_m(q;q)_n(\beta;q)_{m+n}}{(\beta;q)_m(\beta;q)_n(q;q)_{m+n}} \times \frac{\left(\beta^{-2}q^{-m-n},\beta^{-1};q\right)_k\left(1-\beta^{-2}q^{2k-m-n}\right)}{\left(q,\beta^{-1}q^{1-m-n};q\right)_k\left(1-\beta^{-2}q^{-m-n}\right)}\left(\beta^2q^{-1}\right)^k.$$

8.13 Give alternate derivations of (8.6.3) and (8.6.4) by using the q-integral representation (7.4.7) of $C_n(x;\beta|q)$ and the q-integral formula (2.10.19) for an ${}_8\phi_7$ series.

8.14 By equating the coefficients of $e^{i(m+n-2k)\theta}$ on both sides of the linearization formula (8.5.1) show that

(i)

$$\begin{aligned}{}_4\phi_3&\left[\begin{matrix}q^{-k},q^{-m},\beta,\beta q^{n-k}\\ \beta^{-1}q^{1-k},\beta^{-1}q^{1-m},q^{1+n-k}\end{matrix};q,\frac{q^2}{\beta^2}\right] = \frac{\left(q^{-m-n},\beta^{-1}q^{1-n};q\right)_k}{\left(q^{-n},\beta^{-1}q^{1-m-n};q\right)_k}\\ &\times\Phi\left[\begin{matrix}\beta, & q^{-k}, & q^{-m}, & q^{-n}, & \beta^{-1}q^{-m-n},\\ & \beta^{-1}q^{1-k}, & \beta^{-1}q^{1-m}, & \beta^{-1}q^{1-n}, & \beta^{-2}q^{1-m-n},\end{matrix}\right.\\ &\qquad\left.\begin{matrix}q^{k-m-n}: & \beta^{-2}q^{1-m-n}, & \beta^{-2}q^{2-m-n}, & \beta^{-1}q^{2-m-n}\\ \beta^{-1}q^{1+k-m-n}: & q^{1-m-n}, & q^{-m-n}, & \beta^{-1}q^{-m-n}\end{matrix};q,q^2;\frac{q}{\beta}\right],\end{aligned}$$

for $k=0,1,\ldots,n$, and

(ii)

$$\begin{aligned}{}_4\phi_3&\left[\begin{matrix}q^{k-m-n},q^{-n},\beta,\beta q^{k-n}\\ \beta^{-1}q^{1-n},\beta^{-1}q^{1+k-m-n},q^{1+k-n}\end{matrix};q,\frac{q^2}{\beta^2}\right]\\ &= \frac{\left(q^{m+1},\beta^{-1}q^{1-k};q\right)_n}{\left(q^{-k},\beta q^m;q\right)_n}\left(\beta q^{-1}\right)^n\\ &\times\Phi\left[\begin{matrix}\beta, & q^{-k}, & q^{-m}, & q^{-n}, & \beta^{-1}q^{-m-n},\\ & \beta^{-1}q^{1-k}, & \beta^{-1}q^{1-m}, & \beta^{-1}q^{1-n}, & \beta^{-2}q^{1-m-n},\end{matrix}\right.\\ &\qquad\left.\begin{matrix}q^{k-m-n}: & \beta^{-2}q^{1-m-n}, & \beta^{-2}q^{2-m-n}, & \beta^{-1}q^{2-m-n}\\ \beta^{-1}q^{1+k-m-n}: & q^{1-m-n}, & q^{-m-n}, & \beta^{-1}q^{-m-n}\end{matrix};q,q^2;\frac{q}{\beta}\right]\end{aligned}$$

for $k=n,n+1,\ldots,n+m$, where Φ is the bibasic series defined in §3.9. (Gasper [1985])

8.15 Show that, by analytic continuation, it follows from Ex. 8.14 that

$$\begin{aligned}{}_4\phi_3&\left[\begin{matrix}a, & b, & c, & d\\ & bq/a, & cq/a, & dq/a\end{matrix};q,\frac{q^2}{a^2}\right]\\ &= \frac{(a/d,bq/d,cq/d,abc/d;q)_\infty}{(q/d,ab/d,ac/d,bcq/d;q)_\infty}\\ &\times {}_{12}W_{11}(bc/d;(bcq/ad)^{\frac12},-(bcq/ad)^{\frac12},q(bc/ad)^{\frac12},-q(bc/ad)^{\frac12},\\ &\qquad ab/d,ac/d,a,b,c;q,q/a),\end{aligned}$$

where at least one of a, b, c is of the form $q^{-n}, n = 0, 1, \ldots$.
(Gasper [1985])

8.16 From Ex. 8.15 deduce that

$$ {}_{10}W_9\left(a; q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q, a^2/b, a^2/c, b, c; q, q/a\right) = \frac{\left(aq, a^2/b, a^2/c, aq/bc; q\right)_\infty}{\left(a^2, aq/b, aq/c, a^2/bc; q\right)_\infty} {}_4\phi_3\left[\begin{matrix} a, & b, & c, & bc/a \\ & bq/a, & cq/a, & bcq/a^2 \end{matrix}; q, \frac{q^2}{a^2}\right] $$

where one of a, b, c is of the form $q^{-n}, n = 0, 1, \ldots$.

8.17 Prove that

$$ \left\{ {}_4\phi_3\left[\begin{matrix} a^2, b^2, abz, ab/z \\ a^2b^2q, -ab, -abq \end{matrix}; q^2, q^2\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, abz, ab/z \\ a^2b^2, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right] $$

and

$$ \left\{ {}_4\phi_3\left[\begin{matrix} a^2, b^2, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -a^2b^2 \end{matrix}; q, q\right]\right\}^2 = {}_5\phi_4\left[\begin{matrix} a^4, b^4, a^2b^2, a^2b^2a^2, a^2b^2/z^2 \\ a^4b^4, a^2b^2q, -a^2b^2q, -a^2b^2 \end{matrix}; q^2, q^2\right] $$

when the series terminate.
(Gasper [1989b])

8.18 With the notation of Ex. 7.37 prove that

$$ D_q\left[\left(q^{1-\nu}e^{i\theta}, q^{1-\nu}e^{-i\theta}; q\right)_{2\nu} U_n(\cos\theta)\right] \le 0 $$

when $\nu > -\frac{1}{2}, \lambda \ge 0$, $0 < q < 1, \theta$ is real and $n = 1, \ldots, r$.
(Gasper [1989b])

8.19 Prove the expansion formulas

(i) $$ \mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty \omega^{-\nu}}{(q^\nu;q)_\infty(-q\omega^2;q^2)_\infty} \times \sum_{n=0}^{\infty} i^n(1-q^{\nu+n})q^{n^2/4} J^{(2)}_{\nu+n}(2\omega;q) C_m(x;q^\nu|q), $$

(ii) $$ \mathcal{E}_q(x; i\omega) = \frac{(-\omega^2;q^2)_\infty}{(-i\omega;q^{1/2})_\infty} \sum_{n=0}^{\infty} q^{n(n-2\alpha-1)/4}(i\omega)^n \times \frac{(-q^{(\alpha+\beta+1)/2};q^{1/2})_n(q^{\alpha+\beta+1};q)_n}{(q^{\alpha+\beta+1};q)_{2n}} \times {}_2\phi_1\left[\begin{matrix} q^{(\beta+n+1)/2}, -q^{(\alpha+n+1)/2} \\ q^{(\alpha+\beta+2n+2)/2} \end{matrix}; q^{1/2}, i\omega\right] P_n^{(\alpha,\beta)}(x|q). $$

(See Ismail and Zhang [1994]) for (i), and Ismail, Rahman and Zhang [1996] for (ii).)

8.20 Prove the following linearization formula

$$p_m(x;a,q/a,-b,-q/b|q)p_n(x;a,q/a,-b,-q/b|q)$$
$$= \sum_{k=0}^{2\min(m,n)} C_k p_{m+n-k}(x;a,q/a,-b,-q/b|q),$$

where

$$C_k = \frac{(q;q)_{2m}(q;q)_{2n}(q;q)_{m+n}}{(q;q)_m(q;q)_n(q;q)_{2m+2n}} b^{-k} q^{(3m+3n+1)k-3\binom{k}{2}} \frac{1-q^{2m+2n+1-2k}}{1-q^{2m+2n+1-k}}$$
$$\times \frac{(q^{-m},q^{-m},q^{-n},q^{-n},q^{-2m-2n},-bq^{-m-n}/a,-abq^{-m-n-1};q)_k}{(q,q^{-2m},q^{-2n},q^{-m-n};q)_k}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-k},q^{-m},q^{-n},-bq^{m+n+1-k}/a \\ q^{1+m-k},q^{1+n-k},-bq^{-m-n}/a \end{matrix};q,q\right]$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-k},q^{-m},q^{-n},-abq^{m+n-k} \\ q^{1+m-k},q^{1+n-k},-abq^{-m-n-1} \end{matrix};q,q\right],$$

with $a,b>0$.

(Koelink and Van der Jeugt [1998], after transforming their formula to a symmetric form in m and n)

8.21 Prove the following convolution identity for the Askey-Wilson polynomials:

$$\sum_{k=0}^{m+n} b^{m-k} \begin{bmatrix} m+n \\ k \end{bmatrix}_q \frac{(b^2;q)_n(a^2b^2c^2q^{m+n-1};q)_n(c^2;q)_k}{(c^2,b^2c^2q^{k-1};q)_k(b^2c^2q^{2k};q)_{m+n-k}}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-k},b^2c^2q^{k-1},q^{-n},a^2b^2q^{n-1} \\ b^2,a^2b^2c^2q^{m+n-1},q^{-m-n} \end{matrix};q,q\right]$$
$$\times p_k\Big(\frac{w_2+w_2^{-1}}{2};bw_1,b/w_1,cs,c/s|q\Big)p_{m+n-k}\Big(\frac{w_1+w_1^{-1}}{2};at,a/t,bcsq^k,bcq^k/s|q\Big)$$
$$= p_n\Big(\frac{w_1+w_1^{-1}}{2};at,a/t,bw_2,b/w_2|q\Big)p_m\Big(\frac{w_2+w_2^{-1}}{2};abtq^n,abq^n/t,cs,c/s|q\Big).$$

(Koelink and Van der Jeugt [1998])

8.22 Show that

$${}_8W_7\Big(\gamma\tilde{a}\tilde{b}\tilde{c}/q;az,a/z,\gamma\tilde{a},\gamma\tilde{b},\gamma\tilde{c};q,q/\gamma\tilde{d}\Big)$$
$$= \frac{(\gamma\tilde{a}\tilde{b}\tilde{c},bc,bq/d,cq/d;q)_\infty}{(abcq/d,q/\gamma\tilde{d},qax\gamma/\tilde{d},qa\gamma/x\tilde{d};q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{\Big(1-abcq^{2n}/d\Big)\Big(abc/d,ab,ac;q\Big)_n}{\Big(1-abc/d\Big)(q,cq/d,bq/d;q)_n}(-ad)^{-n}q^{\binom{n+1}{2}}$$
$$\times r_n\Big(\frac{\gamma+\gamma^{-1}}{2};\tilde{a},\tilde{b},\tilde{c},q/\tilde{d}|q\Big)r_n\Big(\frac{z+z^{-1}}{2};a,b,c,q/d|q\Big),$$

where $\tilde{a}=\sqrt{abcd/q}$, $\tilde{a}\tilde{b}=ab$, $\tilde{a}\tilde{c}=ac$, $\tilde{a}\tilde{d}=ad$.

(Stokman [2002])

8.23 Prove the following product formula for the continuous q-Jacobi polynomials:

$$\begin{aligned} &r_n(x;a,aq,-c,-cq|q^2)r_n(y;a,aq,-c,-cq|q^2)\\ &=\int_{-1}^{1} K(x,y,z)r_n(z;a,aq,-c,-cq|q^2)dz, \end{aligned}$$

where $x=\cos\theta$, $y=\cos\phi$, $0\le\theta,\ \phi\le\pi$, $0<q<1$, $0\le a<c<1$, and

$$\begin{aligned} K(x,y,z) &= \frac{(q^2,a^2,c^2;q)_\infty (q,ac,a/c,ae^{i\theta},ae^{-i\theta},ae^{i\phi},ae^{-i\phi};q)_\infty}{4\pi^2(a^2;q)_\infty(a^2c^2;q^2)_\infty}\\ &\times \frac{(e^{2i\psi},e^{-2i\psi};q^2)_\infty}{(ae^{i\theta+i\phi+i\psi},ae^{i\theta+i\phi-i\psi},ae^{i\psi-i\theta-i\phi},ae^{-i\theta-i\phi-i\psi};q^2)_\infty\sin\psi}\\ &\times\int_{-1}^{1}\frac{h(\tau;1,-1,q^{1/2},-q^{1/2},-\sqrt{ac}\,e^{i(\theta+\phi)/2},-\sqrt{ac}\,e^{-i(\theta+\phi)/2})}{h(\tau;\sqrt{c}e^{i\psi/2},\sqrt{c}e^{-i\psi/2},-\sqrt{c}e^{i\psi/2},-\sqrt{c}e^{-i\psi/2},\sqrt{\frac{a}{c}}e^{i(\theta-\phi)/2},\sqrt{\frac{a}{c}}e^{i(\phi-\theta)/2})}\\ &\times\frac{d\tau}{\sqrt{1-\tau^2}}. \end{aligned}$$

(Rahman [1986d])

8.24 Let m,n be non-negative integers with $n\ge m$. Prove the following linearization formula:

$$\begin{aligned} &r_n(x,q^{1/2},a,-b,-q^{1/2}|q)r_{n-m}(x;q^{1/2},a,-b,-q^{1/2}|q)\\ &=\sum_{k=0}^{2(n-m)} c_k r_{m+k}(x;q^{1/2},a,-b,-q^{1/2}|q), \end{aligned}$$

where

$$\begin{aligned} c_k &= \frac{(ab;q)_{2m}(ab;q)_{2n-2m}(q,-aq^{1/2},bq^{1/2},-ab;q)_n}{(abq;q)_{2n}(q,-aq^{1/2},bq^{1/2},-ab;q)_m(-q,aq^{1/2},-bq^{1/2},ab;q)_{n-m}}\\ &\times\frac{(1-abq^{2m+2k})(abq^{2m},-q^{m+1},-bq^{m+1/2},a/b,a^2b^2q^{2n},q^{2m-2n};q)_k}{(1-ab)(q,-abq^m,-aq^{m+1/2},b^2q^{2m+1},q^{2m-2n+1}/ab,abq^{2n+1};q)_k}q^{n-m+k/2}a^{-k}\\ &\times{}_{10}W_9\Big(b^2q^{2m};b^2,b^2q^{2n+1},q^{2m-2n+1}/a^2,abq^{2m+k},abq^{2m+k+1},q^{1-k},q^{-k};q^2,q^2\Big). \end{aligned}$$

(Rahman [1981])

8.25 Let the *little q-Legendre function* be defined by

$$p_\nu(q^x;a,b;q) = {}_2\phi_1\left[\begin{matrix} q^{-\nu},abq^{\nu+1}\\ aq\end{matrix};q,q^{x+1}\right],\quad \nu\in\mathcal{C},$$

for x a nonnegative integer. Derive the following addition formula:

$$\begin{aligned}
&p_\nu(q^z;1,1;q)\ W_y(q^z;q^x,q)\\
&= W_y(q^z;q^x,q)\ p_\nu(q^y,1,1;q)\ p_\nu(q^{x+y};1,1;q)\\
&\quad+\sum_{k=1}^{\infty}\frac{(q^{-\nu},q^{\nu+1},q^{x+y+1};q)_k}{(q,q;q)_k}(-1)^k q^{yk+\binom{k+1}{2}}\\
&\quad\times\ p_{\nu-k}(q^y;q^k,q^k;q)\ p_{\nu-k}(q^{x+y};q^k,q^k;q)\ W_{y+k}(q^z;q^x,q)\\
&\quad+\sum_{k=1}^{y}\frac{(q;q)_y(q^{-\nu},q^{\nu+1};q)_k}{(q,q;q)_k(q;q)_{y-k}}(-1)^k q^{(x+y+1)k-\binom{k}{2}}\\
&\quad\times\ p_{\nu-k}(q^{y-k};q^k,q^k;q)\ p_{\nu-k}(q^{x+y-k};q^k,q^k;q)\ W_{y+k}(q^z;q^x,q),
\end{aligned}$$

where $W_n(x;a,q)$ is the Wall polynomial, defined in Notes §7.3. (Rahman and Tariq M. Qazi [1999])

8.26 The *associated Askey-Wilson polynomials* $r_n^\alpha(x)=r_n^\alpha(x;a,b,c,d|q)$ satisfy the 3-term recurrence relation

$$(2x-a-a^{-1}+A_{n+\alpha}+C_{n+\alpha})\ r_n^\alpha(x)$$

$$=A_{n+\alpha}r_{n+1}^\alpha(x)+C_{n+\alpha}r_{n-1}^\alpha(x),$$

where $n=0,1,2,\ldots$, $r_{-1}^\alpha(x)=0$, $r_0^\alpha(x)=1$, α is the association parameter, and

$$\begin{aligned}
A_\lambda &= \frac{a^{-1}(1-abq^\lambda)(1-acq^\lambda)(1-adq^\lambda)(1-abcdq^{\lambda-1})}{(1-abcdq^{2\lambda-1})(1-abcdq^{2\lambda})},\\
C_\lambda &= \frac{a(1-bcq^{\lambda-1})(1-bdq^{\lambda-1})(1-cdq^{\lambda-1})(1-q^\lambda)}{(1-abcdq^{2\lambda-2})(1-abcdq^{2\lambda-1})}.
\end{aligned}$$

Verify that, with $x=\cos\theta$,

$$\begin{aligned}
r_n^\alpha(x)&=\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},abcdq^{2\alpha-1},ae^{i\theta},ae^{-i\theta};q)_k}{(q,abq^\alpha,acq^\alpha,adq^\alpha,abcdq^{\alpha-1};q)_k}q^k\\
&\times{}_{10}W_9\Big(abcdq^{2\alpha+k-2};q^\alpha,bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1},q^{k+1},abcdq^{2\alpha+n+k-1},q^{k-n};q,a^2\Big).
\end{aligned}$$

(Ismail and Rahman [1991], Rahman [2001])

8.27 By applying Ex. 2.20 to the ${}_{10}W_9$ series in Ex. 8.26, show that

$$\begin{aligned}
r_n^\alpha(x)&=\frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-n\alpha}\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_k}q^k\\
&\qquad\times\sum_{j=0}^{k}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_j}{(q,abcdq^{2\alpha-2},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_j}q^j.
\end{aligned}$$

Hence show that

$$r_n^\alpha(x) = \frac{(abcdq^{2\alpha-1}, q^{\alpha+1}; q)_n}{(q, abcdq^{\alpha-1}; q)_n} q^{-n\alpha} \times \int_{-1}^{1} K(x,y) r_n(y; aq^{\alpha/2}, bq^{\alpha/2}, cq^{\alpha/2}, dq^{\alpha/2}|q)dy,$$

where

$$K(x,y) = \frac{(q,q,q,abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}, bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha; q)_\infty}{4\pi^2(q^{\alpha+1}, abcdq^{2\alpha-2}; q)_\infty} \times \frac{(e^{2i\phi}, e^{2i\phi}; q)_\infty (1-y^2)^{-1/2}}{h(y; q^{\alpha/2}e^{i\theta}, q^{\alpha/2}e^{-i\theta})} \times \int_0^\pi \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty h(\cos\psi; q^{\frac{\alpha+1}{2}}e^{i\theta}, q^{\frac{\alpha+1}{2}}e^{-i\theta})}{h(\cos\psi; aq^{\frac{\alpha-1}{2}}, bq^{\frac{\alpha-1}{2}}, cq^{\frac{\alpha-1}{2}}, dq^{\frac{\alpha-1}{2}}, q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})} d\psi,$$

$$x = \cos\theta, \quad y = \cos\phi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le \pi, \quad 0 < q < 1,$$

$$\max(|aq^{\frac{\alpha-1}{2}}|, |bq^{\frac{\alpha-1}{2}}|, |cq^{\frac{\alpha-1}{2}}|, |dq^{\frac{\alpha-1}{2}}|) < 1, \ \mathrm{Re}(\alpha) > 0.$$

(Rahman [1996b, 2001])

8.28 The *associated q-ultraspherical polynomials* satisfy the 3-term recurrence relation

$$2x\, C_n^\alpha(x;\beta|q) = \frac{1-\alpha q^{n+1}}{1-\alpha\beta q^n} C_{n+1}^\alpha(x;\beta|q) + \frac{1-\alpha\beta^2 q^{n-1}}{1-\alpha\beta q^n} C_{n-1}^\alpha(x;\beta|q), \ n \ge 0,$$

with $C_{-1}^\alpha(x;\beta|q) = 0$, $C_0^\alpha(x;\beta|q) = 1$. Prove the following linearization formula

$$C_m^\alpha(x;\beta|q) C_n^\alpha(x;\beta|q) = \sum_{k=0}^{\min(m,n)} A_k^{(m,n)} C_{m+n-2k}^\alpha(x;\beta|q),$$

where

$$A_k^{(m,n)} = \frac{(q;q)_m (q;q)_n (\alpha q;q)_{m+n-2k} (\alpha\beta;q)_{m-k} (\alpha\beta;q)_{n-k} (\alpha\beta;q)_k}{(\alpha q;q)_m (\alpha q;q)_n (\alpha\beta^2;q)_{m+n-2k} (q;q)_{m-k} (q;q)_{n-k} (q;q)_k} \times \frac{(\alpha\beta^2;q)_{m+n-k} (q^{k-m-n}/\alpha;q)_k}{(\alpha\beta q;q)_{m+n-k} (q^{k-m-n};q)_k} \alpha^k \frac{(1-\alpha\beta q^{m+n-2k})}{(1-\alpha\beta)} \times {}_{10}W_9\Big(q^{k-m-n-1}; q^{k-m-n}/\alpha\beta, q^{k-m-n}/\alpha\beta, \alpha, \alpha\beta^2/q, q^{k-m}, q^{k-n}, q^{-k}; q, q\Big).$$

(Qazi and Rahman [2003])

8.29 A multivariable extension of the Askey-Wilson polynomials in (7.5.2) is given by

$$P_{\mathbf{n}}(\mathbf{x}|q) = P_{\mathbf{n}}(\mathbf{x}; a, b, c, d, a_2, a_3, \ldots, a_s|q) = \left[\prod_{k=1}^{s-1} p_{n_k}(x_k; aA_{2,k}q^{N_{k-1}}, bA_{2,k}q^{N_{k-1}}, a_{k+1}e^{i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}}|q)\right] \times p_{n_s}(x_s; aA_{2,s}q^{N_{s-1}}, bA_{2,s}q^{N_{s-1}}, c, d|q),$$

where $x_k = \cos\theta_k$, $\mathbf{x} = (x_1, \ldots, x_s)$, $\mathbf{n} = (n_1, \ldots, n_s)$, $N_k = \sum_{j=1}^{k} n_j$,

$$A_{j,k} = \prod_{i=j}^{k} a_i,\, A_{k+1,k} = 1,\ A_k = A_{1,k},\ 1 \le j \le k \le s.$$

Prove that these polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} \cdots \int_{-1}^{1} P_{\mathbf{n}}(\mathbf{x}|q) P_{\mathbf{m}}(\mathbf{x}|q) \rho(\mathbf{x}|q) dx_1 \cdots dx_s = \lambda_{\mathbf{n}}(q) \delta_{\mathbf{n},\mathbf{m}}$$

with $\max(|q|, |a|, |b|, |c|, |d|, |a_2|, \ldots, |a_s|) < 1$, $\delta_{\mathbf{n},\mathbf{m}} = \prod_{j=1}^{s} \delta_{n_j,m_j}$,

$$\begin{aligned}
&\rho(\mathbf{x}|q) = \rho(\mathbf{x}; a, b, c, d, a_2, a_3, \ldots, a_s|q) \\
&= (ae^{i\theta_1}, ae^{-i\theta_1}, be^{i\theta_1}, be^{-i\theta_1}; q)_\infty^{-1} \\
&\times \left[\prod_{k=1}^{s-1} \frac{(e^{2i\theta_k}, e^{-2i\theta_k}; q)_\infty (1 - x_k^2)^{-1/2}}{(a_{k+1}e^{i\theta_{k+1}+i\theta_k}, a_{k+1}e^{i\theta_{k+1}-i\theta_k}, a_{k+1}e^{i\theta_k - i\theta_{k+1}}, a_{k+1}e^{-i\theta_{k+1}-i\theta_k}; q)_\infty} \right] \\
&\times \frac{(e^{2i\theta_s}, e^{-2i\theta_s}; q)_\infty (1 - x_s^2)^{-1/2}}{(ce^{i\theta_s}, ce^{-i\theta_s}, de^{i\theta_s}, de^{-i\theta_s}; q)_\infty},
\end{aligned}$$

$$\begin{aligned}
\lambda_{\mathbf{n}}(q) &= \lambda_{\mathbf{n}}(a, b, c, d, a_2, a_3, \ldots, a_s|q) \\
&= (2\pi)^s \left[\prod_{k=1}^{s} \frac{(q, A_{k+1}^2 q^{N_k + N_{k-1} - 1}; q)_{n_k} (A_{k+1}^2 q^{2N_k}; q)_\infty}{(q, A_k^2 q^{N_k + N_{k-1}}, a_{k+1}^2 q^{n_k}; q)_\infty} \right] \\
&\quad \times (acA_{2,s}q^{N_s}, adA_{2,s}q^{N_s}, bcA_{2,s}q^{N_s}, bdA_{2,s}q^{N_s}; q)_\infty^{-1},
\end{aligned}$$

where $a_1^2 = ab$ and $a_{s+1}^2 = cd$.
(Gasper and Rahman [2003b])

8.30 Prove that

$$p_n(q^k; q, 1; q) = \frac{(q^{k+1}; q)_\infty}{2\pi} \int_0^\pi U_{2n}(\cos\theta)(H_k(\cos\theta|q))^2 (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta.$$

(Koelink [1996])

8.31 Let $\max(|a|, |b|, |c|, |d|) < 1$, $|f| < \min(|a|, |b|, |c|, |d|)$ and let N be a non-negative integer such that $|fq^{-N}| < 1 < |fq^{-N-1}|$. Prove the biorthogonality relation

$$\int_{-1}^{1} R_m(x) S_n(x) v(x; a, b, c, d, f) dx = \frac{g_0(a, b, c, d, f)}{h_n(a, b, c, d, f)} \delta_{m,n}$$

for $m = 0, 1, \ldots$, $n = 0, 1, \ldots, N$, where $v(x; a, b, c, d, f)$ is the weight function defined in (6.4.3), $g_0(a, b, c, d, f)$ is the normalization constant given by (6.4.1),

$$h_n(a, b, c, d, f) = \frac{2\pi(1 - abcdfq^{2n-1})(abcd/q, ab, ac, ad, 1/af, bcdf/q; q)_n}{(1 - abcd/q)(q, cd, bd, bc, a^2bcdf, aq/f; q)_n} q^n,$$

and $R_m(x)$, $S_n(x)$ are the biorthogonal rational functions defined by

$$R_m(x) = R_m(x; a, b, c, d, f)$$
$$= \frac{(a^2bcdf, bc, df, de^{i\theta}, de^{-i\theta}; q)_m}{(ad, d/a, abcdfe^{i\theta}, abcdfe^{-i\theta}; q)_m}$$
$$\times {}_{10}W_9(aq^{-m}/d; q^{1-m}/bd, q^{1-m}/cd, 1/df, abcf, ae^{i\theta}, ae^{-i\theta}, q^{-m}; q, q),$$
$$S_n(x) = S_n(x; a, b, c, d, f)$$
$$= \frac{(aq/f, q/af, de^{i\theta}, de^{-i\theta}; q)_n}{(ad, d/a, qe^{i\theta}/f, qe^{-i\theta}/f; q)_n}$$
$$\times {}_{10}W_9(aq^{-n}/d; q^{1-n}/bd, q^{1-n}/cd, q/df, abcf/q, ae^{i\theta}, ae^{-i\theta}, q^{-n}; q, q)$$

with $x = \cos\theta$.
(Rahman [1991])

Notes

§8.5 For additional nonnegativity results for the coefficients in the linearization of the product of orthogonal polynomials and their applications to convolution structures, Banach algebras, multiplier theory, heat and diffusion equations, maximal principles, stochastic processes, etc., see Askey and Gasper [1977], Gasper [1970, 1971, 1972, 1975a,b, 1976, 1983], Gasper and Rahman [1983b], Gasper and Trebels [1977–2000], Ismail and Mulla [1987], Koornwinder and Schwartz [1997], and Rahman [1986d].

§8.6 and 8.7 The nonnegativity of other Poisson kernels and their applications to probability theory and other fields are considered in Beckmann [1973] and Gasper [1973, 1975a,b, 1976, 1977]. A complicated formula for the Poisson kernel for the Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$ in the most general case is given in Rahman and Verma [1991] (a typo is corrected in Rahman and Suslov [1996b]; also see Askey, Rahman and Suslov [1996] for a nonsymmetric extension of this kernel). Rahman and Tariq M. Qazi [1997a] contains a Poisson kernel for the associated q-ultraspherical polynomials.

§8.8 and 8.9 A historical summary of related inequalities is given in the survey paper Askey and Gasper [1986]. For additional material related to de Branges' proof of the Bieberbach conjecture, see de Branges [1968, 1985, 1986], and de Branges and Trutt [1978], Duren [1983], Gasper [1986], Koornwinder [1984, 1986], and Milin [1977]. Sums of squares are also used in Gasper [1994] to prove that certain entire functions have only real zeros.

§8.10 Additional applications of q-series are given in A.K. Agarwal, Kalnins and Miller [1987], Alder [1969], Andrews, Dyson and Hickerson [1988], Andrews and Forrester [1986], Andrews and Onofri [1984], Askey [1984a,b, 1988a, 1989a,e, 1992–1996], Askey, Koornwinder and Schempp [1984], Askey, Rahman and Suslov [1996], Askey and Suslov [1993a,b], M. N. Atakishiyev, N. M. Atakishiyev and Klimyk [2003], Berndt [1988, 1989], Bhatnagar [1998, 1999], Bhatnagar and Milne [1997], Bhatnagar and Schlosser [1998], Bressoud and Goulden [1985, 1987], W. Chu [1998b], Chung, Kalnins and Miller [1999], Cohen [1988], Comtet [1974], Frenkel and Turaev [1995], Geronimo [1994], Gessel and Krattenthaler [1997], Gustafson [1989–1994b], Gustafson and Krattenthaler [1997], Gustafson and Milne [1986], Gustafson and Rakha [2000], Ismail [1990–2003b], Kadell [1987a–1998], Kirillov [1995], Kirillov and Noumi [1999], Kirillov and Reshetikhin [1989], Koelink [1994–2003], Koelink and Rosengren [2001], Koelink and Stokman [2001a–2003], Koelink and Swarttouw [1994], Koelink and Van der Jeugt [1998, 1999], Koornwinder [1989–1991a, 1992–2003], Koornwinder and Swarttouw [1992], Koornwinder and Touhami [2003], Krattenthaler [1984–2001], Krattenthaler and Rosengren [2003], Krattenthaler and Schlosser [1999], Leininger and Milne [1999a,b], Lilly and Milne [1993], Milne [1980a–2002], Milne and Bhatnagar [1998], Milne and Lilly [1992, 1995], Milne and Newcomb [1996], Milne and Schlosser [2002], Rahman and Suslov [1996b], Rota and Goldman [1969], Rota and Mullin [1970], and Stokman [2002–2003c].

§8.11 Mordell [1917] considered the representation of numbers as the sum of $2r$ squares. Milne [1996, 2002] derived many infinite families of explicit exact formulas for sums of squares. Another proof of (8.11.1) is given in Hirschhorn [1985]. Also see Cooper [2001], Cooper and Lam [2002], and Liu [2001].

Ex. 8.3 For more on classical biorthogonal rational functions, see Rahman and Suslov [1993] and Ismail and Rahman [1996].

Ex. 8.11 A second addition formula for continuous q-ultraspherical polynomials is given by Koornwinder [2003]. Addition formulas for q-Bessel functions are given in Rahman [1988c] and Swarttouw [1992]. For other addition formulas, see Floris [1999], Koelink [1994, 1995a, 1997], Koelink and Swarttouw [1995], and Van Assche and Koornwinder [1991].

Ex. 8.19 A short elementary proof of the formula in (i) was found by Ismail and Stanton [2000], and is reproduced in Suslov [2003]. Other proofs of (ii) are given in Ismail, Rahman and Stanton [1999] and Suslov [2003, §4.8].

Ex. 8.20 The expression given for C_k is symmetric in m and n, but is equal to the one given in Koelink and Van der Jeugt [1998].

Ex. 8.29 This is a q-analogue of the orthogonality relation for the multivariable Wilson polynomials in Tratnik [1989]. Also see the multivariable orthogonal or biorthogonal systems in van Diejen [1996, 1997a, 1999], van Diejen and Stokman [1998, 1999], Gasper and Rahman [2003a,b,c], Rosengren [1999, 2001b], Stokman [1997a,b, 2000, 2001], and Tratnik [1991a,b].

9

LINEAR AND BILINEAR GENERATING FUNCTIONS FOR BASIC ORTHOGONAL POLYNOMIALS

9.1 Introduction

Suppose that a function $F(x, t)$ has a (formal) power series expansion in t of the form

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n. \tag{9.1.1}$$

Then $F(x, t)$ is called a *generating function* for the functions $f_n(x)$. A useful extension of (9.1.1) is the *bilinear generating function*

$$H(x, y, t) = \sum_{n=0}^{\infty} a_n f_n(x) g_n(y) t^n. \tag{9.1.2}$$

We saw some examples of generating functions in Chapters 7 and 8. Some important linear generating functions for the classical orthogonal polynomials are listed in Koekoek and Swarttouw [1998]. In this chapter we restrict ourselves entirely to generating and bilinear generating functions for basic hypergeometric orthogonal polynomials. Of the many uses of generating functions the one that is most commonly applied is Darboux's method to find the asymptotic properties of the corresponding orthogonal polynomials which, in turn, are essential to determining their orthogonality measures. Darboux's method, as described in Ismail and Wilson [1982], is as follows: If $f(t) = \sum_{n=0}^{\infty} f_n t^n$ and $g(t) = \sum_{n=0}^{\infty} g_n t^n$ are analytic in $|t| < r$ and $f(t) - g(t)$ is continuous in $|t| \le r$, then $f_n = g_n + O(r^{-n})$. In a slight generalization of this theorem Ismail and Wilson state, further, that if $f(t)$ and $g(t)$ depend on parameters $a_1, \ldots, a_m$ and $f(t) - g(t)$ is a continuous function of t, $a_1, \ldots, a_m$ for $|t| \le r$ and $a_1, \ldots, a_m$ restricted to compact sets, then the conclusion holds uniformly with respect to the parameters. As far as the bilinear generating functions are concerned one of the most useful is the Poisson kernel $K_t(x, y)$ defined in (8.6.1). A related kernel, the so-called Christoffel-Darboux kernel, is defined for orthonormal polynomials $p_n(x)$ by

$$\sum_{k=0}^{n} p_k(x) p_k(y) = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y}, \tag{9.1.3}$$

where k_n is the coefficient of x^n in $p_n(x)$. (9.1.3) is a fundamental identity in the theory of general orthogonal polynomials, see Szegő [1975], Chihara [1978], Dunkl and Xu [2001], Nevai [1979], and Temme [1996]. Some evaluations of Poisson kernels were given in Chapter 8. Here we shall derive some important bilinear generating functions for basic orthogonal polynomials and give some significant applications.

9.2 The little q-Jacobi polynomials

For the little q-Jacobi polynomials, defined in (7.3.1), we shall first prove that

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} t^n q^{\binom{n}{2}} p_n(x;a,b;q) = \frac{(-t;q)_\infty}{(-xt;q)_\infty}\, {}_2\phi_3 \left[\begin{matrix} -xt, bxq \\ -t, 0, 0 \end{matrix} ; q, -aqt \right]. \tag{9.2.1}$$

Let $G(x,t)$ denote the sum of the series in (9.2.1). Using (1.4.5) and (7.3.1) we have

$$p_n(x;a,b;q) = \frac{(x^{-1};q)_n}{(aq;q)_n} (-x)^n q^{-\binom{n}{2}}\, {}_2\phi_1(q^{-n}, bxq; xq^{1-n}; q, aq^{n+1}), \tag{9.2.2}$$

which, when substituted into the left hand side of (9.2.1), gives

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(bxq;q)_k (x^{-1};q)_{n-k}}{(q;q)_k (q;q)_{n-k}} (-xt)^n (aq^n/x)^k \\ &= \sum_{k=0}^{\infty} \frac{(bxq;q)_k}{(q;q)_k} (-at)^k q^{k^2} \sum_{n=0}^{\infty} \frac{(x^{-1};q)_n}{(q;q)_n} (-xtq^k)^n \\ &= \sum_{k=0}^{\infty} \frac{(bxq;q)_k}{(q;q)_k} \frac{(-tq^k;q)_\infty}{(-xtq^k;q)_\infty} (-at)^k q^{k^2}, \end{aligned} \tag{9.2.3}$$

from which (9.2.1) follows immediately. Note that while the $q \to 1^-$ limit of

$$\sum_{n=0}^{\infty} \frac{(q^{a+1};q)_n}{(q;q)_n} t^n q^{\binom{n}{2}} p_n\left(\frac{1-x}{2}; q^\alpha, q^\beta; q\right)$$

is $\sum_{n=0}^{\infty} P_n^{\alpha,\beta}(x) t^n$, which has the value $2^{\alpha+\beta} R^{-1} (1-t+R)^{-\alpha} (1+t+R)^{-\beta}$ with $R = (1-2xt+t^2)^{\frac{1}{2}}$, see Szegő [1975, (4.4.5)], the last expression on the right side of (9.2.3) does not give this limit directly. Nevertheless, it can be used to derive an asymptotic formula for $p_n(x;a,b;q)$ by applying Darboux's method. Observe that, as a function of t, the pole of $G(x,t)$ that is closest to the origin is at $-x^{-1}$, the next one at $-(qx)^{-1}$, the next at $-(q^2x)^{-1}$, and so on. Clearly,

$$\lim_{t \to -(xq^k)^{-1}} (1+xtq^k) G(x,t) = \frac{(bxq;q)_k}{(q;q)_k} \frac{(x^{-1};q)_\infty}{(q;q)_\infty} (a/x)^k,$$

and so a suitable comparison function for asymptotic purposes is

$$\sum_{k=0}^{\infty} \frac{(bxq;q)_k}{(q;q)_k} \frac{(x^{-1};q)_\infty}{(q;q)_\infty} \frac{(a/x)^k}{1+xtq^k} = \frac{(x^{-1};q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{(abq^{m+1};q)_\infty}{(aq^m/x;q)_\infty} (-xt)^m. \tag{9.2.4}$$

Combining (9.2.4) and the left side of (9.2.1) we find, by Darboux's theorem, as extended by Ismail and Wilson [1982], that

$$p_n(x;a,b;q) \sim \frac{(x^{-1};q)_\infty}{(aq;q)_\infty}(-x)^n q^{-\binom{n}{2}}, \qquad x \neq 0,1,q,q^2,\ldots,$$

uniformly for x, a and b in compact sets.

Now we shall reconsider the Poisson kernel for the little q-Jacobi polynomials given in Ex. 8.3, which has the right positivity properties but suffers from the flaw that it does not directly lead to the known $q \to 1^-$ limit:

$$\sum_{n=0}^\infty \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n n!}{(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n} t^n P_n^{(\alpha,\beta)}(\cos 2\theta) P_n^{(\alpha,\beta)}(\cos 2\phi)$$
$$= \frac{1-t}{(1+t)^{\alpha+\beta+2}} F_4\left[\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1,\beta+1; \frac{a^2}{\kappa^2}, \frac{b^2}{\kappa^2}\right], \quad (9.2.5)$$

where $a = \sin\theta\sin\phi$, $b = \cos\theta\cos\phi$, $\kappa = \frac{1}{2}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)$, $0 \le \theta, \phi \le \pi$, $0 < t < 1$, see Bailey [1935, p. 102]. First, use the product formula Ex. 8.1(i) to obtain the following expression

$$K_t(x,y) = \sum_{n=0}^\infty \frac{1-abq^{2n+1}}{1-ab} \frac{(abq,aq;q)_n}{(q,bq;q)_n} (t/aq)^n \, p_n(x;a,b;q)\, p_n(y;a,b;q)$$
$$= \sum_{r=0}^\infty \sum_{s=0}^{\min(x,y)} \frac{(abq^2;q)_{2r+2s}(q^{-x},q^{-y})_s}{(q,aq;q)_r(q,bq;q)_s} t^{r+s} a^{-s} q^{(x+y)(r+s)-s^2-2rs}$$
$$\times \sum_{n=0}^\infty \frac{1-abq^{2(n+r+s)+1}}{1-abq^{2(r+s)+1}} \frac{(abq^{2r+2s+1};q)_s}{(q;q)_n} q^{\binom{n}{2}}(-t)^n. \quad (9.2.6)$$

What is needed to compute now is a series of the form

$$\sum_{n=0}^\infty \frac{1-aq^{2n}}{1-a} \frac{(a;q)_n}{(q;q)_n} q^{\binom{n}{2}} (-t)^n \equiv h_t(a), \text{ say.} \quad (9.2.7)$$

Without the $q^{\binom{n}{2}}$ factor the computation is trivial via the q-binomial theorem (1.3.2). But with this factor the best one can do seems to be to consider this as the $b \to \infty$ limit of the very-well-poised series

$$_4W_3(a;b;q,t/b)$$

which, by (3.4.8), transforms to

$$\frac{(t,aq^2t^2;q)_\infty}{(qt^2,atq^2;q)_\infty}\; {}_8W_7\Big(aqt;(aq)^{\frac{1}{2}},-(aq)^{\frac{1}{2}},qa^{\frac{1}{2}},-qa^{\frac{1}{2}},bqt;q,t/b\Big)$$
$$= (1-t)\frac{(-qta^{\frac{1}{2}},-tq^{\frac{3}{2}}a^{\frac{1}{2}},t(aq)^{\frac{1}{2}}/b,tqa^{\frac{1}{2}}/b;q)_\infty}{(-tq^{\frac{1}{2}},-tq,t/b,atq^{\frac{3}{2}}/b;q)_\infty}$$
$$\times\, {}_8W_7\Big(atq^{\frac{1}{2}}/b;tq^{\frac{1}{2}},(aq)^{\frac{1}{2}},qa^{\frac{1}{2}},-a^{\frac{1}{2}}/b,-(aq)^{\frac{1}{2}}/b;q,tq^{\frac{1}{2}}\Big), \quad (9.2.8)$$

from (2.10.1). Since $0 < t < 1$, it follows by letting $b \to \infty$ that

$$h_t(a) = (1-t)\frac{(-tqa^{\frac{1}{2}}, -tq^{\frac{3}{2}}a^{\frac{1}{2}}; q)_\infty}{(-tq^{\frac{1}{2}}, -tq; q)_\infty} \times {}_3\phi_2\left[\begin{matrix}(aq)^{\frac{1}{2}}, qa^{\frac{1}{2}}, tq^{\frac{1}{2}} \\ -qta^{\frac{1}{2}}, -tq^{\frac{3}{2}}a^{\frac{1}{2}}\end{matrix}; q, tq^{\frac{1}{2}}\right]. \tag{9.2.9}$$

So, if we replace a and b in (9.2.6) by q^α and q^β, respectively, and use (I.6) and (I.25), then $K_t(x,y)$ of (9.2.6) can be written in a more suggestive form

$$\begin{aligned} K_t(x,y) = {} & \frac{(1-t)}{(-tq^{\frac{1}{2}}; q^{\frac{1}{2}})_{\alpha+\beta+2}} \\ & \times \sum_{r=0}^{\infty} \sum_{s=0}^{\min(x,y)} \frac{\left(q^{\frac{1}{2}(\alpha+\beta+2)}, q^{\frac{1}{2}(\alpha+\beta+3)}, -q^{\frac{1}{2}(\alpha+\beta+2)}, -q^{\frac{1}{2}(\alpha+\beta+3)}; q\right)_{r+s}}{\left(-tq^{\frac{1}{2}(\alpha+\beta+3)}, -tq^{\frac{1}{2}(\alpha+\beta+4)}; q\right)_{r+s}} \\ & \times \frac{(q^{-x}, q^{-y}; q)_s (tq^{x+y})^{r+s} q^{-s^2-2rs-2s}}{(q, q^{\alpha+1}; q)_r (q, q^{\beta+1}; q)_s} \\ & \times {}_3\phi_2\left[\begin{matrix} q^{r+s+\frac{1}{2}(\alpha+\beta+2)}, q^{r+s+\frac{1}{2}(\alpha+\beta+3)}, tq^{\frac{1}{2}} \\ -tq^{r+s+\frac{1}{2}(\alpha+\beta+3)}, -tq^{r+s+\frac{1}{2}(\alpha+\beta+4)} \end{matrix}; q, tq^{\frac{1}{2}}\right]. \end{aligned} \tag{9.2.10}$$

It is obvious that the $q \to 1^-$ limit of (9.2.10) is indeed (9.2.5) and that the expression on the right side of (9.2.10) is nonnegative when $0 < t < 1$ and $\alpha, \beta > -1$.

9.3 A generating function for Askey-Wilson polynomials

There are many different generating functions for the Askey-Wilson polynomials $r_n(x; a, b, c, d|q)$ defined in (8.4.4), of which

$$G_t(x; a, b, c, d|q) = \sum_{n=0}^{\infty} \frac{(abcd/q; q)_n}{(q; q)_n} t^n r_n(x; a, b, c, d|q), \quad |t| < 1, \tag{9.3.1}$$

is one of the simplest. By Ex. 7.34,

$$G_t(x; a, b, c, d|q) = D^{-1}(\theta) \int_{qe^{i\theta}/d}^{qe^{-i\theta}/d} \frac{(due^{i\theta}, due^{-i\theta}, abcdu/q; q)_\infty}{(dau/q, dbu/q, dcu/q; q)_\infty} \times {}_3\phi_2\left[\begin{matrix} abcd/q, bc, q/u \\ ad, abcdu/q \end{matrix}; q, adut/q\right] d_q u, \tag{9.3.2}$$

where $D(\theta)$ is defined in Ex. 7.34. However, by (3.4.1),

$$\begin{aligned} {}_3\phi_2\left[\begin{matrix} abcd/q, bc, q/u \\ ad, abcdu/q \end{matrix}; q, adut/q\right] = {} & \frac{(abcdt/q; q)_\infty}{(t; q)_\infty} \\ & \times {}_5\phi_4\left[\begin{matrix} (abcd/q)^{\frac{1}{2}}, -(abcd/q)^{\frac{1}{2}}, (abcd)^{\frac{1}{2}}, -(abcd)^{\frac{1}{2}}, adu/q \\ ad, abcdu/q, abcdt/q, q/t \end{matrix}; q, q\right] \\ & + \frac{(abcd/q, adt, abcdut/q, adu/q; q)_\infty}{(ad, abcdu/q, adut/q, 1/t; q)_\infty} \end{aligned}$$

$$\times {}_5\phi_4 \left[\begin{matrix} t(abcd/q)^{\frac{1}{2}}, -t(abcd/q)^{\frac{1}{2}}, t(abcd)^{\frac{1}{2}}, -t(abcd)^{\frac{1}{2}}, adut/q \\ adt, abcdut/q, abcdt^2/q, qt \end{matrix} ; q, q \right]. \tag{9.3.3}$$

Substituting this into (9.3.2), using (2.10.18) and simplifying the coefficients, we find that

$$\begin{aligned} G_t(x; a, b, c, d|q) &= \frac{(abcdt/q; q)_\infty}{(t; q)_\infty} \\ &\times {}_6\phi_5 \left[\begin{matrix} (abcd/q)^{\frac{1}{2}}, -(abcd/q)^{\frac{1}{2}}, (abcd)^{\frac{1}{2}}, -(abcd)^{\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad, abcdt/q, q/t \end{matrix} ; q, q \right] \\ &+ \frac{(abcd/q, abt, act, adt, ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(ab, ac, ad, ate^{i\theta}, ate^{-i\theta}, t^{-1}; q)_\infty} \\ &\times {}_6\phi_5 \left[\begin{matrix} t(abcd/q)^{\frac{1}{2}}, -t(abcd/q)^{\frac{1}{2}}, t(abcd)^{\frac{1}{2}}, -t(abcd)^{\frac{1}{2}}, ate^{i\theta}, ate^{-i\theta} \\ abt, act, adt, abcdt^2/q, qt \end{matrix} ; q, q \right]. \end{aligned} \tag{9.3.4}$$

The generating function given in Ex. 7.34 is a special case of (9.3.4).

A more difficult problem is to evaluate the sum of the series

$$H_t(x; a, b, c, d|q) = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n}{(q; q)_n} t^n \ r_n(x; a, b, c, d|q), \tag{9.3.5}$$

where α is an arbitrary parameter. For $\alpha = ab$ this would, in particular, give a q-analogue of the generating function of Jacobi polynomials given in the previous section. Note that

$$\begin{aligned} H_t(x) &\equiv H_t(x; a, b, c, d|q) \\ &= \frac{(\alpha; q)_\infty}{(abcd/q; q)_\infty} \sum_{m=0}^{\infty} \frac{(abcd/\alpha q; q)_m}{(q; q)_m} \alpha^m G_{tq^m}(x; a, b, c, d|q). \end{aligned} \tag{9.3.6}$$

So, use of (9.3.4) in (9.3.6) gives

$$\begin{aligned} H_t(x) &= \frac{(\alpha, abcdt/q; q)_\infty}{(t, abcd/q; q)_\infty} \sum_{m=0}^{\infty} \frac{(t, abcd/\alpha q; q)_m}{(q, abcdt/q; q)_m} \alpha^m \\ &\times {}_6\phi_5 \left[\begin{matrix} (abcd/q)^{\frac{1}{2}}, -(abcd/q)^{\frac{1}{2}}, (abcd)^{\frac{1}{2}}, -(abcd)^{\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad, abcdtq^{m-1}, q^{1-m}/t \end{matrix} ; q, q \right] \\ &+ \frac{(\alpha, abt, act, adt, ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(ab, ac, ad, ate^{i\theta}, ate^{-i\theta}, t^{-1}; q)_\infty} \sum_{m=0}^{\infty} \frac{(abcd/\alpha q; q)_m}{(q; q)_m} \\ &\times \frac{(ate^{i\theta}, ate^{-i\theta}; q)_m q^{\binom{m+1}{2}} (-\alpha t)^m}{(abt, act, adt, qt; q)_m} \\ &\times {}_6\phi_5 \left[\begin{matrix} tq^m(abcd/q)^{\frac{1}{2}}, -tq^m(abcd/q)^{\frac{1}{2}}, tq^m(abcd)^{\frac{1}{2}}, -tq^m(abcd)^{\frac{1}{2}}, \\ abtq^m, actq^m, adtq^m, tq^{m+1}, \end{matrix} \right. \\ &\qquad \left. \begin{matrix} atq^m e^{i\theta}, atq^m e^{-i\theta} \\ abcdt^2 q^{2m+1} \end{matrix} ; q, q \right]. \end{aligned} \tag{9.3.7}$$

Since, by (1.4.5)

$$
\begin{aligned}
&{}_2\phi_1(abcd/\alpha q, tq^{-n}; abcdtq^{n-1}; q, \alpha q^n) \\
&= \frac{(\alpha t q^n, abcdq^{n-1}; q)_\infty}{(\alpha q^n, abcdtq^{n-1}; q)_\infty} {}_2\phi_1(q^{-n}, abcd/\alpha q; abcdq^{n-1}; q, \alpha t q^n),
\end{aligned}
\tag{9.3.8}
$$

the first term on the right side of (9.3.7) becomes

$$
\begin{aligned}
&\frac{(\alpha t; q)_\infty}{(t;q)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, abcd/\alpha q, ae^{i\theta}, ae^{-i\theta}; q)_n q^{\binom{n}{2}}}{(q, ab, ac, ad, \alpha t, q/t; q)_n} (-\alpha t)^n \\
&\times {}_7\phi_6 \left[\begin{matrix} q^n(abcd/q)^{\frac{1}{2}}, -q^n(abcd/q)^{\frac{1}{2}}, q^n(abcd)^{\frac{1}{2}}, -q^n(abcd)^{\frac{1}{2}}, \alpha q^n, \\ abq^n, acq^n, adq^n, abcdq^{2n-1}, \end{matrix} \right. \\
&\qquad \left. \begin{matrix} aq^n e^{i\theta}, aq^n e^{-i\theta} \\ \alpha t q^n, q^{n+1}/t \end{matrix} ; q, q \right].
\end{aligned}
\tag{9.3.9}
$$

Interchange of the order of summation followed by the use of (1.4.5) and simplification in the second term of (9.3.7) gives

$$
\begin{aligned}
&\frac{(\alpha, \alpha t^2, abcdt/q, abt, act, adt, ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(ab, ac, ad, \alpha t, abcdt^2/q, t^{-1}, ate^{i\theta}, ate^{-i\theta}; q)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(abcd/\alpha q, t^{-1}; q)_n}{(q, abcdt/q; q)_n} (\alpha t^2)^n \\
&\times {}_7\phi_6 \left[\begin{matrix} \alpha t, t(abcd/q)^{\frac{1}{2}}, -t(abcd/q)^{\frac{1}{2}}, t(abcd)^{\frac{1}{2}}, -t(abcd)^{\frac{1}{2}}, ate^{i\theta}, ate^{-i\theta} \\ abt, act, adt, \alpha t^2, abcdtq^{n-1}, tq^{1-n} \end{matrix} ; q, q \right].
\end{aligned}
\tag{9.3.10}
$$

In the special case $b = aq^{\frac{1}{2}}$, $d = cq^{\frac{1}{2}}$, $c \to -c$ and $\alpha = ab = a^2 q^{\frac{1}{2}}$, both ${}_7\phi_6$ series above become balanced ${}_4\phi_3$'s, which along with their coefficients, can be combined via (2.10.10). Denoting the sum of (9.3.9) and (9.3.10) in this combination by $G_t(x; a, c|q)$, we get

$$
\begin{aligned}
&G_t(x; a, c|q) \\
&= \frac{(a^2 t, a^2 tq^{\frac{1}{2}}, a^2 cte^{i\theta}, a^2 cte^{-i\theta}; q)_\infty}{(actq^{\frac{1}{2}}, a^3 ctq^{\frac{1}{2}}, ate^{i\theta}, ate^{-i\theta}; q)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(ac, acq^{\frac{1}{2}}, c^2 q^{-\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}; q)_n (a^3 ctq^{\frac{1}{2}}; q)_{2n}}{(q, a^2 t, a^2 tq^{\frac{1}{2}}, a^2 ctq^{\frac{1}{2}} e^{i\theta}, a^2 ctq^{\frac{1}{2}} e^{-i\theta}; q)_n (a^2 c^2; q)_{2n}} (a^2 t^2 q^{\frac{1}{2}})^n \\
&\times {}_8W_7\left(a^3 ctq^{2n-\frac{1}{2}}; atq^{\frac{1}{2}}/c, acq^n, acq^{n+\frac{1}{2}}, aq^n e^{i\theta}, aq^n e^{-i\theta}; q, act\right).
\end{aligned}
\tag{9.3.11}
$$

Unfortunately, neither the ${}_8W_7$ series inside nor the outside series over n can be summed exactly except in the limit $q \to 1^-$. If we replace a and c by $q^{\frac{1}{2}(\alpha+\frac{1}{2})}$ and $q^{\frac{1}{2}(\beta+\frac{1}{2})}$, respectively, in (9.3.11), then

$$
\lim_{q \to 1^-} G_t(x; q^{\frac{1}{2}(\alpha+\frac{1}{2})}, q^{\frac{1}{2}(\beta+\frac{1}{2})}|q)
$$

$$= (1-t)^{\beta+1} R^{-\alpha-\beta-1} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \left[\frac{2t^2(1-x)}{R^2}\right]^n$$
$$\times {}_2F_1 \left[\begin{matrix} n+\frac{1}{2}(\alpha+\beta+1), n+\frac{1}{2}(\alpha+\beta+2) \\ \alpha+\beta+1+2n \end{matrix} ; \frac{2t(1-x)}{R^2} \right]. \quad (9.3.12)$$

By the summation formula Erdélyi [1953 Vol. I. 2.8(6)] we can evaluate the above ${}_2F_1$ series, and then do the summation over n in (9.3.12) by the binomial theorem (1.3.1) to obtain the well-known generating function Szegő [1975, (4.4.5)] for the Jacobi polynomials.

9.4 A bilinear sum for the Askey-Wilson polynomials I

We shall now compute the sum

$$F(x,y|q) := \sum_{n=0}^{\infty} \frac{(f,g;q)_n}{(abcd/f, abcd/g;q)_n} (abcd/fg)^n k_n r_n(x) r_n(y), \quad (9.4.1)$$

where $r_n(x)$ is the Askey-Wilson polynomial given in (8.4.4),

$$k_n = \kappa(a,b,c,d|q) \Big(\int_{-1}^{1} [r_n(x)]^2 w(x) dx \Big)^{-1}, \quad (9.4.2)$$

and f and g are arbitrary parameters such that the series in (9.4.1) has a convergent sum. By Ex. 7.34,

$$r_n(\cos\theta) = B^{-1}(\theta) \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q;q)_\infty (cd, q/u;q)_n}{(bau/q, bcu/q, bdu/q;q)_\infty (ab, abcdu/q;q)_n} \times (abu/q)^n d_q u, \quad (9.4.3)$$

$$r_n(\cos\phi) = C^{-1}(\phi) \int_{qe^{i\phi}/c}^{qe^{-i\phi}/c} \frac{(cve^{i\phi}, cve^{-i\phi}, abcdv/q;q)_\infty (bd, q/v;q)_n}{(cav/q, cbv/q, cdv/q;q)_\infty (ac, abcdv/q;q)_n} \times (acv/q)^n d_q v, \quad (9.4.4)$$

where

$$B(\theta) = -\frac{iq(1-q)}{2b} (q, ac, ad, cd;q)_\infty h(\cos\theta; b) w(\cos\theta; a,b,c,d|q), \quad (9.4.5)$$

and

$$C(\phi) = -\frac{iq(1-q)}{2c} (q, ab, ad, bd;q)_\infty h(\cos\phi; c) w(\cos\phi; a,b,c,d|q). \quad (9.4.6)$$

Hence

$$F(x,y|q) = B^{-1}(\theta) C^{-1}(\phi) \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcdu/q;q)_\infty}{(bau/q, bcu/q, bdu/q;q)_\infty}$$
$$\times \int_{qe^{i\phi}/c}^{qe^{-i\phi}/c} \frac{(cve^{i\phi}, cve^{-i\phi}, abcdv/q;q)_\infty}{(cav/q, cbv/q, cdv/q;q)_\infty}$$
$$\times {}_8W_7(abcd/q; ad, f, g, q/u, q/v; q, ab^2c^2duv/fgq^2) d_q u \, d_q v. \quad (9.4.7)$$

However, by (2.10.10) the above ${}_8W_7$ equals

$$\frac{(abcd, bc/f, bc/g, abcd/fg; q)_\infty}{(bc, abcd/f, abcd/g, bc/fg; q)_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix} ad, f, g, abcduv/q^2 \\ abcdu/q, abcdv/q, qfg/bc \end{matrix}; q, q\right]$$
$$+ \frac{(abcd, ad, f, g, abcduv/q^2, ab^2c^2du/fgq, ab^2c^2dv/fgq; q)_\infty}{(bc, abcd/f, abcd/g, abcdu/q, abcdv/q, ab^2c^2duv/fgq^2, fg/bc; q)_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix} bc/f, bc/g, abcd/fg, ab^2c^2duv/fgq^2 \\ ab^2c^2du/fgq, ab^2c^2dv/fgq, bcq/fg \end{matrix}; q, q\right], \tag{9.4.8}$$

which, when substituted into (9.4.7), leads to the sum of two terms, say, $F = F_1 + F_2$, where

$$F_1(x, y|q) = \frac{(abcd, bc/f, bc/g, abcd/fg; q)_\infty}{(bc, abcd/f, abcd/g, bc/fg; q)_\infty} B^{-1}(\theta)C^{-1}(\phi)$$
$$\times \sum_{n=0}^{\infty} \frac{(ad, f, g; q)_n}{(q, qfg/bc; q)_n} q^n$$
$$\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcduq^{n-1}; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty}$$
$$\times \int_{qe^{i\phi}/c}^{qe^{-i\phi}/c} \frac{(cve^{i\phi}, cve^{-i\phi}, abcdvq^{n-1}, abcduvq^{-2}; q)_\infty}{(cav/q, cbv/q, cdv/q, abcduvq^{n-2}; q)_\infty} d_qv\, d_qu, \tag{9.4.9}$$

and

$$F_2(x, y|q) = \frac{(abcd, ad, f, g; q)_\infty}{(bc, abcd/f, abcd/g, fg/bc; q)_\infty} B^{-1}(\theta)C^{-1}(\phi)$$
$$\times \sum_{n=0}^{\infty} \frac{(bc/f, bc/g, abcd/fg; q)_n}{(q, bcq/fg; q)_n} q^n$$
$$\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, ab^2c^2duq^{n-1}/fg; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty}$$
$$\times \int_{qe^{i\phi}/c}^{qe^{-i\phi}/c} \frac{(cve^{i\phi}, cve^{-i\phi}, ab^2c^2dvq^{n-1}/fg, abcduv/q^2; q)_\infty}{(cav/q, cbv/q, cdv/q, ab^2c^2duvq^{n-2}/fg; q)_\infty}$$
$$\times d_qv\, d_qu \tag{9.4.10}$$

with $x = \cos\theta,\ y = \cos\phi$. The q-integral over v on the right side of (9.4.9) can be expressed as a terminating ${}_8\phi_7$ series via (2.10.19), which can then be transformed to a balanced ${}_4\phi_3$ series that can be transformed back into a different ${}_8\phi_7$ series. The final expression for this q-integral turns out to be

$$C(\phi)\frac{(de^{i\phi}, de^{-i\phi}, bdu/q; q)_n}{(ad, bd, d/a; q)_n}$$
$$\times {}_8W_7(aq^{-n}/d; ae^{i\phi}, ae^{-i\phi}, q^{1-n}/bd, q^{-n}, abu/q; q, q^2/adu). \tag{9.4.11}$$

The q-integral over u in (9.4.9) then has the form

$$\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, abcduq^{n-1}; q)_\infty}{(bauq^{k-1}, bcu/q, bduq^{n-k-1}; q)_\infty}\, d_q u,$$

which, by (2.10.18), gives

$$B(\theta)\frac{(ae^{i\theta}, ae^{-i\theta}; q)_k (de^{i\theta}, de^{-i\theta}; q)_{n-k}}{(ad;q)_n (ac;q)_k (cd;q)_{n-k}}. \tag{9.4.12}$$

Substitution of (9.4.11) and (9.4.12) into (9.4.9) then gives

$$\begin{aligned} F_1(x,y|q) &= \frac{(abcd, bc/f, bc/g, abcd/fg; q)_\infty}{(bc, abcd/f, abcd/g, bc/fg; q)_\infty} \\ &\times \sum_{n=0}^\infty \frac{(f, g, de^{i\theta}, de^{-i\theta}, de^{i\phi}, de^{-i\phi}; q)_n}{(q, ad, bd, cd, d/a, qfg/bc; q)_n} q^n \\ &\times {}_{10}W_9\Big(aq^{-n}/d; ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}, q^{1-n}/bd, q^{1-n}/cd, q^{-n}; q, \frac{bcq}{ad}\Big). \end{aligned} \tag{9.4.13}$$

The q-integral over v in (9.4.10) equals

$$\begin{aligned} &C(\phi)\frac{(abdue^{-i\phi}/q, ab^2cdq^n e^{-i\phi}/fg; q)_\infty}{(ab^2cduq^{n-1}e^{-i\phi}/fg, abde^{-i\phi}; q)_\infty} \\ &\times {}_8W_7(abde^{-i\phi}/q; ae^{-i\phi}, be^{-i\phi}, de^{-i\phi}, q/u, fgq^{-n}/bc; q, ab^2cduq^{n-1}e^{i\phi}/fg), \end{aligned} \tag{9.4.14}$$

which is a bit more troublesome than the previous case because the ${}_8W_7$ series is nonterminating unless fg/bc is of the form q^{-k}, $k = 0, 1, \ldots$, which cannot be the case because of the factor $(bc/fg;q)_\infty$ in the denominator of $F_1(x,y|q)$ and of the factor $(fg/bc;q)_\infty$ in the denominator of $F_2(x,y|q)$. So either we split up this ${}_8W_7$ series into a pair of balanced ${}_4\phi_3$ series via (2.10.10) and get bogged down in a long and tedious computation, or seek an alternative shorter method. In fact, by (6.3.9) the expression in (9.4.14) can be written as

$$\begin{aligned} &\frac{C(\phi)(q, ae^{i\phi}, be^{i\phi}, de^{i\phi}, \frac{ab^2cq^n}{fg}, \frac{b^2cdq^n}{fg}, \frac{abcdq^n}{fg}, abu/q, bdu/q, adu/q; q)_\infty}{2\pi\big(ab, ad, bd, ab^2cduq^{n-1}e^{i\phi}/fg; q\big)_\infty} \\ &\times \int_{-1}^1 w\Big(z; e^{\frac12 i\phi}\Big(\frac{ab}{d}\Big)^{\frac12}, e^{\frac12 i\phi}\Big(\frac{bd}{a}\Big)^{\frac12}, e^{\frac12 i\phi}\Big(\frac{ad}{b}\Big)^{\frac12}, e^{-\frac12 i\phi}\frac{u(abd)^{\frac12}}{q}\,|\, q\Big) \\ &\times \frac{h\big(z; e^{-\frac12 i\phi}(abd)^{\frac12}\big)}{h\big(z; bc(abd)^{\frac12}q^n e^{-\frac12 i\phi}/fg\big)} dz, \end{aligned} \tag{9.4.15}$$

assuming, for the time being, that

$$\max\Big(\Big|\Big(\frac{ab}{d}\Big)^{\frac12}\Big|, \Big|\Big(\frac{bd}{a}\Big)^{\frac12}\Big|, \Big|\Big(\frac{ad}{b}\Big)^{\frac12}\Big|, \Big|\frac{bc}{fg}\Big|\Big) < 1. \tag{9.4.16}$$

If we make the further assumption that $ad = bc$, then the q-integral over u reduces to

$$\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, ab^2c^2duq^{n-1}/fg; q)_\infty d_q u}{(ab^2cduq^{n-1}e^{i\phi}/fg, q^{-1}(abd)^{\frac{1}{2}}ue^{i\psi-\frac{1}{2}i\phi}, q^{-1}(abd)^{\frac{1}{2}}ue^{-i\psi-\frac{1}{2}i\phi}; q)_\infty},$$

which sums to

$$B(\theta)\frac{|(ae^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty|^2 (ce^{-i\phi}; q)_\infty}{\big(ac, cd, ad, abcdq^n e^{i\phi+i\theta}/fg, abcdq^n e^{i\phi-i\theta}/fg; q\big)_\infty} \times \frac{h\big(z; bc^2q^n(abd)^{\frac{1}{2}}e^{\frac{1}{2}i\phi}/fg\big)}{h\big(z; e^{i\theta-\frac{1}{2}i\phi}(ad/b)^{\frac{1}{2}}, e^{-i\theta-\frac{1}{2}i\phi}(ad/b)^{\frac{1}{2}}\big)}. \tag{9.4.17}$$

So the q-integral in u over the expression in (9.4.14) equals

$$\frac{B(\theta)C(\phi)(q, ae^{i\phi}, be^{i\phi}, ce^{-i\phi}, de^{i\phi}; q)_\infty |(ae^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty|^2}{2\pi(ab, ac, ad, ad, bd, cd; q)_\infty}$$
$$\times \frac{(abcdq^n/fg, ab^2cq^nfg, b^2cdq^n/fg; q)_\infty}{(abcdq^n e^{i\phi+i\theta}/fg, abcdq^n e^{i\phi-i\theta}/fg; q)_\infty}$$
$$\times \int_{-1}^{1} \frac{h(z; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{h\big(z; (ab/d)^{\frac{1}{2}}e^{\frac{1}{2}i\phi}, (ad/b)^{\frac{1}{2}}e^{\frac{1}{2}i\phi}, (bd/a)^{\frac{1}{2}}e^{\frac{1}{2}i\phi}\big)}$$
$$\times \frac{h(z; (abd)^{\frac{1}{2}}e^{-i\phi/2}, bc^2(abd)^{\frac{1}{2}}q^n e^{\frac{1}{2}i\phi}/fg)}{h(z; (ad/b)^{\frac{1}{2}}e^{i\theta-\frac{1}{2}i\phi}, (ad/b)^{\frac{1}{2}}e^{-i\theta-\frac{1}{2}i\phi}, bc(abd)^{\frac{1}{2}}q^n e^{-\frac{1}{2}i\phi}/fg)}$$
$$\times \frac{dz}{\sqrt{1-z^2}}, \tag{9.4.18}$$

with $ad = bc$. Since this integral is balanced we have, by (6.4.11), as its value the sum of multiples of two balanced very-well-poised ${}_{10}W_9$ series. Combining this result with (9.4.13), we then find the following bilinear summation formula

$$\sum_{n=0}^{\infty} \frac{1-b^2c^2q^{2n-1}}{1-b^2c^2q^{-1}} \frac{(b^2c^2q^{-1}, ab, ac, f, g; q)_n}{(q, bc^2a^{-1}, b^2ca^{-1}, b^2c^2f^{-1}, b^2c^2g^{-1}; q)_n} \Big(\frac{b^2c^2}{fga^2}\Big)^n$$
$$\times r_n(x; a, b, c, bca^{-1}|q)\; r_n(y; a, b, c, bca^{-1}|q)$$
$$= \frac{(b^2c^2, bc/f, bc/g, b^2c^2/fg; q)_\infty}{(bc, b^2c^2/f, b^2c^2/g, bc/fg; q)_\infty} \sum_{n=0}^{\infty} \frac{(f, g; q)_n |(bca^{-1}e^{i\theta}, bca^{-1}e^{i\phi}; q)_n|^2}{(q, bc, bc^2a^{-1}, b^2ca^{-1}, bca^{-2}, fgq/bc; q)_n} q^n$$
$$\times {}_{10}W_9\Big(\frac{a^2q^{-n}}{bc}; ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}, ab^{-1}c^{-2}q^{1-n}, ac^{-1}b^{-2}q^{1-n}, q^{-n}; q, q\Big)$$
$$+ \frac{(b^2c^2, f, g, b^2c^2/fg, b^2c^3/afg; q)_\infty}{(ac, b^2ca^{-1}, bc^2a^{-1}, bc, c/a, b^2c^2/f, b^2c^2/g, fg/bc, ab^3c^2/fg; q)_\infty}$$
$$\times \frac{\Big|\Big(ce^{i\theta}, bca^{-1}e^{i\phi}, b^3c^2e^{i\theta}/fg, ab^2c^2e^{i\phi}/fg; q\Big)_\infty\Big|^2}{\Big|\Big(b^2c^2e^{i\theta+i\phi}/fg, b^2c^2e^{i\theta-i\phi}/fg; q\Big)_\infty\Big|^2}$$

$$\times \sum_{n=0}^{\infty} \frac{\left(bc/f, bc/g, ab^3c^2/fg; q\right)_n \left|\left(b^2c^2e^{i\theta+i\phi}/fg, b^2c^2e^{i\theta-i\phi}/fg; q\right)_n\right|^2 q^n}{\left(q, b^2c^3/afg, bcq/fg; q\right)_n \left|\left(b^3c^2e^{i\theta}/fg, ab^2c^2e^{i\phi}/fg; q\right)_n\right|^2}$$

$$\times {}_{10}W_9\Big(\frac{ab^3c^2q^{n-1}}{fg}; \frac{b^3c^3q^{n-1}}{fg}, \frac{b^2c^2q^n}{fg}, \frac{ab^2cq^n}{fg}, ae^{i\theta}, ae^{-i\theta}, be^{i\phi}, be^{-i\phi}; q, q\Big)$$

$$+ \frac{(b^2c^2, f, g, b^2c^2/fg, ab^2c/fg; q)_\infty}{\left(ab, ac, bc, b^2ca^{-1}, a/c, b^2c^2/f, b^2c^2/g, fg/bc, \frac{b^3c^4}{afg}; q\right)_\infty}$$

$$\times \left|\left(ae^{i\theta}, be^{i\phi}, b^3c^3e^{i\theta}/afg, b^2c^3e^{i\phi}/fg; q\right)_\infty\right|^2$$

$$\times \sum_{n=0}^{\infty} \frac{\left(bc/f, bc/g, b^3c^4/afg; q\right)_n \left|\left(b^2c^2e^{i\theta+i\phi}/afg, b^2c^2e^{i\theta-i\phi}/fg; q\right)_n\right|^2 q^n}{\left(q, abc^2fg, bcq/fg; q\right)_n \left|\left(b^3c^3/afge^{i\theta}, b^2c^3e^{i\phi}/fg; q\right)_n\right|^2}$$

$$\times {}_{10}W_9\Big(\frac{b^3c^4q^{n-1}}{afg}; \frac{b^2c^2q^n}{fg}, \frac{b^3c^3q^{n-1}}{fg}, \frac{b^2c^3q^n}{afg}, ce^{i\theta}, ce^{-i\theta}, bce^{i\phi}/a, bce^{-i\phi}/a; q, q\Big). \tag{9.4.19}$$

Note that by analytic continuation we may now remove the restrictions in (9.4.16) and require only that no zero factors appear in the denominators on the right side of (9.4.19). Note also that we have tacitly assumed that a, b, c and the product fg are real, although it is not necessary. An application of this formula is given in Rahman [1999]. For a more complicated formula without the restriction $ad = bc$, see Rahman [2000b].

9.5 A bilinear sum for the Askey-Wilson polynomials II

Let $\{w_k\}_{k=0}^{\infty}$ be an arbitrary complex sequence such that $\sum_{k=0}^{\infty} |w_k| < \infty$. Suppose

$$G_1(x, y|q) := \sum_{n=0}^{\infty} \nu_n k_n r_n(x) r_n(y), \tag{9.5.1}$$

where $r_n(x) = r_n(x; a, b, c, d|q)$,

$$\nu_n = \frac{(bc, ad; q)_n}{(abcd; q)_{2n}} (-ad)^n q^{\binom{n}{2}} \sum_{j=0}^{\infty} \frac{(adq^n, adq^n; q)_j}{(q, abcdq^{2n}; q)_j} w_{n+j}, \tag{9.5.2}$$

and k_n is the normalization constant given in (9.4.2). We shall prove that

$$G_1(x, y|q) = \sum_{n=0}^{\infty} \frac{(de^{i\theta}, de^{-i\theta}, de^{i\phi}, de^{-i\phi}; q)_n}{(q, bd, cd, d/a; q)_n} w_n$$
$$\times {}_{10}W_9(aq^{-n}/d; q^{1-n}/bd, q^{1-n}/cd, q^{-n}, ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}; q, bcq/ad). \tag{9.5.3}$$

The proof is actually quite simple. Use (8.3.4) to write

$$
\begin{aligned}
&r_n(x)r_n(y)\\
&= \frac{(bd, cd; q)_n}{(ac, ab; q)_n}\left(\frac{a}{d}\right)^n \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}; q)_k}{(q, ad; q)_k}\\
&\quad \times \frac{(de^{i\theta}, de^{-i\theta}, de^{i\phi}, de^{-i\phi}; q)_k}{(ad, bd, cd, d/a; q)_k} q^k\\
&\quad \times {}_{10}W_9(aq^{-k}/d; q^{1-k}/bd, q^{1-k}/cd, q^{-k}, ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}; q, bcq/ad).
\end{aligned} \tag{9.5.4}
$$

Observe that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1-abcdq^{2n-1}}{1-abcdq^{-1}} \frac{(abcdq^{-1}, ad; q)_n}{(q, bc; q)_n}(ad)^{-n} \cdot \frac{(bc, ad; q)_n}{(abcd; q)_{2n}}(-ad)^n q^{\binom{n}{2}}\\
&\quad \times \frac{(q^{-n}, abcq^{n-1}; q)_k}{(ad, ad; q)_k} \sum_{j=0}^{\infty} \frac{(adq^n, adq^n; q)_j}{(q, abcdq^{2n}; q)_j} w_{n+j}\\
&= \sum_{n=0}^{\infty} \frac{1-abcdq^{2k+2n-1}}{1-abcdq^{-1}} \frac{(abcdq^{-1}; q)_{2k+n}}{(q; q)_n (ad, ad; q)_k}(-1)^n q^{\binom{n}{2}-k}\\
&\quad \times \sum_{j=0}^{\infty} \frac{(ad, ad; q)_{k+n+j}}{(q; q)_j (abcd; q)_{2n+2k+j}} w_{n+k+j}\\
&= \frac{(abcd; q)_{2k}}{(ad, ad; q)_k} q^{-k} \sum_{m=k}^{\infty} \frac{(ad, ad; q)_m w_m}{(q; q)_{m-k}(abcd; q)_{m+k}}\, {}_4W_3(abcdq^{2k-1}; q^{k-m}; q, q^{m-k})\\
&= w_k q^{-k},
\end{aligned}
$$

by (2.3.4), which immediately gives (9.5.3).

9.6 A bilinear sum for the Askey-Wilson polynomials III

We shall now consider another bilinear sum, namely,

$$
G_2(x, y|q) = \sum_{n=0}^{\infty} \pi_n k_n r_n(x) r_n(y), \tag{9.6.1}
$$

where

$$
\pi_n = \frac{(bc, ad/f; q)_n}{(ad, bcf; q)_n} f^n \sum_{k=0}^{\infty} \frac{\sigma_k}{(q, bcfq^n, fq^{1-n}/ad; q)_k}, \tag{9.6.2}
$$

$f \neq 0$ being an arbitrary complex parameter and $\{\sigma_k\}_{k=0}^{\infty}$ an arbitrary complex sequence such that $\sum_{k=0}^{\infty} |\sigma_k| < \infty$. Changing the order of summation in (9.6.1), we can write it as

$$
G_2(x, y|q) = \sum_{j=0}^{\infty} \frac{\sigma_j}{(q, bcf, qf/ad; q)_j}
$$

$$\times \sum_{n=0}^{\infty} \frac{(bc, adq^{-j}/f;q)_n}{(ad, bcfq^j;q)_n} (fq^j)^n k_n r_n(x) r_n(y). \tag{9.6.3}$$

In the case $ad = bc$, which is what we shall assume to be true, the sum over n in (9.6.3) is the special case of the left side of (9.4.19) with f and g replaced by bc and bcq^{-j}/f, respectively. Note that in this special case the first term on the right side of (9.4.19) vanishes while the other two double series become single series. Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(bcq^{-j}/f;q)_n}{(bcfq^j;q)_n} (fq^j)^n k_n r_n(x) r_n(y) \\
&= \frac{(b^2c^2, f, cf/a;q)_\infty |(ce^{i\theta}, bca^{-1}e^{i\phi}, bfe^{i\phi}, afe^{i\phi};q)_\infty|^2}{(ac, bc, bc^2a^{-1}, b^2ca^{-1}, ca^{-1}, bcf, abf;q)_\infty |(fe^{i\theta+\phi}, fe^{i\theta-i\phi};q)_\infty|^2} \\
&\quad \times \frac{(abf, bcf;q)_j |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_j|^2}{(f, cf/a;q)_j |(afe^{i\phi}, bfe^{i\theta};q)_j|^2} \\
&\quad \times {}_{10}W_9\left(abfq^{j-1}; bcfq^{j-1}, fq^j, \frac{afq^j}{c}, ae^{i\theta}, ae^{-i\theta}, be^{i\phi}, be^{-i\phi}; q, q\right) \\
&\quad + \frac{(b^2c^2, f, af/c;q)_\infty |(ae^{i\theta}, be^{i\phi}, bcfa^{-1}e^{i\theta}, cfe^{i\phi};q)_\infty|^2}{(ab, ac, bc, b^2ca^{-1}, ac^{-1}, bcf, bc^2fa^{-1};q)_\infty |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_\infty|^2} \\
&\quad \times \frac{(bcf, bc^2fa^{-1};q)_j |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_j|^2}{(f, af/c;q)_j |(bcfa^{-1}e^{i\theta}, cfe^{i\phi};q)_j|^2} \\
&\quad \times {}_{10}W_9\Big(bc^2fa^{-1}q^{j-1}; bcfq^{j-1}, fq^j, \frac{cfq^j}{a}, \\
&\qquad\qquad ce^{i\theta}, ce^{-i\theta}, bca^{-1}e^{i\phi}, bca^{-1}e^{-i\phi}; q, q\Big).
\end{aligned} \tag{9.6.4}$$

Hence,

$$\begin{aligned}
&G_2(x,y|q) \\
&= \frac{(b^2c^2, f, cf/a;q)_\infty |(ce^{i\theta}, bca^{-1}e^{i\phi}, bfe^{i\theta} afe^{i\phi};q)_\infty|^2}{(ac, bc, bc^2a^{-1}, b^2ca^{-1}, c/a, bcf, abf;q)_\infty |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_\infty|^2} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(abf;q)_j |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_j|^2 \sigma_j}{(q, f, cf/a, qf/ac;q)_j |(bfe^{i\theta}, afe^{i\phi};q)_j|^2} \\
&\quad \times {}_{10}W_9(abfq^{j-1}; bcfq^{j-1}, fq^j, afq^j/c, ae^{i\theta}, ae^{-i\theta}, be^{i\phi}, be^{-i\phi}; q, q) \\
&\quad + \frac{(b^2c^2, f, af/c;q)_\infty |(ae^{i\theta}, be^{i\phi}, bcfa^{-1}e^{i\theta}, cfe^{i\phi};q)_\infty|^2}{(ab, ac, bc, b^2ca^{-1}, a/c, bcf, bc^2fa^{-1};q)_\infty |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_\infty|^2} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(bc^2fa^{-1};q)_j |(fe^{i\theta+i\phi}, fe^{i\theta-i\phi};q)_j|^2 \sigma_j}{(q, f, qf/ac, af/c;q)_j |(bcfa^{-1}e^{i\theta}, cfe^{i\phi};q)_j|^2} \\
&\quad \times {}_{10}W_9\left(bc^2fa^{-1}q^{j-1}; bcfq^{j-1}, fq^j, \frac{cfq^j}{a}, ce^{i\theta}, ce^{-i\phi}, bca^{-1}e^{i\phi}, bca^{-1}e^{-i\phi}; q, q\right).
\end{aligned} \tag{9.6.5}$$

Exercises

9.1 Show that

$$\sum_{n=0}^{\infty} \frac{(cq;q)_n}{(q,bq;q)_n} P_n(x;a,b,c;q)t^n$$
$$= {}_2\phi_1(aq/x,0;aq;q,qxt)\ {}_1\phi_1(bx/c;bq;q,cqt),$$

when $P_n(x;a,b,c;q)$ is the big q-Jacobi polynomial defined in (7.3.10). Prove also that

$$P_n(x;a,b,c;q) \sim \frac{(aq/x,cq/x;q)_\infty}{(aq,cq;q)_\infty} x^n$$

as $n \to \infty$, for fixed $x \neq 0$, aq^{m+1}, cq^{m+1}, $m = 0, 1, \ldots$, uniformly for x, a, b and c in compact sets.
(Ismail and Wilson [1982])

9.2 Prove that

$$p_n(q^m;a,b;q) \sim q^{\binom{n-m}{2}}(-aq)^{n-m}(bq^{m+1};q)_\infty/(aq;q)_\infty$$

as $n \to \infty$, $m = 0, 1, \ldots$, uniformly for a and b in compact sets.

9.3 Show that

(i)
$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta;q)_n}{(q,bq;q)_n} q^{\binom{n}{2}} p_n(x;a,b;q)t^n$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha,\beta;q)_{m+n}(x^{-1};q)_m}{(q,aq;q)_n(q,bq;q)_m} q^{n^2} a^n (-xt)^{m+n},$$

(ii)
$$\sum_{n=0}^{\infty} \frac{(abq;q)_n}{(q;q)_n} q^{\binom{n}{2}}(qt)^n p_n(x;a,b;q)$$
$$= \frac{(qt(ab)^{\frac12};q^{\frac12})_\infty}{(tq^{\frac12};q)_\infty} \sum_{k=0}^{\infty} \frac{((abq)^{\frac12},-(abq)^{\frac12},q(ab)^{\frac12},-q(ab)^{\frac12};q)_k}{(q,aq,qt(ab)^{\frac12},qt(abq)^{\frac12};q)_k}$$
$$\times (-xtq)^k\, {}_3\phi_2\left[\begin{matrix}(abq)^{\frac12}q^k,(ab)^{\frac12}q^{k+1},-tq^{\frac12}\\ t(ab)^{\frac12}q^{k+1},t(ab)^{\frac12}q^{k+\frac32}\end{matrix};q,-tq^{\frac12}\right].$$

9.4 For the q-Hahn polynomials defined in (7.2.21) derive the following generating functions:

(i)
$$\sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q,bq;q)_n} Q_n(x;a,b,N;q)t^n$$
$$= {}_2\phi_1(q^{x-N},0;bq;q,tq^{-x})\ {}_1\phi_1(q^{-x};aq;q,aqt),$$

(ii)

$$\sum_{n=0}^{N} \frac{(aq, q^{-N}; q)_n}{(q;q)_n} q^{-\binom{n}{2}} Q_n(x; a, b, N; q) t^n$$
$$= {}_2\phi_1(q^{-x}, bq^{N-x+1}; 0; q, -atq^{x-N+1})\ {}_2\phi_0(q^{x-N}, aq^{x+1}; -; q, -tq^{-x}).$$

(See Koekoek and Swarttouw [1998])

9.5 For the dual q-Hahn polynomials defined in (7.2.23), prove that

(i)
$$\sum_{n=0}^{N} \frac{(q^{-N}; q)_n}{(q;q)_n} R_n(\mu(x); b, c, N; q) t^n$$
$$= {}_2\phi_1(q^{-x}, c^{-1}q^{-x}; bq; q, bctq^{x+1})(tq^{-N}; q)_{N-x},$$

(ii)
$$\sum_{n=0}^{N} \frac{(q^{-N}, bq; q)_n}{(q, c^{-1}q^{-N}; q)_n} R_n(\mu(x); b, c, N; q) t^n$$
$$= {}_2\phi_1(q^{x-N}, bq^{x+1}; c^{-1}q^{-N}; q, tq^{-x})(bcqt; q)_x.$$

(See Koekoek and Swarttouw [1998])

9.6 *Al-Salam–Chihara polynomials* $Q_n(x; a, b|q)$ are the $c = d = 0$ special case of the Askey-Wilson polynomials defined in (7.5.2). Prove that

(i)
$$\sum_{n=0}^{\infty} \frac{Q_n(x; a, b|q) t^n}{(q;q)_n} = \frac{(at, bt; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty},$$

(ii)
$$\sum_{n=0}^{\infty} \frac{(c;q)_n}{(q, ab; q)_n} Q_n(x; a, b|q) t^n$$
$$= \frac{(at, ct/a; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}\ {}_3\phi_2\left[\begin{matrix} ab/c, ae^{i\theta}, ae^{-i\theta} \\ ab, at \end{matrix}; q, ct/a\right],$$

where $x = \cos\theta$, $|ct/a| < 1$, $|t| < 1$.
(See Al-Salam and Chihara [1976], Askey and Ismail [1984], and Koekoek and Swarttouw [1998].)

9.7 Using the recurrence relation of $C_n^\alpha(x; \beta|q)$ in Ex. 8.28 show that

$$\sum_{n=0}^{\infty} C_n^\alpha(x; \beta|q) t^n$$
$$= \frac{1-\alpha}{(1 - 2xt + t^2)}\ {}_3\phi_2\left[\begin{matrix} \beta te^{i\theta}, \beta te^{-i\theta}, q \\ qte^{i\theta}, qte^{-i\theta} \end{matrix}; q, \alpha\right].$$

Deduce that

$$C_n^\alpha(x; \beta|q) = \sum_{k=0}^{n} \frac{1-\alpha}{1-\alpha q^k} \beta^k C_k(x; q/\beta|q) C_{n-k}(x; \beta|q).$$

(Bustoz and Ismail [1982])

9.8 Prove that

$$\sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\beta^2;q)_n} C_n(x;\beta|q)t^n$$
$$= \frac{(\lambda, \beta t e^{i\theta}, \beta t e^{-i\theta};q)_\infty}{(\beta^2, te^{i\theta}, te^{-i\theta};q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} \beta^2/\lambda, te^{i\theta}, te^{-i\theta} \\ \beta t e^{i\theta}, \beta t e^{-i\theta} \end{matrix} ;q, \lambda \right],$$

where $|\lambda| < 1$, $|t| < 1$. Using Ex. 7.35 show that

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x|q)t^n$$
$$= \frac{\left(q, q^{(\alpha+\beta+2)/2}, q^{(\alpha-\beta)/2}, q^\beta; q\right)_\infty \left(q^{(2\alpha+3)/4}e^{i\theta}, q^{2\alpha+3)/4}e^{-i\theta}; q^{1/2}\right)_\infty}{2\pi\left(q^{\alpha+1}, q^{\alpha+\beta+1}; q\right)_\infty}$$
$$\times \int_{-1}^{1} \frac{h\left(y; 1, -1, q^{1/2}, -q^{1/2}, tq^{(5\alpha+\beta+5)/4}\right)}{h\left(y; q^{(\alpha+\beta+1)/4}, q^{(\alpha+\beta+3)/4}, q^{(\alpha-\beta)/4}e^{i\theta}, q^{(\alpha-\beta)/4}e^{-i\theta}, tq^{(\alpha+\beta+1)/4}\right)}$$
$$\times \, {}_3\phi_2 \left[\begin{matrix} q^{\alpha+1}, tq^{(3\alpha-\beta+3)/4}e^{i\phi}, tq^{(3\alpha-\beta+3)/4}e^{-i\phi} \\ tq^{(5\alpha+\beta+5)/4}e^{i\phi}, tq^{(5\alpha+\beta+5)/4}e^{-i\phi} \end{matrix} ;q, q^\beta \right] \frac{dy}{\sqrt{1-y^2}},$$

where $y = \cos\phi$, provided $0 < \beta < \alpha$.

9.9 Show that

$$\sum_{n=0}^{\infty} \frac{p_n(x;a,b,c,d|q)}{(q,ab,cd;q)_n} t^n$$
$$= {}_2\phi_1(a/z, b/z; ab; q, zt) \; {}_2\phi_1(cz, dz; cd; q, t/z),$$

where $x = \frac{1}{2}(z + z^{-1})$, $|tz| < 1$, $|tz^{-1}| < 1$, and deduce the asymptotic formula (7.5.13).
(Ismail and Wilson [1982])

9.10 (i) Prove that, for $\alpha\beta = ab$,

$$\sum_{n=0}^{\infty} \frac{(\alpha t/a)^n}{(q,ab;q)_n} Q_n(x;a,b|q) Q_n(y;\alpha,\beta|q)$$
$$= \frac{\left(bt/a, \alpha^2 te^{i\theta}/a, \alpha^2 te^{-i\theta}/a, \alpha te^{i\phi}, \alpha te^{-i\phi}; q\right)_\infty}{\left(\alpha^2 t, \alpha te^{i\theta+i\phi}/a, \alpha te^{i\theta-i\phi}/a, \alpha te^{i\phi-i\theta}/a, \alpha te^{-i\theta-i\phi}/a; q\right)_\infty}$$
$$\times \, {}_8W_7(\alpha^2 t/q; ae^{i\theta}, ae^{-i\theta}, \alpha e^{i\phi}, \alpha e^{-i\phi}, \alpha t/\beta; q, bt/a), \quad |bt/a| < 1.$$

(Askey, Rahman and Suslov [1996]; Koelink [1995a])

(ii) Show that the continuous q-Hermite polynomial defined in Ex. 1.28 is related to the Al-Salam–Chihara polynomial by

$$H_n(x|q^2) = Q_n(x; q^{\frac{1}{2}}, -q^{\frac{1}{2}}|q).$$

(iii) Using (i), (ii) and the summation formula Ex. 2.17(ii) deduce the q-Mehler's formula

$$\sum_{n=0}^{\infty} H_n(x|q)H_n(y|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{(t^2;q)_\infty}{(te^{i\theta+i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, te^{-i\theta-i\phi};q)_\infty}, \quad |t| < 1,$$

where $x = \cos\theta$, $y = \cos\phi$.
(See Carlitz [1957a])

9.11 Use Ex. 7.33(ii) to prove that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1-q^{m+\frac{1}{2}}}{1-q^{n+1}} q^{\frac{1}{2}n} C_m(\cos\theta; q^{\frac{1}{2}}|q) C_m(\cos\phi; q^{\frac{1}{2}}|q)$$
$$= \left| \frac{\left(qe^{i(\theta+\phi)}, qe^{i(\theta-\phi)};q\right)_\infty}{\left(q^{\frac{1}{2}}e^{i(\theta+\phi)}, q^{\frac{1}{2}}e^{i(\theta-\phi)};q\right)_\infty} \right|^2 .$$

(Ismail and Rahman [2002b])

9.12 Show that

$$\sum_{n=0}^{\infty} \frac{(q,\gamma;q)_n}{(\beta^2,\beta^2;q)_n} C_n(\cos\theta;\beta|q) C_n(\cos\phi;\beta|q) t^n$$
$$= \frac{(\beta,\gamma,t^2,\beta te^{i(\theta+\phi)},\beta te^{i(\theta-\phi)},\beta te^{i(\phi-\theta)},\beta te^{-i(\theta+\phi)};q)_\infty}{(\beta^2,\beta^2,\beta t^2, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)};q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(\beta^2/\gamma, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)};q)_n (\beta t^2;q)_{2n}}{(q,\beta te^{i(\theta+\phi)},\beta te^{i(\theta-\phi)},\beta te^{i(\phi-\theta)},\beta te^{-i(\theta+\phi)};q)_n (t^2;q)_{2n}} \gamma^n$$
$$\times {}_8W_7(\beta t^2 q^{2n-1};\beta, tq^n e^{i(\theta+\phi)}, tq^n e^{i(\theta-\phi)}, tq^n e^{i(\phi-\theta)}, tq^n e^{-i(\theta+\phi)};q,\beta),$$

where $\max(|t|,|\beta|,|\gamma|) < 1$ and $0 \le \theta,\phi \le \pi$.

(Ismail, Masson and Suslov [1997])

9.13 Prove that

$$\sum_{n=0}^{\infty} i^n \frac{(1-q^{n+\nu})(q;q)_n}{(q^{2\nu};q)_n} q^{n(n+2\nu)/4}$$
$$\times J^{(2)}_{\nu+n}(b;q) C_n(\cos\theta;q^\nu|q) C_n(\cos\phi;q^\nu|q)$$
$$= \frac{(-b^2/4;q^2)_\infty (q^\nu,q^\nu,-q;q)_\infty}{(q,q^{2\nu},-q^{\nu+1};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n/4+\nu)}}{(q;q)_n} \left(\frac{b}{2}\right)^{n+\nu}$$

$$\times \Big(- iq^{(1-\nu-n)/2}e^{i(\theta+\phi)}, -iq^{(1-\nu-n)/2}e^{-i(\theta+\phi)}; q\Big)_n$$
$$\times \frac{\Big(iq^{(1+3\nu+n)/2}e^{i(\theta-\phi)}, iq^{(1+3\nu+n)/2}e^{i(\phi-\theta)}; q\Big)_\infty}{\Big(iq^{(1+\nu+n)/2}e^{i(\theta-\phi)}, iq^{(1+\nu+n)/2}e^{i(\phi-\theta)}; q\Big)_\infty}$$
$$\times {}_8W_7\Big(- q^\nu; q^\nu, iq^{(1+\nu+n)/2}e^{i(\theta+\phi)}, iq^{(1+\nu+n)/2}e^{-i(\theta+\phi)},$$
$$iq^{(1-\nu-n)/2}e^{i(\phi-\theta)}, iq^{(1-\nu-n)/2}e^{i(\theta-\phi)}; q, q^\nu\Big).$$

(Ismail, Masson and Suslov [1997])

9.14 Show that

$$\sum_{n=0}^{\infty} \frac{1-q^{\nu+n}}{1-q^\nu} q^{n(n+\nu)/2} J^{(2)}_{\nu+n}(a(1-q)x;q) J^{(2)}_{\nu+n}(b(1-q)x;q) C_n(\cos\psi; q^\nu|q)$$
$$= \Big(\frac{abx^2}{4}\Big)^\nu \Gamma_q^{-1}(\nu+1)\Big(-\frac{a^2(1-q)^2x^2}{4}; q\Big)_\infty$$
$$\times {}_2\phi_1\Big(\frac{b}{a}q^{(\nu+1)/2}e^{i\psi}, \frac{b}{a}q^{(\nu+1)/2}e^{-i\psi}; q^{\nu+1}; q, -\frac{a^2(1-q)^2x^2}{4}\Big),$$

where $\operatorname{Re}\nu > 0$, $0 < b < a$, and $0 \le \psi \le \pi$.
(Rahman [1988c])

9.15 If $k = 0, 1, 2, \ldots$, and $\alpha\beta = ab$, prove that

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n (ab;q)_{n+k}} Q_n(\cos\theta; a, b|q) Q_{n+k}(\cos\phi; \alpha, \beta|q)$$
$$= e^{ik\phi} \frac{(\alpha e^{-i\phi}, \beta e^{-i\phi}, ate^{i\phi}, bte^{i\phi}; q)_\infty}{(\alpha\beta, te^{i(\theta+\phi)}, te^{i(\phi-\theta)}, e^{-2i\phi}; q)_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix}\alpha e^{i\phi}, \beta e^{i\phi}, te^{i(\theta+\phi)}, te^{i(\phi-\theta)} \\ qe^{2i\phi}, ate^{i\phi}, bte^{i\phi}\end{matrix}; q, q^{k+1}\right]$$
$$+ e^{-ik\phi} \frac{(\alpha e^{i\phi}, \beta e^{i\phi}, ate^{-i\phi}, bte^{-i\phi}; q)_\infty}{(\alpha\beta, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)}, e^{2i\phi}; q)_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix}\alpha e^{-i\phi}, \beta e^{-i\phi}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)} \\ qe^{-2i\phi}, ate^{-i\phi}, bte^{-i\phi}\end{matrix}; q, q^{k+1}\right],$$

where $0 \le \theta \le \pi$ and $0 < \phi < \pi$.
(Ismail and Stanton [1997])

9.16 For $k = 0, 1, 2, \ldots$, show that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n+k}(\cos\theta;\beta|q)C_n(\cos\phi;\gamma|q)t^n \\
&= e^{ik\theta}\frac{(\beta,\beta e^{-2i\theta},\gamma te^{i(\theta+\phi)},\gamma te^{i(\theta-\phi)};q)_\infty}{(q,te^{i(\theta+\phi)},te^{i(\theta-\phi)},e^{-2i\theta};q)_\infty} \\
&\quad\times {}_4\phi_3\left[\begin{matrix} q/\beta, qe^{2i\theta}/\beta, te^{i(\theta+\phi)}, te^{i(\theta-\phi)} \\ qe^{2i\theta}, \gamma te^{i(\theta+\phi)}, \gamma te^{i(\theta-\phi)} \end{matrix}; q, \beta^2 q^k\right] \\
&\quad + e^{-ik\theta}\frac{(\beta,\beta e^{2i\theta},\gamma te^{i(\phi-\theta)},\gamma te^{-i(\theta+\phi)};q)_\infty}{(q,te^{i(\phi-\theta)},te^{-i(\theta+\phi)},e^{2i\theta};q)_\infty} \\
&\quad\times {}_4\phi_3\left[\begin{matrix} q/\beta, qe^{-2i\theta}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)} \\ qe^{-2i\theta}, \gamma te^{i(\phi-\theta)}, \gamma te^{-i(\theta+\phi)} \end{matrix}; q, \beta^2 q^k\right],
\end{aligned}$$

where $0 \le \phi \le \pi$ and $0 < \theta < \pi$.
(Ismail and Stanton [1997])

9.17 (i) Use (9.5.1)–(9.5.3) to prove that

$$\begin{aligned}
&\sum_{n=0}^{N} \frac{(q^{-N},\alpha\beta\gamma\delta q^{N-1};q)_n}{(\alpha\beta,-\gamma\delta;q)_n}\frac{(1-abcdq^{2n-1})(abcdq^{-1},ab,ac,-ab;q)_n}{(1-abcdq^{-1})(q,bd;q)_n(abcd;q)_{2n}} \\
&\times q^{\binom{n+1}{2}}(d/a)^n\, {}_4\phi_3\left[\begin{matrix} q^{n-N}, \alpha\beta\gamma\delta q^{n+N-1}, abq^n, -cdq^n \\ abcdq^{2n}, \alpha\beta q^n, -\gamma\delta q^n \end{matrix}; q, q\right] \\
&\times r_n(x;a,b,c,d|q)\ r_n(y;a,b,c,d|q) \\
&= \frac{(-ab,\gamma\delta;q)_N}{(ab,-\gamma\delta;q)_N}(-1)^N \sum_{m=0}^{N} \frac{(q^{-N},\alpha\beta\gamma\delta q^{N-1},-ab;q)_m|(de^{i\theta},de^{i\phi};q)_m|^2 q^m}{(q,ad,ad,bd,-\alpha\beta,\gamma\delta,d/a;q)_m} \\
&\times {}_{10}W_9(aq^{-m}/d; q^{1-m}/bd, q^{1-m}/cd, q^{-m}, ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}; q, \frac{bcq}{ad}).
\end{aligned}$$

(ii) From (i) deduce the projection formula

$$P_m^{(a,b)}(x;q) = \sum_{n=0}^{m} g(n,m)P_n^{(\alpha,\beta)}(x;q),$$

where $P_m^{(a,b)}(x;q)$ is the continuous q-Jacobi polynomial defined in (7.5.25), and

$$\begin{aligned}
g(n,m) &= \frac{(q^{a+1},-q^{b+1};q)_m(q^{\alpha+\beta+1},-q;q)_n}{(q,-q;q)_m(q^{\alpha+\beta+1};q)_{2n}} \\
&\times \frac{(q^{m+a+b+1};q)_n q^{n(n-m)}}{(q;q)_{m-n}(q^{a+1},-q^{b+1};q)_n} \\
&\times {}_4\phi_3\left[\begin{matrix} q^{n-m}, q^{n+m+a+b+1}, q^{\alpha+1+n}, -q^{\beta+1+n} \\ q^{\alpha+\beta+2+2n}, q^{a+n+1}, -q^{b+n+1} \end{matrix}; q, q\right].
\end{aligned}$$

(Rahman [1985])

9.18 In (9.5.2) and (9.5.3) take $ad = bc$ and

$$w_n = \frac{(\alpha, \beta, \gamma, \delta; q)_n}{(bc, bc, bcq/f, \alpha\beta\gamma\delta f/b^3c^3; q)_n}.$$

Also, in (9.6.2) and (9.6.3) take $ad = bc$ and

$$\sigma_n = \frac{(\alpha f/bc, \beta f/bc, \gamma f/bc, \delta f/bc; q)_n}{(\alpha\beta\gamma\delta f^2/b^4c^4; q)_n} q^n.$$

Choosing appropriate multiples of (9.5.1) and (9.6.1), show that

$$\sum_{n=0}^{\infty} \frac{(1-b^2c^2q^{2n-1})(b^2c^2q^{-1}, ab, ac, \alpha, \beta, \gamma, \delta; q)_n (\alpha\beta\gamma f/bc; q)_{2n}}{(1-b^2c^2q^{-1})\left(q, bc^2a^{-1}, b^2ca^{-1}, \frac{\beta\gamma f}{bc}, \frac{\alpha\gamma f}{bc}, \frac{\alpha\beta f}{bc}, \frac{\alpha\beta\gamma\delta f}{b^3c^3}; q\right)_n (b^2c^2; q)_{2n}}$$
$$\times {}_8W_7\left(\frac{\alpha\beta\gamma f q^{2n-1}}{bc}; \frac{\alpha\beta\gamma f}{b^3c^3}, \frac{b^2c^2q^n}{\delta}, \alpha q^n, \beta q^n, \gamma q^n; q, \frac{\delta f}{bc}\right)\left(\frac{f}{a^2}\right)^n$$
$$\times r_n(x; a, b, c, bca^{-1}|q)\ r_n(y; a, b, c, bca^{-1}|q)$$
$$= \frac{(\alpha f/bc, \beta f/bc, \gamma f/bc, \alpha\beta\gamma f/bc; q)_\infty}{(\alpha\beta f/bc, \alpha\gamma f/bc, \beta\gamma f/bc, f/bc; q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(\alpha, \beta, \gamma, \delta; q)_n |(bca^{-1}e^{i\theta}, bca^{-1}e^{i\phi}; q)_n|^2 q^n}{(q, bc, bc, b^2ca^{-1}, bc^2a^{-1}, bca^{-2}, \alpha\beta\gamma\delta f/b^3c^3, bcq/f; q)_n}$$
$$\times {}_{10}W_9\left(a^2q^{-n}/bc; aq^{1-n}/b^2c, aq^{1-n}/bc^2, q^{-n}, ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, ae^{-i\phi}; q, q\right)$$
$$+ \frac{(\alpha, \beta, \gamma, \delta, \alpha\beta\gamma f/bc, \alpha\beta\gamma\delta f^2/b^4c^4, f, af/c; q)_\infty}{(\alpha\beta f/bc, \alpha\gamma f/bc, \beta\gamma f/bc, \delta f/bc, ab, ac, bc, b^2c/a; q)_\infty}$$
$$\times \frac{|(ae^{i\theta}, be^{i\phi}, bcfe^{i\theta}/a, cfe^{i\phi}; q)_\infty|^2}{(a/c, bc^2f/a, bc/f, \alpha\beta\gamma\delta f/b^3c^3; q)_\infty |(fe^{i(\theta+\phi)}, fe^{i(\theta-\phi)}; q)_\infty|^2}$$
$$\times \sum_{n=0}^{\infty} \frac{(\alpha f/bc, \beta f/bc, \gamma f/bc, \delta f/bc, bc^2f/a; q)_n}{(q, f, qf/bc, af/c, \alpha\beta\gamma\delta f^2/b^4c^4; q)_n} \frac{|(fe^{i(\theta+\phi)}, fe^{i(\theta+\phi)}; q)_n|^2 q^n}{|(cfe^{i\phi}, bcfe^{i\theta}/a; q)_n|^2}$$
$$\times {}_{10}W_9(bc^2fq^{n-1}/a; fq^n, bcfq^{n-1}, cfq^n/a, ce^{i\theta}, ce^{-i\theta}, bce^{i\phi}/a, bce^{-i\phi}/a; q, q)$$
$$+ \frac{(\alpha, \beta, \gamma, \delta, \alpha\beta\gamma f/bc, f, cf/a, \alpha\beta\gamma\delta f^2/b^4c^4; q)_\infty}{(\alpha\beta f/bc, \alpha\gamma f/bc, \beta\gamma f/bc, \delta f/bc, ac, bc, b^2ca^{-1}, bc^2a^{-1}; q)_\infty}$$
$$\times \frac{|(ce^{i\theta}, bce^{i\phi}/a, bfe^{i\theta}, afe^{i\phi}; q)_\infty|^2}{(c/a, abf, bc/f, \alpha\beta\gamma\delta f/b^3c^3; q)_\infty |(fe^{i(\theta+\phi)}, fe^{i(\theta-\phi)}; q)_\infty|^2}$$
$$\times \sum_{n=0}^{\infty} \frac{(\alpha f/bc, \beta f/bc, \gamma f/bc, \delta f/bc, abf; q)_n}{(q, f, qf/bc, cf/a, \alpha\beta\gamma\delta f^2/b^4c^4; q)_n} \frac{|(fe^{i(\theta+\phi)}, fe^{i(\theta-\phi)}; q)_n|^2 q^n}{|(afe^{i\phi}, bfe^{i\theta}; q)_n|^2}$$
$$\times {}_{10}W_9(abfq^{n-1}; fq^n, bcfq^{n-1}, afq^n/c, ae^{i\theta}, ae^{-i\theta}, be^{i\phi}, be^{-i\phi}; q, q).$$

(Rahman [1985])

9.19 Show that the special case $\alpha = bc = -\gamma$, $\beta = bcq^{\frac{1}{2}} = -\delta$, $f = t$ of the formula in Ex. 9.18 gives the Poisson kernel for the special Askey-Wilson polynomials $r_n(x; a, b, c, bca^{-1}|q)$:

$$\sum_{n=0}^{\infty} \frac{(1-b^2c^2q^{2n-1})(b^2c^2q^{-1}, ab, ac; q)_n}{(1-b^2c^2q^{-1})(q, bc^2a^{-1}, b^2ca^{-1}; q)_n}(t/a^2)^n$$

$$\times r_n(x; a, b, c, bca^{-1}|q)(r_n(y; a, b, c, bca^{-1}|q))$$

$$= (1-t^2)\frac{(bcqt; q)_\infty}{(t/bc; q)_\infty} \sum_{n=0}^{\infty} \frac{(bcq^{\frac{1}{2}}, -bc, -bcq^{\frac{1}{2}}; q)_n |(bca^{-1}e^{i\theta}, bca^{-1}e^{i\phi}; q)_n|^2 q^n}{(q, bc^2a^{-1}, b^2ca^{-1}, bc, bca^{-2}, bcqt, bcq/t; q)_n}$$

$$\times {}_{10}W_9(a^2q^{-n}/bc; aq^{1-n}/b^2c, aq^{1-n}/bc^2, q^{-n}, ae^{i\theta}, ae^{-i\theta}, ae^{i\phi}, a^{-i\phi}; q, q)$$

$$+ \frac{(t, dt/c, b^2c^2; q)_\infty |(ae^{i\theta}, be^{i\phi}, cte^{i\phi}, bca^{-1}te^{i\theta}; q)_\infty|^2}{(ab, ac, bc, ac^{-1}, b^2ca^{-1}, bc^2a^{-1}t, bc/t; q)_\infty |(te^{i(\theta+\phi)}, te^{i(\phi-\theta)}; q)_\infty|^2}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t, tq^{\frac{1}{2}}, -tq^{\frac{1}{2}}, bc^2a^{-1}t; q)_n |(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_n|^2}{(q, qt/bc, qt^2, at/c; q)_n |(cte^{i\phi}, tbca^{-1}e^{i\theta}; q)_n|^2} q^n$$

$$\times {}_{10}W_9(bc^2tq^{n-1}/a; tq^n, bctq^{n-1}, ctq^n/a, bce^{i\phi}/a, bce^{-i\phi}/a, ce^{i\theta}, ce^{-i\theta}; q, q)$$

$$+ \frac{(t, ct/a, b^2c^2; q)_\infty |(ce^{i\theta}, bca^{-1}e^{i\phi}, ate^{i\phi}, bte^{i\theta}; q)_\infty|^2}{(ab, bc, b^2ca^{-1}, bc^2a^{-1}, abt, bc/t; q)_\infty |(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_\infty|^2}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t, tq^{\frac{1}{2}}, -tq^{\frac{1}{2}}, abt; q)_n |(te^{i(\theta+\phi)}, te^{i(\phi-\theta)}; q)_n|^2}{(q, qt/bc, qt^2ct/a; q)_n |(ate^{i\phi}, bte^{i\theta}; q)_n|^2} q^n$$

$$\times {}_{10}W_9(abtq^{n-1}; tq^n, bctq^{n-1}, atq^n/c, ae^{i\theta}, ae^{-i\theta}, be^{i\phi}, be^{-i\phi}; q, q),$$

where $0 < t < 1$.
(Rahman [1985]; Gasper and Rahman [1986])

9.20 Prove that

$$\sum_{n=-\infty}^{\infty} q^{-n}\, {}_2\phi_1(ste^{i\theta}, ste^{-i\theta}; s^2; q, -q^nt^{-2})\, {}_2\phi_1(se^{i\theta}/t, se^{-i\theta}/t; s^2; q, -q^n)$$

$$= \frac{(1-qt^2)}{\left(1 - \frac{2qt}{s}\cos\theta + \frac{q^2t^2}{s^2}\right)} \frac{\left(q, q, -q^2t^2/s^2, -s^2/qt^2; q\right)_\infty}{(-qt^2, -t^{-2}, -1, -1, -q, -q, s^2, s^2, s^{-2}, qs^{-2}; q)_\infty}$$

$$\times \frac{h(\cos\theta; st, -st, qst, s/t, -qt/s)}{h(\cos\theta; q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q, -q, qt/s)}$$

$$\times {}_8W_7\left(qt^2; q, ste^{i\theta}, ste^{-i\theta}, qte^{i\theta}/s, qte^{-i\theta}/s; q, q\right),$$

where $|t| < 1$, $0 \le \theta \le \pi$, and appropriate analytic continuations of the ${}_2\phi_1$ series are used as necessary.
(Koelink and Stokman [2003])

9.21 The continuous q-Hermite polynomial $H_n(\cos\theta|q)$ in base q^{-1} becomes

$$h_n(\sinh\theta|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k(k-n)} e^{(n-2k)\theta}.$$

Show that

(i) $$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} h_n(\sinh\theta|q) t^n = \frac{1}{(te^{-\theta}, -te^{\theta}; q)_\infty},$$

(ii) $$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} h_n(\sinh\theta|q) h_n(\sinh\phi|q)}{(q;q)_n} t^n$$
$$= (te^{\theta-\phi}, te^{\phi-\theta}, -te^{\theta+\phi}, -te^{-\theta-\phi}; q)_\infty / (t^2;q)_\infty.$$

(Ismail and Masson [1994])

9.22 (i) For $|\alpha/\beta| < 1$, prove that

$$\mathcal{E}_q(\cos\theta;\alpha) = \frac{(q, \alpha^2/\beta^2; q)_\infty (q\beta^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \times \frac{1}{2\pi} \int_0^{\pi} \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{h\left(\cos\psi; \frac{\alpha}{\beta} e^{i\theta}, \frac{\alpha}{\beta} e^{-i\theta}\right)} \mathcal{E}_q(\cos\psi;\beta) d\psi.$$

(ii) Prove that

$$\mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta) = \frac{(\alpha^2;q^2)_\infty}{(q\beta^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-\alpha\beta^{-1}q^{(1-n)/2};q)_n}{(\gamma;q)_n} \beta^n q^{n^2/4} C_n(\cos\theta;\gamma|q) \times {}_2\phi_1\left[\begin{matrix} -\beta\alpha^{-1}q^{(n+1)/2}, -\gamma\beta\alpha^{-1}q^{(n+1)/2} \\ \gamma q^{n+1} \end{matrix}; q, \alpha^2\right].$$

(See Bustoz and Suslov [1998] for (i), and Ismail and Stanton [2000, (5.8)] for an equivalent form of (ii).)

9.23 Prove that

$$\sum_{n=0}^{\infty} \frac{(abcdq^{-1}, ac, ad, a'b', a'c', a'd'; q)_n (bb'cdt;q)_{2n}}{(q, cd, bc't, b'ct, bd't, b'dt; q)_n (abcdq^{-1};q)_{2n}} \left(\frac{t}{aa'}\right)^n$$
$$\times {}_8W_7\left(bb'cdtq^{2n-1}; bcq^n, bdq^n, b'c'q^n, b'd'q^n, bt/a'; q, a't/b\right)$$
$$\times r_n(x;a,b,c,d|q)\; r_n(y;a',b',c',d'|q)$$
$$= \frac{(bb'cdt, dt/c', bte^{i\phi}, bte^{-i\phi}, cte^{i\phi}, cte^{-i\phi}, b'te^{i\theta}, b'te^{-i\theta}, c'te^{i\theta}, c'te^{-i\theta}; q)_\infty}{(bc't, b'ct, bd't, b'dt, bb't, cc't, te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(\phi-\theta)}, te^{-i(\theta+\phi)}; q)_\infty}$$
$$\times {}_8W_7\left(bb't/q; be^{i\theta}, be^{-i\theta}, b'e^{i\phi}, b'e^{-i\phi}, bt/a'; q, a't/b\right)$$
$$\times {}_8W_7\left(cc't/q; ce^{i\theta}, ce^{-i\theta}, c'e^{i\phi}, c'e^{-i\phi}, c't/d; q, d't/c\right),$$

where $ab = a'b'$, $cd = c'd'$, $x = \cos\theta$, and $y = \cos\phi$.

(Koelink and Van der Jeugt [1999])

Notes

§9.3 The special case $\alpha = ab$, $b = aq^{\frac{1}{2}}$, $d = cq^{\frac{1}{2}}$, $c \to -c$ of (9.3.7), in which case the expression on the right side of (9.3.7) can be combined into a sum of ${}_8W_7$ series, was given by Ismail, Masson and Suslov [1997]. However, formula (3.6) in their paper does not directly lead to the generating function $\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n$ when one sets $a = q^{\alpha/2+1/4}$, $c = q^{\beta/2+1/4}$ and takes the limit $q \to 1^-$.

Ex. 9.14 A product formula for Jackson's q-Bessel function $J_\nu^{(2)}(x;q)$, which is obtainable from this formula, is used in Rahman [2000a] to evaluate a Weber-Schafheitlin type integral for these functions.

Ex. 9.18 This is a q-analogue of Feldheim's [1941] bilinear generating function for the Jacobi polynomials.

Ex. 9.20 Rosengren [2003e] computed a more general bilinear generating function in an elementary manner. Bustoz and Suslov [1998] gave a bilateral bilinear sum that generalizes the classical Poisson kernel for the Fourier series:

$$\sum_{n=-\infty}^{\infty} t^{|n|} e^{in(x-y)} = \frac{1-t^2}{1-2t\cos(x-y)+t^2}, \quad 0 \le t < 1.$$

Ex. 9.21 The continuous q-Hermite polynomials in base q^{-1} were introduced by Askey [1989b] who also gave an orthogonality measure for those polynomials on $(-\infty, \infty)$ and computed the orthogonality relation. Ismail and Masson [1994] gave a detailed account of the family of extremal measures for these polynomials. See also Carlitz [1963b], Ismail and Masson [1993], and Ismail [1993].

10

q-SERIES IN TWO OR MORE VARIABLES

10.1 Introduction

The main objective of this chapter is to consider q-analogues of Appell's four well-known functions F_1, F_2, F_3 and F_4. We start out with Jackson's [1942] $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$ and $\Phi^{(4)}$ functions, defined in terms of double hypergeometric series, which are q-analogues of the Appell functions. It turns out that not all of Jackson's q-Appell functions have the properties that enable them to have transformation and reduction formulas analogous to those for the Appell functions. Also, starting with a q-analogue of the function on one side of a hypergeometric transformation of a reduction formula may lead to a different q-analogue of the formula than starting with a q-analogue of the function on the other side of the formula. We find, further, that the alternative approach of using the q-integral representations of these q-Appell functions can be very fruitful. For example, it immediately leads to the fact that a general $\Phi^{(1)}$ series is indeed equal to a multiple of a ${}_3\phi_2$ series (see (10.3.4) below). The q-integral approach can be used to derive q-analogues of the Appell functions that are quite different from the ones given by Jackson. In the last section we give a completely different q-analogue of F_1, based on the so-called q-quadratic lattice, which has a representation in terms of an Askey-Wilson type integral. We do not attempt to consider Askey-Wilson type q-analogues of F_2 and F_3 because these are probably the least interesting of the four Appell functions and nothing seems to be known about these analogues. Instances of Askey-Wilson type q-analogues of F_4 have already occurred in Chapter 8 (product of two q-Jacobi polynomials) and then in Chapter 9 (§9.5, §9.6, Ex. 9.18, and Ex. 9.19). Since in this chapter we will be mainly concerned with deriving and applying q-integral representations of q-Appell functions, in many of the formulas it will be necessary to denote the parameters by powers of q. Once a formula has been derived via the q-integral techniques, if it does not contain any q-integrals or q-gamma functions, then the reader may simplify the formula by replacing the q^a, q^b, etc. powers of q by a, b, etc., as we did in the formulas in the exercises for this chapter.

10.2 q-Appell and other basic double hypergeometric series

In order to obtain q-analogues of the four Appell double hypergeometric series (see Appell and Kampé de Fériet [1926])

$$F_1(a;b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \tag{10.2.1}$$

$$F_2(a;b,b';c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_m(c')_n} x^m y^n, \tag{10.2.2}$$

$$F_3(a,a';b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \tag{10.2.3}$$

$$F_4(a,b;c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(c')_n} x^m y^n, \tag{10.2.4}$$

Jackson [1942] replaced each shifted factorial by a corresponding q-shifted factorial giving the functions

$$\Phi^{(1)}(a;b,b';c;q;x,y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m(b';q)_n}{(q;q)_m(q;q)_n(c;q)_{m+n}} x^m y^n, \tag{10.2.5}$$

$$\Phi^{(2)}(a;b,b';c,c';q;x,y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m(b';q)_n}{(q;q)_m(q;q)_n(c;q)_m(c';q)_n} x^m y^n, \tag{10.2.6}$$

$$\Phi^{(3)}(a,a';b,b';c;q;x,y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_m(a';q)_n(b;q)_m(b';q)_n}{(q;q)_m(q;q)_n(c;q)_{m+n}} x^m y^n, \tag{10.2.7}$$

$$\Phi^{(4)}(a,b;c,c';q;x,y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_{m+n}}{(q;q)_m(q;q)_n(c;q)_m(c';q)_n} x^m y^n. \tag{10.2.8}$$

The series in (10.2.5)–(10.2.8) are absolutely convergent when $|q|$, $|x|$, $|y| < 1$, by the comparison test with the series $\sum_{m,n=0}^{\infty} x^m y^n$, and then their sums are called the q-Appell functions. Jackson also considered q-analogues of the Appell series with y^n replaced by $y^n q^{\binom{n}{2}}$ in each term of the series.

Similarly, one could obtain a q-analogue of the general double hypergeometric series

$$F^{A:B;C}_{D:E;F}\left[\begin{matrix} a_A : b_B; c_C \\ d_D : e_E; f_F \end{matrix}; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a_A)_{m+n}(b_B)_m(c_C)_n}{m!\,n!\,(d_D)_{m+n}(e_E)_m(f_F)_n} x^m y^n, \tag{10.2.9}$$

where we use the contracted notations defined in §3.7 by replacing each shifted factorial by a q-shifted factorial, such as in Exton [1977, p. 36] and Srivastava [1982, p. 278]. However, such a basic double series might not be of the same form as its confluent limit cases or as the double series obtained by inverting the base as in (1.2.24) or in Ex. 1.4(i). Therefore, with the same motivation as given on p. 5 for our definition of the ${}_r\phi_s$ series hypergeometric series, we define the q-analogue of (10.2.9) by

$$\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} a_A : b_B; c_C \\ d_D : e_E; f_F \end{matrix}; q; x, y\right]$$
$$= \sum_{m,n=0}^{\infty} \frac{(a_A;q)_{m+n}(b_B;q)_m(c_C;q)_n}{(d_D;q)_{m+n}(q,e_E;q)_m(q,f_F;q)_n}$$
$$\times \left[(-1)^{m+n}q^{\binom{m+n}{2}}\right]^{D-A}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+E-B}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+F-C} x^m y^n, \tag{10.2.10}$$

where $q \neq 0$ when $\min(D-A, 1+E-B, 1+F-C) < 0$. This double series converges absolutely for $|x|$, $|y| < 1$ when $\min(D-A, 1+E-B, 1+F-C) \geq 0$ and $|q| < 1$. Also, confluent limit cases of (10.2.10) have the same form as the series in (10.2.10), and if one of the parameters $b_1, \ldots, b_B, c_1, \ldots c_C$ equals 1, then the double series in (10.2.10) reduces to a single series of the form in (1.2.22). Since the series in (10.2.9) is called a Kampé de Fériet series when $B = C$ and $E = F$, the series in (10.2.10) can be called a q-Kampé de Fériet series when $B = C$ and $E = F$.

10.3 An integral representation for $\Phi^{(1)}(q^a; q^b, q^{b'}; q^c; q; x, y)$

Since, by (1.11.7) and (1.10.13)

$$\frac{(q^a;q)_{m+n}}{(q^c;q)_{m+n}} = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 t^{a+m+n-1} \frac{(qt;q)_\infty}{(tq^{c-a};q)_\infty} d_q t, \tag{10.3.1}$$

for $0 < \text{Re } a < \text{Re } c$, we find that

$$\Phi^{(1)}(q^a; q^b, q^{b'}; q^c; q; x, y) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)}$$
$$\times \int_0^1 t^{a-1} \frac{(qt, xtq^b, ytq^{b'};q)_\infty}{(xt, yt, tq^{c-a};q)_\infty} d_q t, \tag{10.3.2}$$

which was given by Jackson [1942, p. 81]. This is a q-analogue of the well-known integral representation of F_1:

$$F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'} dt, \tag{10.3.3}$$

$0 < \text{Re } a < \text{Re } c$. It follows from (10.3.2) and the definition of Jackson's q-integral (1.11.3) that

$$\Phi^{(1)}(q^a; q^b, q^{b'}; q^c; q; x, y) = \frac{(q^a, xq^b, yq^{b'};q)_\infty}{(q^c, x, y;q)_\infty} {}_3\phi_2\left[\begin{matrix} q^{c-a}, x, y \\ xq^b, yq^{b'} \end{matrix}; q, q^a\right]. \tag{10.3.4}$$

Andrews [1972] gave an independent derivation of (10.3.4).

Some reduction formulas, analogous to the ones for F_1 listed, for example, in Erdélyi [1953, Vol. I], can be derived from (10.3.4).

Case I. Let $y = xq^b$. Then (10.3.4) and (1.4.1) give

$$\begin{aligned}
&\Phi^{(1)}(q^a;q^b,q^{b'};q^c;q;x,xq^b)\\
&= \frac{(q^a, xq^{b+b'};q)_\infty}{(q^c,x;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^{c-a}, x \\ xq^{b+b'} \end{matrix}; q, q^a\right]\\
&= {}_2\phi_1\left[\begin{matrix} q^a, q^{b+b'} \\ q^c \end{matrix}; q, x\right],
\end{aligned} \tag{10.3.5}$$

which is a q-analogue of the reduction formula

$$F_1(a;b,b';c;x,x) = {}_2F_1(a,b+b';c;x). \tag{10.3.6}$$

Case II. Let $y = q^{c-a-b'}$. Then (10.3.4) and (1.4.1) give

$$\begin{aligned}
&\Phi^{(1)}(q^a;q^b,q^{b'};q^c;q;x,q^{c-a-b'})\\
&= \frac{(q^a,q^{c-a},xq^b;q)_\infty}{(q^c,q^{c-a-b'},x;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} x, q^{c-a-b'} \\ xq^b \end{matrix}; q, q^a\right]\\
&= \frac{(q^{c-b'},q^{c-a};q)_\infty}{(q^c,q^{c-a-b'};q)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^{c-b'} \end{matrix}; q, x\right]\\
&= \frac{\Gamma_q(c)\Gamma_q(c-a-b')}{\Gamma_q(c-a)\Gamma_q(c-b')}\, {}_2\phi_1(q^a,q^b;q^{c-b'};q,x),
\end{aligned} \tag{10.3.7}$$

which yields a q-analogue of

$$F_1(a;b,b';c;x,1) = \frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}\, {}_2F_1(a,b;c-b';x). \tag{10.3.8}$$

Case III. Let $c = b+b'$. The ${}_3\phi_2$ series of (10.3.4) in this case becomes a series of type II, and hence by (3.2.7)

$$\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{b+b'-a}, x, y \\ xq^b, yq^{b'} \end{matrix}; q, q^a\right]\\
&= \frac{(q^{b+b'},yq^{a-b};q)_\infty}{(q^a,yq^{b'};q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{b+b'-a}, q^b, xq^b/y \\ xq^b, q^{b+b'} \end{matrix}; q, yq^{a-b}\right].
\end{aligned}$$

So (10.3.4) gives

$$\begin{aligned}
&\Phi^{(1)}(q^a;q^b,q^{b'};q^{b+b'};q;x,y)\\
&= \frac{(xq^b,yq^{a-b};q)_\infty}{(x,y;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{b+b'-a}, q^b, xq^b/y \\ q^{b+b'}, xq^b \end{matrix}; q, yq^{a-b}\right],
\end{aligned} \tag{10.3.9}$$

provided $|yq^{a-b}| < 1$, which is a q-analogue of the reduction formula

$$F_1(a;b,b';b+b';x,y) = (1-x)^{-b}(1-y)^{b-a}\, {}_2F_1\Big(b, b+b'-a; b+b'; \frac{y-x}{1-x}\Big). \tag{10.3.10}$$

10.4 Formulas for $\Phi^{(2)}(q^a;q^b,q^{b'};q^c,q^{c'};q;x,y)$

Integral representation. Since it follows from (10.2.6) that

$$\begin{aligned}&\Phi^{(2)}(q^a;q^b,q^{b'};q^c,q^{c'};q;x,y)\\&=\sum_{m=0}^{\infty}\frac{(q^a,q^b;q)_m}{(q,q^c;q)_m}x^m{}_2\phi_1(q^{a+m},q^{b'};q^{c'};q,y),\end{aligned}\tag{10.4.1}$$

and, by (1.11.9),

$${}_2\phi_1(q^{a+m},q^{b'};q^{c'};q,y)=\frac{\Gamma_q(c')}{\Gamma_q(b')\Gamma_q(c'-b')}\int_0^1\frac{(qu,uyq^{a+m};q)_\infty}{(uy,uq^{c'-b'};q)_\infty}u^{b'-1}d_qu,\tag{10.4.2}$$

when $0<\text{Re } b'<\text{Re } c'$, we find that

$$\begin{aligned}&\Phi^{(2)}(q^a;q^b,q^{b'};q^c,q^{c'};q;x,y)\\&=\frac{\Gamma_q(c')}{\Gamma_q(b')\Gamma_q(c'-b')}\int_0^1\frac{(qu,uyq^a;q)_\infty}{(uy,uq^{c'-b'};q)_\infty}{}_3\phi_2\left[\begin{matrix}q^a,q^b,0\\q^c,uyq^a\end{matrix};q,x\right]u^{b'-1}d_qu,\end{aligned}\tag{10.4.3}$$

provided $0<\text{Re } b'<\text{Re } c'$. By (1.11.7)

$$\frac{(q^b;q)_n}{(q^c;q)_n}=\frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 v^{b+n-1}\frac{(vq;q)_\infty}{(vq^{c-b};q)_\infty}d_qv,\tag{10.4.4}$$

when $0<\text{Re } b<\text{Re } c$. Thus, by combining (10.4.3) and (10.4.4) we obtain the following integral representation:

$$\begin{aligned}&\Phi^{(2)}(q^a;q^b,q^{b'};q^c,q^{c'};q;x,y)\\&=B_q^{-1}(b,c-b)B_q^{-1}(b',c'-b')\int_0^1\int_0^1 u^{b'-1}v^{b-1}\frac{(qu,qv,uyq^a;q)_\infty}{(uy,uq^{c'-b'},vq^{c-b};q)_\infty}\\&\quad\times{}_2\phi_1(q^a,0;uyq^a;q,xv)\,d_qu\,d_qv,\end{aligned}\tag{10.4.5}$$

when $0<\text{Re } b<\text{Re } c$ and $0<\text{Re } b'<\text{Re } c'$, which is a q-analogue of

$$\begin{aligned}&F_2(a;b,b';c,c';x,y)\\&=B^{-1}(b,c-b)B^{-1}(b',c'-b')\\&\times\int_0^1\int_0^1 u^{b'-1}v^{b-1}(1-u)^{c'-b'-1}(1-v)^{c-b-1}(1-uy-vx)^{-a}du\,dv,\end{aligned}\tag{10.4.6}$$

since

$$\lim_{q\to1^-}{}_2\phi_1(q^a,0;uyq^a;q,xv)=(1-uy)^a(1-uy-vx)^{-a},\quad |xv|<1.$$

Transformation formulas. To derive q-analogues of the transformation formulas for F_2 note that

$${}_2\phi_1(q^{a+m},q^{b'};q^{c'};q,y)=\frac{(yq^{a+m};q)_\infty}{(y;q)_\infty}{}_2\phi_2(q^{a+m},q^{c'-b'};q^{c'},yq^{a+m};q,yq^{b'})\tag{10.4.7}$$

by (1.5.4), and hence (10.4.1) and (10.4.7) give

$$\begin{aligned}&\Phi^{(2)}(q^a; q^b, q^{b'}; q^c, q^{c'}; q; x, y)\\&= \frac{(yq^a;q)_\infty}{(y;q)_\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^a;q)_{m+n}(q^b;q)_m(q^{c'-b'};q)_n}{(yq^a;q)_{m+n}(q,q^c;q)_m(q,q^{c'};q)_n} x^m(-y)^n q^{nb'+\binom{n}{2}},\end{aligned} \tag{10.4.8}$$

which is a q-analogue of Erdélyi [1953, Vol. I, 5.11 (7)]. Similarly, we get a q-analogue of Erdélyi [1953, Vol. I., 5.11 (6)] by interchanging $x \leftrightarrow y$, $b \leftrightarrow b'$, $c \leftrightarrow c'$.

Reduction formulas. Let $c' = a$. Then

$$\begin{aligned}{}_2\phi_1(q^{a+m}, q^{b'}; q^a; q, y) &= \frac{(yq^{b'+m};q)_\infty}{(y;q)_\infty}\, {}_2\phi_1(q^{-m}, q^{a-b'}; q^a; q, yq^{b'+m})\\&= \frac{(yq^{b'+m};q)_\infty (q^{b'};q)_m}{(y;q)_\infty (q^a;q)_m}\, {}_3\phi_2\left[\begin{matrix} q^{-m}, q^{a-b'}, y \\ q^{1-m-b'}, 0\end{matrix}; q, q\right],\end{aligned} \tag{10.4.9}$$

by (III.2) and (III.7), and hence, with a bit of simplification, we find that

$$\begin{aligned}&\Phi^{(2)}(q^a; q^b, q^{b'}; q^c, q^a; q; x, y)\\&= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^b;q)_{m+n}(q^{a-b'}, y;q)_m(q^{b'};q)_n}{(q;q)_m(q;q)_n(q^c, yq^{b'};q)_{m+n}} x^{m+n} q^{mb'}.\end{aligned} \tag{10.4.10}$$

This can be written in the notation of (10.2.10) in the form

$$\begin{aligned}&\Phi^{(2)}(q^a; q^b, q^{b'}; q^c, q^a; q; x, y)\\&= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} \Phi^{2:2;1}_{2:1;0}\left[\begin{matrix} q^b, 0 : q^{a-b'}, y; q^{b'} \\ q^c, yq^{b'} : 0; - \end{matrix}; q; xq^{b'}, x\right],\end{aligned} \tag{10.4.11}$$

which is a q-analogue of Bailey [1935, 9.5(6)].

If we specialize further by setting $c = c' = a$, then we can use the transformation

$$\begin{aligned}&{}_2\phi_1(q^{a+m}, q^{b'}; q^a; q, y)\\&= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} {}_2\phi_2(q^{-m}, q^{b'}; q^a, yq^{b'}; q, yq^{a+m})\end{aligned} \tag{10.4.12}$$

to get

$$\begin{aligned}&\Phi^{(2)}(q^a; q^b, q^{b'}; q^a, q^a; q; x, y)\\&= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} \sum_{m=0}^{\infty} \frac{(q^b;q)_m}{(q;q)_m} x^m \sum_{n=0}^{\infty} \frac{(q^{-m}, q^{b'};q)_n}{(q, q^a, yq^{b'};q)_n} (-y)^n q^{(a+m)n+\binom{n}{2}}\\&= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^b;q)_{m+n}(q^{b'};q)_n (xy)^n x^m}{(q, q^a, yq^{b'};q)_n (q;q)_m} q^{n(n-1)+an}\end{aligned}$$

$$= \frac{(yq^{b'};q)_\infty}{(y;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^b,q^{b'};q)_n}{(q,q^a,yq^{b'};q)_n}(xy)^n q^{n(n-1)+an} \sum_{m=0}^{\infty} \frac{(q^{b+n};q)_m}{(q;q)_m} x^m$$

$$= \frac{(xq^b,yq^{b'};q)_\infty}{(x,y;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^b,q^{b'};q)_n (xy)^n}{(q,q^a,yq^{b'},xq^b;q)_n} q^{n(n-1)+an}$$

$$= \frac{(xq^b,yq^{b'};q)_\infty}{(x,y;q)_\infty} \, {}_2\phi_3 \left[\begin{matrix} q^b, q^{b'} \\ q^a, xq^b, yq^{b'} \end{matrix} ; q, q^a xy \right], \tag{10.4.13}$$

which is a q-analogue of Bailey [1935, 9.5(7)].

Finally, let $c = b$. Then (10.2.6) gives

$$\Phi^{(2)}(q^a;q^b,q^{b'};q^b,q^{c'};q;x,y)$$
$$= \sum_{n=0}^{\infty} \frac{(q^a,q^{b'};q)_n}{(q,q^{c'};q)_n} \sum_{m=0}^{\infty} \frac{(q^{a+n};q)_m}{(q;q)_m} x^m$$
$$= \frac{(xq^a;q)_\infty}{(x;q)_\infty} {}_3\phi_2 \left[\begin{matrix} q^a, q^{b'}, 0 \\ q^{c'}, xq^a \end{matrix} ; q, y \right], \tag{10.4.14}$$

which is a q-analogue of Bailey [1935, 9.5(3)].

10.5 Formulas for $\Phi^{(3)}(q^a,q^{a'};q^b,q^{b'};q^c;q;x,y)$

From (10.2.7) it follows that

$$\Phi^{(3)}(q^a,q^{a'};q^b,q^{b'};q^c;q;x,y)$$
$$= \sum_{m=0}^{\infty} \frac{(q^a,q^b;q)_m}{(q,q^c;q)_m} x^m \, {}_2\phi_1(q^{a'},q^{b'};q^{c+m};q,y). \tag{10.5.1}$$

Using

$$\int_0^1 u^{b+m-1} \frac{(qu;q)_\infty}{(uq^{c-b+n};q)_\infty} d_q u = \frac{\Gamma_q(b)\Gamma_q(c-b)}{\Gamma_q(c)} \frac{(q^b;q)_m (q^{c-b};q)_n}{(q^c;q)_{m+n}}, \tag{10.5.2}$$

for $0 < \text{Re } b < \text{Re } c$, and

$$\int_0^1 v^{b'+n-1} \frac{(qv;q)_\infty}{(vq^{c-b-b'};q)_\infty} d_q v = \frac{\Gamma_q(b')\Gamma_q(c-b-b')}{\Gamma_q(c-b)} \frac{(q^{b'};q)_n}{(q^{c-b};q)_n}, \tag{10.5.3}$$

provided $0 < \text{Re } b' < \text{Re } (c-b)$, we obtain the following integral representation

$$\Phi^{(3)}(q^a,q^{a'};q^b,q^{b'};q^c;q;x,y)$$
$$= \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c-b-b')} \int_0^1 \int_0^1 u^{b-1} v^{b'-1} \frac{(qu,qv,xuq^a;q)_\infty}{(uq^{c-b},vq^{c-b-b'},xu;q)_\infty}$$
$$\times \, {}_2\phi_1(q^{a'},uq^{c-b};0;q,vy) \, d_q u \, d_q v. \tag{10.5.4}$$

Since

$$\lim_{q\to 1^-} {}_2\phi_1(q^{a'},uq^{c-b};0;q,vy) = (1-vy(1-u))^{-a'}, \tag{10.5.5}$$

(10.5.4) is a q-analogue of Bailey [1935, 9.3(3)].

The function $\Phi^{(3)}(q^a, q^{a'}; q^b, q^{b'}; q^c; q; x, y)$ does not seem to have any useful transformation formulas, just as $F_3(a, a'; b, b'; c; x, y)$ does not, as was noted by Bailey [1935, §9.3].

Reduction formulas. Suppose $a + a' = c$. Then

$$\begin{aligned}
&\Phi^{(3)}(q^a, q^{c-a}; q^b, q^{b'}; q^c; q; x, y) \\
&= \sum_{m=0}^{\infty} \frac{(q^a, q^b; q)_m}{(q, q^c; q)_m} x^m \; {}_2\phi_1(q^{c-a}, q^{b'}; q^{c+m}; y) \\
&= \frac{(yq^{b'}; q)_\infty}{(y; q)_\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^a; q)_{m+n} (q^b; q)_m (q^{b'}; q)_n}{(q^c; q)_{m+n} (q; q)_m (q, yq^{b'}; q)_n} x^m (-y)^n q^{n(c-a)+\binom{n}{2}},
\end{aligned} \tag{10.5.6}$$

by (1.5.4), which is a q-analogue of Bailey [1935, 9.5(4)]

$$F_1(a; b, b', c; x, y) = (1 - y)^{-b'} F_3\Big(a, c - a; b, b'; c; x, \frac{y}{y - 1}\Big). \tag{10.5.7}$$

When $a + a' = b + b' = c$, the right hand side of (10.5.6) can be reduced to two ${}_4\phi_3$ series in the following way.

Using

$$\sum_{m=0}^{\infty} \frac{(t; q)_m}{(q; q)_m} x^m = \frac{(xt; q)_\infty}{(x; q)_\infty}, \quad \sum_{n=0}^{\infty} \frac{(tq^{c-a-b}; q)_n}{(q; q)_n} y^n = \frac{(ytq^{c-a-b}; q)_\infty}{(y; q)_\infty},$$

and, by Ex. 5.14,

$$\begin{aligned}
&\int_{q^a}^{q^b} \frac{(tq^{1-a}, tq^{1-b}; q)_\infty}{(t, tq^{c-a-b}; q)_\infty} (t; q)_m (tq^{c-a-b}; q)_n \, d_q t \\
&= \frac{q^b (1 - q)(q, q^{a-b}, q^{1+b-a}, q^c; q)_\infty}{(q^a, q^b, q^{c-a}, q^{c-b}; q)_\infty} \frac{(q^a, q^b; q)_m (q^{c-a}, q^{c-b}; q)_n}{(q^c; q)_{m+n}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\Phi^{(3)}(q^a, q^{c-a}; q^b, q^{c-b}; q^c; q; x, y) \\
&= \frac{q^{-b} (q^a, q^b, q^{c-a}, q^{c-b}; q)_\infty}{(1 - q)(q, q^{a-b}, q^{1+b-a}, q^c; q)_\infty (x, y; q)_\infty} \\
&\quad \times \int_{q^a}^{q^b} \frac{(tq^{1-a}, tq^{1-b}, xt, ytq^{c-a-b}; q)_\infty}{(t, tq^{c-a-b}; q)_\infty} d_q t \\
&= \frac{\Gamma_q(c)\Gamma_q(a - b)}{\Gamma_q(a)\Gamma_q(c - b)} \frac{(xq^b, yq^{c-a}; q)_\infty}{(x, y; q)_\infty} {}_4\phi_3 \left[\begin{matrix} q^b, q^{c-a}, 0, 0 \\ q^{1+b-a}, xq^b, yq^{c-a} \end{matrix} ; q, q \right] \\
&\quad + \frac{\Gamma_q(c)\Gamma_q(b - a)}{\Gamma_q(b)\Gamma_q(c - a)} \frac{(xq^a, yq^{c-b}; q)_\infty}{(x, y; q)_\infty} {}_4\phi_3 \left[\begin{matrix} q^a, q^{c-b}, 0, 0 \\ q^{1+a-b}, xq^a, yq^{c-b} \end{matrix} ; q, q \right].
\end{aligned} \tag{10.5.8}$$

The limit of the right side of (10.5.8) as $q \to 1^-$ is

$$(1 - x)^{-a} (1 - y)^{b-c} \Big\{ \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} {}_2F_1\Big(a, c - b; 1 + a - b; \frac{1}{(1 - x)(1 - y)}\Big)$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-a)}((1-x)(1-y))^{a-b}\,{}_2F_1\Big(b;c-a;1+b-a;\frac{1}{(1-x)(1-y)}\Big)\Big\}$$
$$=(1-y)^{a+b-c}{}_2F_1(a,b;c;x+y-xy), \tag{10.5.9}$$

by Bailey [1935, 1.4(1)]. So we can regard (10.5.8) as a q-analogue of the formula

$$F_3(a,c-a;b,c-b;c;x,y)=(1-y)^{a+b-c}{}_2F_1(a,b;c;x+y-xy), \tag{10.5.10}$$

Bailey [1935, 9.5(5)].

If instead of starting with $\Phi^{(3)}(q^a,q^{c-a};q^b,q^{c-b};q^c;q;x,y)$ we started with $\Phi^{(1)}(q^a;q^b,q^{c-b};q^c;q;x,y)$ we would obtain a different q-analogue of (10.5.10). For,

$$\begin{aligned}\Phi^{(1)}&(q^a;q^b,q^{c-b};q^c;q;x,y)\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^a;q)_{m+n}(q^b;q)_m(q^{c-b};q)_n}{(q^c;q)_{m+n}(q;q)_m(q;q)_n}x^my^n\\ &=\sum_{m=0}^{\infty}\frac{(q^q,q^b;q)_m}{(q,q^c;q)_m}x^m{}_2\phi_1(q^{a+m},q^{c-b};q^{c+m};q,y)\\ &=\frac{(yq^{c-b};q)_\infty}{(y;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^a,q^b;q)_m(q^{c-a},q^{c-b};q)_n}{(q^c;q)_{m+n}(q;q)_m(q,yq^{c-b};q)_n}x^m(-y)^nq^{\binom{n}{2}+(a+m)n}\end{aligned} \tag{10.5.11}$$

by (1.5.4), and

$$\begin{aligned}\Phi^{(1)}&(q^a;q^b,q^{c-b};q^c;q;x,y)\\ &=\frac{(q^a,xq^b,yq^{c-b};q)_\infty}{(q^c,x,y;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{c-a},x,y\\ xq^b,yq^{c-b}\end{matrix};q,q^a\right]\\ &=\frac{(xq^{a+b-c},yq^{c-b};q)_\infty}{(x,y;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{c-a},q^{c-b},yq^{c-b}/x\\ q^c,yq^{c-b}\end{matrix};q,xq^{a+b-c}\right],\end{aligned} \tag{10.5.12}$$

by (10.3.4) and (3.2.7), and hence

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}&\frac{(q^a,q^b;q)_m(q^{c-a},q^{c-b};q)_n}{(q^c;q)_{m+n}(q;q)_m(q,yq^{c-b};q)_n}x^m(-y)^nq^{\binom{n}{2}+(a+m)n}\\ &=\frac{(xq^{a+b-c};q)_\infty}{(x;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{c-a},q^{c-b},yq^{c-b}/x\\ q^c,yq^{c-b}\end{matrix};q,xq^{a+b-c}\right].\end{aligned} \tag{10.5.13}$$

If we take the limit $q\to 1^-$ of (10.5.13) and replace y by $y/(y-1)$ we obtain (10.5.10). This is an example of a situation where one gets different q-analogues by starting with opposite sides of the same formula.

10.6 Formulas for a q-analogue of F_4

Appell's F_4 function is probably the most important of the four Appell functions because of its applications in the theory of classical orthogonal polynomials. It has the integral representation

$$
\begin{aligned}
&F_4(a,b;c,c';x(1-y),y(1-x)) \\
&= \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)} \int_0^1 \int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1} \\
&\quad \times (1-ux)^{1+a-c-c'}(1-vy)^{1+b-c-c'}(1-ux-vy)^{c+c'-a-b-1}du\,dv,
\end{aligned}
\tag{10.6.1}
$$

which was derived by Burchnall and Chaundy [1940] from their expansion

$$
\begin{aligned}
&F_4(a,b;c,c';x(1-y),y(1-x)) \\
&= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1+a+b-c-c')_r}{r!(c)_r(c')_r} x^r y^r {}_2F_1(a+r,b+r;c+r;x) \\
&\quad \times {}_2F_1(a+r,b+r;c'+r;y).
\end{aligned}
\tag{10.6.2}
$$

Jackson's $\Phi^{(4)}$, given by (10.2.8), does not seem to be useful in any applications that we have come across, and it does not have any formulas with its arguments x and y replaced by $x(1-y)$ and $y(1-x)$, respectively, so our approach will be to transform (10.6.1) and (10.6.2) into forms that we can find q-analogues of. First observe that, since

$$
\begin{aligned}
{}_2F_1(a+r,b+r;c+r;x) &= (1-x)^{-a-r}{}_2F_1\Big(a+r,c-b;c+r;\frac{x}{x-1}\Big), \\
{}_2F_1(a+r,b+r;c'+r;y) &= (1-y)^{-b-r}{}_2F_1\Big(b+r,c'-a;c'+r;\frac{y}{y-1}\Big)
\end{aligned}
$$

by (1.5.5), the expansion formula (10.6.2) transforms to

$$
\begin{aligned}
&F_4(a,b;c,c';x(1-y),y(1-x)) \\
&= (1-x)^{-a}(1-y)^{-b}\sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1+a+b-c-c')_r}{r!(c)_r(c')_r}\Big(\frac{x}{x-1}\Big)^r\Big(\frac{y}{y-1}\Big)^r \\
&\quad \times {}_2F_1\Big(a+r,c-b;c+r;\frac{x}{x-1}\Big){}_2F_1\Big(b+r,c'-a;c'+r;\frac{y}{y-1}\Big).
\end{aligned}
\tag{10.6.3}
$$

Replacing x and y by $x/(x-1)$ and $y/(y-1)$, respectively, (10.6.3) becomes

$$
\begin{aligned}
&F_4\Big(a,b;c,c';-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\Big) \\
&= (1-x)^a(1-y)^b\sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1+a+b-c-c')_r}{r!(c)_r(c')_r}(xy)^r \\
&\quad \times {}_2F_1(a+r,c-b;c+r;x)\ {}_2F_1(b+r,c'-a;c'+r';y).
\end{aligned}
\tag{10.6.4}
$$

Analogously, the integral representation (10.6.1) transforms to

$$
\begin{aligned}
&(1-x)^{-a}(1-y)^{-b}\ F_4\Big(a,b;c,c';-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\Big) \\
&= \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-a)}
\end{aligned}
$$

$$\times \int_0^1 \int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{b-c}(1-vy)^{a-c'}$$
$$\times (1-uvxy)^{c+c'-a-b-1}du\,dv, \tag{10.6.5}$$

where $0 < \text{Re } a < \text{Re } c$ and $0 < \text{Re } b < \text{Re } c'$. Let

$$M(x,y) = M(x,y;a,b,c,c';q) = \sum_{r=0}^{\infty} \frac{(a,b,abq/cc';q)_r}{(q,c,c';q)_r}\left(\frac{cc'xy}{abq}\right)^r$$
$$\times {}_2\phi_1(aq^r, c/b; cq^r; q, x)\ {}_2\phi_1(bq^r, c'/a; c'q^r; q, y), \tag{10.6.6}$$

which, via (10.6.4), is a q-analogue of the left side of (10.6.5) on replacing a, b, c, c' by q^a, q^b, q^c, $q^{c'}$, respectively. Since

$${}_2\phi_1(q^{a+r}, q^{c-b}; q^{c+r}; q, x) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)}\frac{(q^c;q)_r}{(q^a;q)_r}$$
$$\times \int_0^1 u^{a+r-1}\frac{(qu, xuq^{c-b};q)_\infty}{(xu, uq^{c-a};q)_\infty}d_qu, \tag{10.6.7}$$

if $0 < \text{Re } a < \text{Re } c$, and

$${}_2\phi_1(q^{b+r}, q^{c'-a}; q^{c'+r}; q, y) = \frac{\Gamma_q(c')}{\Gamma_q(b)\Gamma_q(c'-b)}\frac{(q^{c'};q)_r}{(q^b;q)_r}$$
$$\times \int_0^1 v^{b+r-1}\frac{(qv, yvq^{c'-a};q)_\infty}{(yv, vq^{c'-b};q)_\infty}d_qv, \tag{10.6.8}$$

if $0 < \text{Re } b < \text{Re } c'$, we have the integral representation

$$M(x,y) = \frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c-a)\Gamma_c(c'-b)}\int_0^1\int_0^1 u^{a-1}v^{b-1}$$
$$\times \frac{(qu, qv, xuq^{c-b}, yvq^{c'-a}, xyuv;q)_\infty}{(xu, yv, uq^{c-a}, vq^{c'-b}, xyuvq^{c+c'-a-b-1};q)_\infty}d_qu\ d_qv, \tag{10.6.9}$$

which is an exact analogue of (10.6.5). However, (10.6.9) is, by the definition of q-integrals, the same as the double sum

$$M(x,y) = \frac{(q^a, q^b, xq^{c-b}, yq^{c'-a}, xy;q)_\infty}{(q^c, q^{c'}, x, y, xyq^{c+c'-a-b-1};q)_\infty}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(xyq^{c+c'-a-b-1};q)_{m+n}(x, q^{c-a};q)_m(y, q^{c'-b};q)_n}{(xy;q)_{m+n}(q, xq^{c-b};q)_m(q, yq^{c'-a};q)_n}q^{am+bn}$$
$$= \frac{(q^a, q^b, xq^{c-b}, yq^{c'-a}, xy;q)_\infty}{(q^c, q^{c'}, x, y, xyq^{c+c'-a-b-1};q)_\infty}$$
$$\times \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} xyq^{c+c'-a-b-1} & : x, q^{c-a} & ; y, q^{c'-b} \\ xy & : xq^{c-b} & ; yq^{c'-a} \end{matrix} ; q; q^a, q^b\right], \tag{10.6.10}$$

by (10.2.10). This double series looks more like the series in (10.2.5) for $\Phi^{(1)}$ than the series (10.2.8) for $\Phi^{(4)}$. Note that the term-by-term $q \to 1^-$ limit can be taken on the expression on the right side of (10.6.9) but not on its double sum in (10.6.10). However, it is in the form given by (10.6.6) that the function $M(x,y)$ seems to be most useful. First of all, when $cc' = abq$ the series in (10.6.6) becomes a product

$$ {}_2\phi_1(a, c/b; c; q, x)\ {}_2\phi_1(b, c'/a; c'; q, y) $$

which corresponds to a q-analogue of the right hand side of the product formula

$$ \begin{aligned} & F_4(a, b; c, a+b+1-c; x(1-y), y(1-x)) \\ &= {}_2F_1(a, b; c; x)\ {}_2F_1(a, b; a+b+1-c; y) \end{aligned} \qquad (10.6.11) $$

in Bailey [1935, 9.6(1)].

We shall now manipulate the series in (10.6.5) in order to obtain an expression that closely resembles the F_4 function. First, it follows by combining the terms in the obvious manner, that

$$ \begin{aligned} M(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a, c/b; q)_m (b, c'/a; q)_n}{(q, c; q)_m (q, c'; q)_n} x^m y^n \\ &\quad \times {}_3\phi_2\left[\begin{matrix} q^{-m}, q^{-n}, abq/cc' \\ bq^{1-m}/c, aq^{1-n}/c' \end{matrix}; q, q\right]. \end{aligned} \qquad (10.6.12) $$

Since the ${}_3\phi_2$ series above is balanced it can be summed by (1.7.2). A bit of simplification then leads to

$$ M(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a, cq^{-n}/b; q)_m (b, c'q^{-m}/a; q)_n}{(q, c; q)_m (q, c'; q)_n} x^m y^n q^{mn}, \qquad (10.6.13) $$

which has a straightforward $q \to 1^-$ limit, but is not very useful because of the dependence of the terms $(cq^{-n}/b; q)_m$ and $(c'q^{-m}/a; q)_n$ on m and n. However, since

$$ \frac{(cq^{-n}/b; q)_m}{(c; q)_m} = {}_2\phi_1(q^{-m}, bq^n; c; q, cq^{m-n}/b) $$

and

$$ \frac{(c'q^{-m}/a; q)_n}{(c'; q)_n} = {}_2\phi_1(q^{-n}, aq^m; c'; q, c'q^{n-m}/a), $$

by (1.5.2), we have

$$ \begin{aligned} M(x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(a;q)_{m+s}(b;q)_{n+r} x^m y^n}{(q;q)_{m-r}(q;q)_{n-s}(q,c;q)_r(q,c';q)_s} \\ &\quad \times (-1)^{r+s}(c/b)^r(c'/a)^s q^{mn-ms-nr+\binom{r}{2}+\binom{n}{2}} \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a,b;q)_{r+s}}{(q,c;q)_r(q,c';q)_s}\left(-\frac{cx}{b}\right)^r\left(-\frac{c'y}{a}\right)^s \\ &\quad \times q^{\binom{r}{2}+\binom{s}{2}-rs}\gamma_{r+s}, \end{aligned} \qquad (10.6.14) $$

where

$$\begin{aligned}\gamma_r &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(aq^r;q)_m(bq^r;q)_n}{(q;q)_m(q;q)_n} x^m y^n q^{mn} \\ &= \frac{(byq^r;q)_\infty}{(y;q)_\infty}\ {}_2\phi_1(aq^r, y; byq^r; q, x) \\ &= \frac{(axq^r, byq^r;q)_\infty}{(x,y;q)_\infty}\ {}_2\phi_2(aq^r, bq^r; axq^r, byq^r; q, xy). \end{aligned} \tag{10.6.15}$$

Since $(xq^{a+r+s}, yq^{b+r+s};q)_\infty/(x,y;q)_\infty$ is simply a q-analogue of the product $(1-x)^{-a-r-s}(1-y)^{-b-r-s}$ and the above ${}_2\phi_2$ series approaches 1 when $q \to 1^-$, in view of (10.6.6), we may introduce the following function

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(q^a, q^b;q)_{r+s}(-xq^{c-b})^r(-yq^{c'-a})^s}{(q,q^c;q)_r(q,q^{c'};q)_s(q^a x, q^b y;q)_{r+s}} q^{\binom{r}{2}+\binom{s}{2}-rs}$$
$$\times\ {}_2\phi_2(q^{a+r+s}, q^{b+r+s}; xq^{a+r+s}, yq^{b+r+s}; q, xy)$$

as an appropriate q-analogue of

$$(1-x)^{-a}(1-y)^{-b}\ F_4\Big(a,b;c,c';-\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\Big).$$

10.7 An Askey-Wilson-type integral representation for a q-analogue of F_1

The q-integral representation for $\Phi^{(1)}(\alpha;\beta,\beta';\gamma;q;x,y)$ given in (10.3.2) (with $\alpha = q^a$, $\beta = q^b$, $\beta' = q^{b'}$, $\gamma = q^c$) is related to the q-linear lattice with respect to which the q-derivative $\mathcal{D}_q$ is as defined in Ex. 1.12. There is, however, a second integral representation of an analogue of F_1 that is related to the q-quadratic lattice $x = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$, with respect to which the q-derivative D_q is the Askey-Wilson derivative defined in §7.7. To make the connection explicit let us first rewrite (10.3.3) in the form

$$\begin{aligned}F_1(\alpha;\beta,\beta';\gamma;x,y) &= \frac{\Gamma(\gamma)2^{1-\gamma}}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^\pi (1-\cos\psi)^{\alpha-\frac{1}{2}}(1+\cos\psi)^{\gamma-\alpha-\frac{1}{2}} \\ &\quad\times\Big(1-\frac{x}{2}+\frac{x}{2}\cos\psi\Big)^{-\beta}\Big(1-\frac{y}{2}+\frac{y}{2}\cos\psi\Big)^{-\beta'} d\psi,\end{aligned} \tag{10.7.1}$$

provided $0 < \text{Re}\ \alpha < \text{Re}\ \gamma$. This suggests the following integral representation

$$\begin{aligned}&\Phi_1(\alpha;\beta,\beta';\gamma;q;x,y) \\ &= \Lambda\int_0^\pi (e^{i\psi}, e^{-i\psi};q^{\frac{1}{2}})_{\frac{\alpha}{2}-\frac{1}{4}}(-e^{i\psi},-e^{-i\psi};q^{\frac{1}{2}})_{\frac{\gamma-\alpha}{2}-\frac{1}{4}} \\ &\quad\times(\lambda e^{i\psi},\lambda e^{-i\psi};q)_{-\beta}(\mu e^{i\psi},\mu e^{-i\psi};q)_{-\beta'}d\psi,\end{aligned} \tag{10.7.2}$$

where Λ is a normalizing constant, and λ, μ are related to x and y by

$$x = -\frac{4\lambda}{(1-\lambda)^2}, \quad y = -\frac{4\mu}{(1-\mu)^2}. \tag{10.7.3}$$

A more general form of the integral in (10.7.2), namely,

$$S(a,b,c,d,f,g;\lambda,\mu) = \int_0^\pi \frac{h(\cos\psi;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}},\lambda,\mu)}{h(\cos\psi;a,b,c,d,f,g)}d\psi \tag{10.7.4}$$

was evaluated in Nassrallah and Rahman [1986]. It was found that

$$\begin{aligned} &S(a,b,c,d,f,g;\lambda,\mu) \\ &= \kappa(a,b,c,d)\frac{(q/abcd,aq/b,aq/c,aq/d,\lambda a,\lambda/a,\mu a,\mu/a;q)_\infty}{(qa^2,q/bc,q/bd,q/cd,af,f/a,ag,g/a;q)_\infty} \\ &\quad\times {}_{10}W_9(a^2;ab,ac,ad,af,ag,aq/\lambda,aq/\mu;q,\lambda\mu q/abcdfg) \\ &\quad+ \text{ idem } (a;f,g), \end{aligned} \tag{10.7.5}$$

with $|\lambda\mu q/abcdfg| < 1$. The proof in Nassrallah and Rahman [1986] is long and tedious. We will give a shorter proof here. By (2.11.7) and (2.10.1)

$$\begin{aligned} \frac{h(z;\lambda,\mu)}{h(z;f,aq)} &= \frac{(\lambda a,\lambda/a,\mu a,\mu/a;q)_\infty}{(qa^2,fa,f/a,q;q)_\infty} \\ &\quad\times {}_8W_7(a^2;af,aq/\lambda,aq/\mu,ae^{i\psi},ae^{-i\psi};q,\lambda\mu/af) \\ &\quad+ \frac{h(z;a,fq)}{h(z;f,aq)}\frac{(\lambda f,\lambda/f,\mu f,\mu/f;q)_\infty}{(qf^2,af,a/f,q;q)_\infty} \\ &\quad\times {}_8W_7(f^2;af,fq/\lambda,fq/\mu,fe^{i\psi},fe^{-i\psi};q,\lambda\mu/af), \end{aligned} \tag{10.7.6}$$

where $z=\cos\psi$. Substituting (10.7.6) into (10.7.4) gives

$$\begin{aligned} &S(a,b,c,d,f,g;\lambda,\mu) \\ &= \frac{(\lambda a,\lambda/a,\mu a,\mu/a;q)_\infty}{(qa^2,fa,f/a,q;q)_\infty}\sum_{n=0}^\infty \frac{1-a^2q^{2n}}{1-a^2}\frac{(a^2,af,aq/\lambda,aq/\mu;q)_n}{(q,aq/f,\lambda a,\mu a;q)_n}\left(\frac{\lambda\mu}{af}\right)^n \\ &\quad\times \int_0^\pi \frac{h(\cos\psi;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}},aq^{n+1})}{h(\cos\psi;b,c,d,g,aq^n)}d\psi \\ &\quad+ \text{ idem } (a;f). \end{aligned} \tag{10.7.7}$$

By (6.3.7) the integral displayed in (10.7.7) equals

$$\begin{aligned} &\frac{2\pi(a^2q^{2n+1},abcdq^n,acdgq^n,abdgq^n,abcgq^n;q)_\infty}{(bc,bd,bg,cd,cg,dg,abq^n,acq^n,adq^n,agq^n,a^2bcdgq^{2n};q)_\infty} \\ &\quad\times {}_8W_7(a^2bcdgq^{2n-1};abq^n,acq^n,adq^n,agq^n,bcdgq^{-1};q,q) \\ &= \frac{2\pi(abcd,q/abcd,aq/b,aq/c,aq/d;q)_\infty}{(ab,ac,ad,bc,bd,cd,ag,g/a,q/bc,q/bd,q/cd;q)_\infty} \\ &\quad\times \frac{(ab,ac,ad,ag;q)_n}{(aq/b,aq/c,aq/d,aq/g;q)_n}\left(\frac{q}{bcdg}\right)^n \\ &\quad- \frac{2\pi q}{bcdg}\frac{2\pi(bcdg/q,q^2/bcdg,gq/b,gq/c,gq/d;q)_\infty}{(q,bc,bd,cd,bg,cg,dg,qg^2;q)_\infty(1-agq^n)\left(1-\frac{aq^n}{g}\right)} \\ &\quad\times {}_8W_7(g^2;bg,cg,dg,agq^n,gq^{-n}/a;q,q/bcdg), \end{aligned} \tag{10.7.8}$$

by (2.11.7) and (2.10.1). So we get

$$
\begin{aligned}
&S(a,b,c,d,f,g;\lambda,\mu)\\
&= \kappa(a,b,c,d)\frac{(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a, \mu a, \mu/a;q)_\infty}{(q/bc, q/bd, q/cd, af, f/a, ga, g/a, qa^2;q)_\infty}\\
&\quad\times {}_{10}W_9\Big(a^2; ab, ac, ad, af, ag, aq/\lambda, aq/\mu; q, \frac{\lambda\mu q}{abcdfg}\Big)\\
&\quad+\ \text{idem } (a;f)\\
&\quad- \frac{2\pi q}{bcdg}\frac{(bcdg/q, q^2/bcdg, gq/b, gq/c, gq/d;q)_\infty}{(q, bc, bd, cd, bg, cg, dg, qg^2;q)_\infty}\\
&\quad\times \sum_{m=0}^{\infty}\frac{1-g^2q^{2m}}{1-g^2}\frac{(g^2, bg, cg, dg, ag, \frac{fgq}{\lambda\mu};q)_m}{(q, gq/b, gq/c, gq/d, gq/a, \frac{\lambda\mu g}{f};q)_m}\Big(\frac{\lambda\mu q}{abcdfg}\Big)^m\\
&\quad\times \bigg\{\frac{(\lambda a, \lambda/a, \mu a, \mu/a, \frac{\lambda\mu g}{f}, \frac{\mu}{fg};q)_\infty}{(fa, f/a, ag, a/g, \frac{\lambda\mu}{af}, a\frac{\lambda\mu}{f};q)_\infty}\\
&\quad\times {}_8W_7\Big(\frac{\lambda\mu a}{qf}; \lambda/f, \mu/f, \lambda\mu/q, agq^m, aq^{-m}/g; q, q\Big) +\ \text{idem } (a;f)\bigg\}.
\end{aligned}
\tag{10.7.9}
$$

Using (2.11.7) to evaluate the expression in { } above and simplifying, we obtain (10.7.5), which can be regarded as an extension of (10.3.4) of Askey-Wilson type.

In the special case $\lambda\mu = abcdfg$ the integral in (10.7.4) was evaluated in (6.4.11). If we replace a, b, c, d, f, g by $q^{\frac{\alpha}{2}-\frac{1}{4}}$, $q^{\frac{\alpha}{2}+\frac{1}{4}}$, $-q^{\frac{\gamma-\alpha}{2}-\frac{1}{4}}$, $-q^{\frac{\gamma-\alpha}{2}+\frac{1}{4}}$, $\lambda q^{-\beta}$ and $\mu q^{-\beta'}$, respectively, then this condition amounts to $\gamma = \beta + \beta'$, and therefore this case corresponds to (10.3.12) and (10.3.13). Note also that if $\lambda = g$ or $\mu = f$, which would imply $\lambda = \mu q^{-\beta'}$ or $\mu = \lambda q^{-\beta}$, then the integral in (10.7.4) becomes an ${}_8\phi_7$ series (see §6.3) and hence an analogue of the Gaussian series ${}_2F_1$. This, then, corresponds to (10.3.5) and (10.3.6).

It can be shown that $\Phi_1(\alpha;\beta,\beta';\gamma;x,y)$ has a q-Appell type double series representation that corresponds to (10.2.5), see Ex. 10.16 for more on the general function $S(a,b,c,d,f,g;\lambda,\mu)$.

Exercises

10.1 Show that

(i) $$\Phi^{(1)}(b'/x; b, b'; bb'; q; x, y) = \frac{(b', bx, b'y/x;q)_\infty}{(bb', x, y;q)_\infty},$$

(ii) $$\Phi^{(1)}(-q/y; b, qx/y^2; -qbx/y; q; x, y) = \frac{(-q, bx;q)_\infty(xq^2/y^2, x^2q^2/y^2;q^2)_\infty}{(x, y, qx/y, -qbx/y;q)_\infty(x;q^2)_\infty}.$$

(Andrews [1972])

10.2 Derive the transformation formulas

(i)
$$\Phi^{(2)}(a;b,b';c,c';q;x,y) = \frac{(b,ax;q)_\infty}{(c,x;q)_\infty} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(c/b,x;q)_m (a,b';q)_n}{(ax;q)_{m+n}(q;q)_m(q,c';q)_n} b^m y^n,$$

(ii)
$$\Phi^{(3)}(a,a';b,b';c;q;x,y) = \frac{(a,bx;q)_\infty}{(c,x;q)_\infty} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(c/a;q)_{m+n}(x;q)_m(a',b';q)_n}{(q,bx;q)_m(q,c/a;q)_n} a^m y^n,$$

and their special cases

(iii)
$$\Phi^{(2)}(a;b,b';c,a;q;x,y) = \frac{(b,ax;q)_\infty}{(c,x;q)_\infty}\ \Phi^{(3)}(c/b,0;x,b';ax;q;b,y),$$

(iv)
$$\Phi^{(3)}(a,a';b,b';aa';q;x,y) = \frac{(a,bx;q)_\infty}{(aa',x;q)_\infty}\ \Phi^{(2)}(a';x,b';bx,0;q;a,y).$$

(Andrews [1972])

10.3 Prove that

$$\Phi^{(1)}(a;b,b;c;q;x,-x) = {}_3\phi_2\left[\begin{matrix} a, aq, b^2 \\ c, cq \end{matrix}; q^2, x^2\right].$$

Deduce the quadratic transformation formula

$${}_3\phi_2\left[\begin{matrix} c/a, x, -x \\ bx, -bx \end{matrix}; q, a\right] = \frac{(c;q)_\infty (x^2;q^2)_\infty}{(a;q)_\infty (b^2x^2;q^2)_\infty}\ {}_3\phi_2\left[\begin{matrix} a, aq, b^2 \\ c, cq \end{matrix}; q^2, x^2\right].$$

10.4 Show that

$$\Phi^{(1)}(bq^{\frac{1}{2}};b,b;b^2;q;x,y) = \frac{(xq^{\frac{1}{2}};q)_\infty}{(y;q)_\infty}\ {}_2\phi_1((by/x)^{\frac{1}{2}}, -(by/x)^{\frac{1}{2}}; -b; q^{\frac{1}{2}}, x).$$

Hence, prove the quadratic transformation formula

$${}_3\phi_2\left[\begin{matrix} x, y, bq^{-\frac{1}{2}} \\ bx, by \end{matrix}; q, bq^{\frac{1}{2}}\right] = \frac{(x,y,b^2;q)_\infty}{(bx,by,bq^{\frac{1}{2}};q)_\infty}\ {}_2\phi_1((by/x)^{\frac{1}{2}}, -(by/x)^{\frac{1}{2}}; -b; q^{\frac{1}{2}}, x).$$

10.5 Derive the quadratic transformation formula

$$\Phi^{(1)}(a;b,b;c;x,y) = \sum_{n=0}^\infty \frac{(a,b^2,xq^{\frac{1}{2}}/y;q)_n}{(q,c,bq^{\frac{1}{2}};q)_n} y^n \times {}_4\phi_3\left[\begin{matrix} q^{-n/2}, -q^{-n/2}, (by/x)^{\frac{1}{2}}, -(by/x)^{\frac{1}{2}} \\ -b, q^{1/4-n/2}(y/x)^{\frac{1}{2}}, -q^{1/4-n/2}(y/x)^{\frac{1}{2}} \end{matrix}; q^{\frac{1}{2}}, q^{\frac{1}{2}}\right].$$

10.6 Prove that

$$\begin{aligned}&\Phi^{(2)}(a;b,b';c,c';q;x,y)\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_{m+n}(c/b;q)_m(c'/b';q)_n}{(ax;q)_{m+n}(q,c;q)_m(q,c';q)_n}(-bx)^m(-b'y)^n q^{\binom{m}{2}+\binom{n}{2}}\\&\quad\times\frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_2\phi_1(aq^{m+n},0;axq^{m+n};q,y).\end{aligned}$$

Note that this gives a q-analogue of Bailey [1935, 9.4(8)].

10.7 Show that a general ${}_3\phi_2$ series with an arbitrary argument is a multiple of Jackson's $\Phi^{(1)}$ series, in particular,

$${}_3\phi_2\left[\begin{matrix}a,b,c\\d,e\end{matrix};q,x\right]=\frac{(ax,b,c;q)_\infty}{(x,d,e;q)_\infty}\,\Phi^{(1)}(x;d/b,e/c;ax;q;b,c).$$

10.8 Show that

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_{m+n}(-xc/b)^m(-yb/a)^n}{(q,c;q)_m(q,b;q)_n(ax,by;q)_{m+n}}q^{\binom{m}{2}+\binom{n}{2}-mn}\\&\quad\times{}_2\phi_2(aq^{m+n},bq^{m+n},axq^{m+n},byq^{m+n};q,xy)\\&=\frac{(x,yb/a;q)_\infty}{(ax,by;q)_\infty}\,\Phi^{(1)}(a;aq/c,c/b;c;q;cxy/aq,x),\end{aligned}$$

and that this is a q-analogue of Bailey's [1935, Ex. 20(ii), p. 102] reduction formula

$$\begin{aligned}&F_4\Big(a,b;c,b;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\Big)\\&=(1-x)^a(1-y)^aF_1(a;1+a-c,c-b;c;xy,x).\end{aligned}$$

10.9 Show that

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_{m+n}(-xa/b)^m(-yb/a)^n}{(q,a;q)_m(q,b;q)_n(ax,by;q)_{m+n}}q^{\binom{m}{2}+\binom{n}{2}-mn}\\&\quad\times{}_2\phi_2(aq^{m+n},bq^{m+n};axq^{m+n},byq^{m+n};q,xy)\\&=\frac{(by/a,ax/b;q)_\infty}{(by,ax;q)_\infty(1-xy/q)},\qquad |xy|<q,\end{aligned}$$

and that this is a q-analogue of Bailey's [1935, Ex. 20(iii), p. 102] formula

$$\begin{aligned}&F_4\Big(a,b;a,b;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\Big)\\&=(1-xy)^{-1}(1-x)^b(1-y)^a.\end{aligned}$$

10.10 Prove that

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_{m+n}(-aqx/b^2)^m(-yb/a)^n}{(q,aq/b;q)_m(q,b;q)_n(ax,by;q)_{m+n}}q^{\binom{m}{2}+\binom{n}{2}-mn}\\&\quad\times{}_2\phi_2(aq^{m+n},bq^{m+n};axq^{m+n},byq^{m+n};q,xy)\\&=\frac{(by/a;q)_\infty}{(by;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}a,b,aq/by\\ax,aq/b\end{matrix};q,xy/b\right]\end{aligned}$$

and that this is a q-analogue of Bailey's [1935, Ex. 20(v), p. 102] reduction formula

$$F_4\Big(a, b; 1+a-b, b; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\Big)$$
$$= (1-y)^a \, {}_2F_1\Big(a, b; 1+a-b; -\frac{x(1-y)}{1-x}\Big).$$

10.11 Using (10.6.15) prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a,b;q)_{m+n}(-cx/b)^m(-qy/c)^n}{(q,c;q)_m(q,aq/c;q)_n(ax,by;q)_{m+n}} q^{\binom{m}{2}+\binom{n}{2}-mn}$$
$$\times \, {}_2\phi_2(aq^{m+n}, bq^{m+n}; axq^{m+n}, byq^{m+n}; q, xy)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m(b;q)_n}{(q,c,ax;q)_m(q,aq/c,by;q)_n}\Big(-\frac{cx}{bq^{\frac12}}\Big)^m\Big(-\frac{yq^{\frac12}}{c}\Big)^n q^{(m-n)^2/2},$$

where $|cx/b| < 1$ and $|y/c| < 1$. This is a q-analogue of Bailey's [1935, Ex. 20(i), p. 102] reduction formula

$$F_4\Big(a, b; c, 1+a-c; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\Big)$$
$$= F_2\Big(a; b, b; c, 1+a-c; \frac{x}{x-1}, \frac{y}{y-1}\Big).$$

10.12 Show that if $f(x,y) = \Phi^{(1)}(a; b, b'; c; q; x, y)$, then

(i) $$(abx - c/q)f(q^2x, qy) + (1-bx)f(qx, y) + (c/q - a)f(qx, qy) + (x-1)f(x,y) = 0,$$

(ii) $$(aby' - c/q)f(qx, q^2y) + (1-b'y)f(x, qy) + (c/q - a)f(qx, qy) + (y-1)f(x,y) = 0.$$

10.13 Show that if $f(x,y) = \Phi^{(2)}(a; b, b'; c, c'; q; x, y)$ then

$$ab'xf(q^2x, qy) - \frac{c}{q}f(q^2x, y) - axf(qx, qy)$$
$$+ \Big(1 + \frac{c}{q} - b'x\Big)f(qx, y) + (x-1)f(x,y) = 0.$$

10.14 Show that if $f(x,y) = \Phi^{(3)}(a, a'; b, b'; c; q; x, y)$ then

$$abxf(q^2x, y) - \frac{c}{q}f(q^2x, qy) + \frac{c}{q}f(qx, qy)$$
$$+ (1 - ax - bx)f(qx, y) + (x-1)f(x,y) = 0.$$

10.15 Show that if $f(x,y) = \Phi^{(4)}(a, b; c, c'; q; x, y)$ then

$$abxf(q^2x, q^2y) - cf(q^2x, y) - (a+b)xf(qx, qy)$$
$$+ (1+c)f(qx, y) + (x-1)f(x,y) = 0.$$

10.16 Using (6.3.3) twice show that

$$
\begin{aligned}
&S(a,b,c,d,f,g;\lambda,\mu)\\
&= \frac{2\pi(\lambda/a,\lambda/f,\mu/b,\mu/g;q)_\infty}{ab(1-q)^2(q,q,q,cd,f/a,aq/f,g/b,qb/g,af,bg;q)_\infty}\\
&\quad\times\int_f^a \frac{(qu/a,qu/f,\lambda u;q)_\infty}{(cu,du,\lambda u/af;q)_\infty}\int_g^b \frac{(qv/b,qv/g,\mu v,cduv;q)_\infty}{(cv,dv,uv,\mu v/bg;q)_\infty}d_qu\ d_qv,\\
&= \frac{2\pi(abcd,\lambda/a,\lambda a,\mu/b,\mu b;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,f/a,fa,g/b,gb;q)_\infty}\\
&\quad\times\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{(ab;q)_{m+n}(ac,ad,\lambda/f;q)_m(bc,bd,\mu/g;q)_n}{(abcd;q)_{m+n}(q,aq/f,\lambda a;q)_m(q,qb/g,\mu b;q)_n}q^{m+n}\\
&\quad+\ \text{idem }(a;f)\\
&\quad+\frac{2\pi(acdg,\lambda/a,\lambda a,\mu/g,\mu g;q)_\infty}{(q,ac,ag,ad,cg,dg,cd,f/a,af,b/g,bg;q)_\infty}\\
&\quad\times\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{(ag;q)_{m+n}(ac,ad,\lambda/f;q)_m(cg,dg,\mu/b;q)_n}{(acdg;q)_{m+n}(q,aq/f,\lambda a;q)_m(q,gq/b,\mu g;q)_n}q^{m+n}\\
&\quad+\ \text{idem }(a;f).
\end{aligned}
$$

10.17 Extend (10.3.4) to

$$
\begin{aligned}
&\Phi_D(a;b_1,\ldots,b_r;c;q;x_1,\ldots,x_r)\\
&= \sum_{n_1,\ldots,n_r=0}^\infty \frac{(a;q)_{n_1+\cdots+n_r}}{(c;q)_{n_1+\cdots+n_r}}\prod_{k=1}^r \frac{(b_k;q)_{n_k}}{(q;q)_{n_k}}x_k^{n_k}\\
&= \frac{(a,b_1x_1,\ldots,b_rx_r;q)_\infty}{(c,x_1,\ldots,x_r;q)_\infty}\ {}_{r+1}\phi_r\left[\begin{matrix}c/a,x_1,\ldots,x_r\\ b_1x_1,\ldots,b_rx_r\end{matrix};q,a\right],
\end{aligned}
$$

where Φ_D is a q-Lauricella function.
(Andrews [1972])

10.18 Prove that

$$
\begin{aligned}
&\Phi^{1:3;3}_{1:2;2}\left[\begin{matrix}q^{-n} & : a/c,c/d,c/e & ;c/a,b/d,b/e\\ bc/de & : a,q^{1-n}/b & ;b,q^{1-n}/a\end{matrix};q;q,q\right]\\
&= \frac{(c,ab/c;q)_n}{(a,b;q)_n},
\end{aligned}
$$

where $n=0,1,\ldots$. Deduce the summation formula

$$
\begin{aligned}
&\sum_{j=0}^\infty\sum_{k=0}^\infty \frac{(a/c,c/d,c/e;q)_j(c/a,b/d,b/e;q)_k}{(bc/de;q)_{j+k}(q,a;q)_j(q,b;q)_k}b^ja^kq^{jk}\\
&= \frac{(c,ab/c;q)_\infty}{(a,b;q)_\infty}.
\end{aligned}
$$

(Gasper [2000])

10.19 Show that

$$\sum_{j,k,m\ge 0} \frac{(\alpha;q)_j(\beta;q)_k(q/\gamma;q)_m(\gamma;q)_{j+k-m}}{(q;q)_j(q;q)_k(q;q)_m(\alpha\beta;q)_{j+k-m}} x^j y^k z^m$$
$$= \frac{(\gamma x, qz/\gamma, \beta y, \gamma y, \alpha xz/\gamma, \beta yz;q)_\infty}{(x, \alpha\beta z/\gamma, \beta\gamma y, y, xz, yz;q)_\infty}$$
$$\times\ {}_8W_7(\beta\gamma y q^{-1}; \beta, \gamma, \gamma y/\alpha, \gamma/z, \beta y/x; q, \alpha xz/\gamma).$$

(Krattenthaler and Rosengren [2003])

10.20 Extend (1.9.6) to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b_1q^{m_1},\ldots,b_rq^{m_r};q)_{n+m}(a,b;q)_n(c,d;q)_m}{(b_1,\ldots,b_r;q)_{m+n}(q,bq;q)_n(q,dq;q)_m}$$
$$\times (a^{-1}q^{1-M})^n(c^{-1}q^{1-M})^m$$
$$= \frac{(q,q,bq/a,dq/c;q)_\infty(b_1/db;q)_{m_1}\cdots(b_r/bd;q)_{m_r}(bd)^M}{(bq,dq,q/a,q/c;q)_\infty(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}},$$

where $m_1,\ldots,m_r$ are nonnegative integers, $M = m_1+\cdots+m_r$, and $|q^{1-M}| < \min(|a|,|c|)$.

(Denis [1988])

Notes

§10.2 B. Srivastava [1995] introduced some elementary bibasic extensions of (10.2.5)–(10.2.8). For a quantum algebra approach to multivariable series, see Floreanini, Lapointe and Vinet [1994]. Sahai [1999] considered the q-Appell functions from the point of view of universal enveloping algebra of $s\ell(2)$. See also Jain [1980a,b] and, for connections of Appell functions with BC_n root systems, Beerends [1992].

§§10.3–10.6 Agarwal [1974] evaluated some q-integrals of the double series in (10.2.5)–(10.2.8). Also see Nassrallah [1990, 1991].

Exercises 10.1–10.2 Upadhyay [1973] gave some transformation and summation formulas for q-Jackson type double series.

Ex. 10.3 This is a q-analogue of a quadratic transformation formula for $F_1(a;b,b;c;x,-x)$ obtained by Ismail and Pitman [2000].

Exercises 10.12–10.15 Agarwal [1954] gave some 3-term contiguous relations satisfied by $\Phi^{(1)},\ldots,\Phi^{(4)}$, including formulas connecting them with the q-derivatives.

Ex. 10.17 For an application of the q-Lauricella series, see Bressoud [1978].

Ex. 10.18 See Van der Jeugt, Pitre and Srinivasa Rao [1994] for more summation formulas of this type. For similar results when $q\to 1$, see Pitre and Van der Jeugt [1996].

11

ELLIPTIC, MODULAR, AND THETA HYPERGEOMETRIC SERIES

11.1 Introduction

Since the series $\sum u_n$ is called a hypergeometric series if $g(n) = u_{n+1}/u_n$ is a rational function of n, and a q-(basic) hypergeometric series if $g(n)$ is a rational function of q^n, it is natural to call this series an *elliptic hypergeometric series* if $g(n)$ is an elliptic (doubly periodic meromorphic) function of n with n considered as a complex variable. One motivation for considering these three classes of series is Weierstrass's theorem that a meromorphic function $f(z)$ which satisfies an algebraic addition theorem of the form

$$P(f(u), f(v), f(u+v)) = 0$$

identically in u and v, where $P(x, y, z)$ is a nonzero polynomial whose coefficients are independent of u and v, is either a rational function of z, a rational function of $e^{\lambda z}$ for some λ, or an elliptic function (see, e.g., Erdélyi [1953, §13.11], Rosengren [2001a, 2003c], and, for a proof, Phragmén [1885]). Elliptic analogues of very-well-poised basic hypergeometric series were introduced rather recently by Frenkel and Turaev [1997] in their work on elliptic $6j$-symbols, which are elliptic solutions of the Yang-Baxter equation found by Baxter [1973], [1982] and Date *et al.* [1986–1988]. Frenkel and Turaev showed that the elliptic $6j$-symbols are multiples of the very-well-poised ${}_{12}v_{11}$ elliptic (modular) hypergeometric series defined in §11.3 (recall from §7.2 that the ordinary q-analogues of Racah's $6j$-symbols are multiples of balanced ${}_4\phi_3$ series, which are transformable to ${}_8\phi_7$ series) and then used the tetrahedral symmetry of the elliptic $6j$-symbols and the finite dimensionality of cusp forms to derive elliptic analogues of Bailey's transformation formula (2.9.1) for terminating ${}_{10}\phi_9$ series and of Jackson's ${}_8\phi_7$ summation formula (2.6.2). This quickly led to a flurry of activity on elliptic hypergeometric series, resulting in several related papers (listed in alphabetical order) by van Diejen and Spiridonov [2000–2003], Gasper and Schlosser [2003], Kajihara and Noumi [2003], Kajiwara, Masuda, Noumi, Ohta and Yamada [2003], Koelink, van Norden and Rosengren [2003], Rosengren [2001a–2003f], Rosengren and Schlosser [2003a,b], Spiridonov [2000–2003c], Spiridonov and Zhedanov [2000a–2003], Warnaar [2002b–2003e], and others. In particular, general elliptic hypergeometric series and their extensions to theta hypergeometric series were introduced and studied by Spiridonov [2002a–2003a], who also considered theta hypergeometric integrals in Spiridonov [2003b]. Transformations of elliptic hypergeometric integrals are considered in Rains [2003b].

We start in §11.2 with the elliptic shifted factorials, Spiridonov's [2002a] "multiplicative" ${}_{r+1}E_r$ theta hypergeometric series notation and its very-well-poised ${}_{r+1}V_r$ special case, and then point out some of their main properties. The "additive" forms of these series ${}_{r+1}e_r$ and ${}_{r+1}v_r$ are presented in §11.3, along with their modular properties, and the Frenkel and Turaev summation and transformation formulas. In §11.4 we present a simpler derivation of the elliptic analogue of Jackson's ${}_8\phi_7$ summation formula via mathematical induction, which is a modification of proofs that were kindly communicated to the authors by Rosengren (in a June 13, 2002, e-mail message), Spiridonov (in a Nov. 22, 2002, e-mail message) and Warnaar (see §6 in his [2002b] paper). Central to this method of proof and, in fact, to all of the formulas in this chapter is the theta function identity given in Ex. 2.16(i) which is simply the elliptic analogue of the trivial identity

$$\begin{aligned}&(1-x\lambda)(1-x/\lambda)(1-\mu\nu)(1-\mu/\nu)-(1-x\nu)(1-x/\nu)(1-\lambda\mu)(1-\mu/\lambda)\\ &\quad=\frac{\mu}{\lambda}(1-x\mu)(1-x/\mu)(1-\lambda\nu)(1-\lambda/\nu). \qquad (11.1.1)\end{aligned}$$

The elliptic analogue of Bailey's ${}_{10}\phi_9$ transformation formula is derived in §11.5, along with some other transformation formulas. Theta hypergeometric extensions of some of the summation and transformation formulas in sections 3.6–3.8 are derived in §11.6. Some multidimensional elliptic hypergeometric series are considered in §11.7, where we present Rosengren's [2003c] elliptic extension of Milne's [1985a] fundamental theorem and related formulas. Many additional formulas involving theta hypergeometric series and elliptic integrals are presented in the exercises at the end of the chapter.

11.2 Elliptic and theta hypergeometric series

Define a modified Jacobi theta function by

$$\theta(x;p)=(x,p/x;p)_\infty\,, \qquad \theta(x_1,\ldots,x_m;p)=\prod_{k=1}^{m}\theta(x_k;p), \qquad (11.2.1)$$

where $x,x_1,\ldots,x_m\neq 0$ and $|p|<1$. Since $\theta(x;p)=(x;p)_\infty(p/x;p)_\infty$ is the product of two infinite p-shifted factorials, analogous to the name of the triple product identity (1.6.1), we will call $\theta(x;p)$ the *double product theta function* (*with argument* x *and nome* p). From (1.6.9) and (1.6.14) it follows that

$$\vartheta_1(x,e^{\pi i\tau})=ie^{-ix}e^{\pi i\tau/4}(p;p)_\infty\theta(e^{2ix};p) \qquad (11.2.2)$$

and

$$[a;\sigma,\tau]=\frac{\theta(q^a;p)}{\theta(q;p)}q^{(1-a)/2} \qquad (11.2.3)$$

with

$$q=e^{2\pi i\sigma}, \quad p=e^{2\pi i\tau}, \qquad (11.2.4)$$

where a,σ,τ are complex numbers such that $\mathrm{Im}\,(\tau)>0$ and $\sigma\neq m+n\tau$ for integer values of m and n. We set $q^a=e^{2\pi i\sigma a}$ and $p^a=e^{2\pi i\tau a}$. Following

Warnaar [2002b], we define an *elliptic* (or *theta*) *shifted factorial* analogue of the q-shifted factorial by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1} \theta(aq^k;p), & n=1,2,\ldots, \\ 1, & n=0, \\ 1/\prod_{k=0}^{-n-1} \theta(aq^{n+k};p), & n=-1,-2,\ldots, \end{cases} \tag{11.2.5}$$

and let

$$(a_1,a_2,\ldots,a_m;q,p)_n = \prod_{k=1}^{m} (a_k;q,p)_n, \tag{11.2.6}$$

where $a, a_1, \ldots, a_m \neq 0$. Analogous to the name q-shifted factorial for $(a;q)_n$, we also call $(a;q,p)_n$ the *q,p-shifted factorial* in order to distinguish it from the *σ,τ-shifted factorial* defined in the next section. Notice that $\theta(x;0)=1-x$ and thus $(a;q,0)_n=(a;q)_n$. Since q is called the base in $(a;q)_n$ and p is called the nome in $\theta(a;p)$, we call q and p in $(a;q,p)_n$ the *base* and *nome*, respectively. Similarly, in order to distinguish the *modular parameters* σ and τ, we call σ the *base modular parameter* and τ the *nome modular parameter.* For the sake of simplicity, we decided not to use Spiridonov's [2002a,b] notation $\theta(a;p;q)_n$ for the elliptic shifted factorial.

Corresponding to Spiridonov [2002a], we formally define an ${}_{r+1}E_r$ *theta hypergeometric series* with base q and nome p by

$$\begin{aligned} &{}_{r+1}E_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{r+1};q,p)_n}{(q,b_1,\ldots,b_r;q,p)_n} z^n, \end{aligned} \tag{11.2.7}$$

where, as elsewhere, it is assumed that the parameters are such that each term in the series is well-defined; in particular, the a's and b's are never zero. Unless stated otherwise, we assume that q and p are independent of each other, but we do not assume that the above series converges or that the numerator parameters $a_1,a_2,\ldots,a_{r+1}$, denominator parameters $b_1,\ldots,b_r$, and the argument z in it are independent of each other or of q and p. Note that if a_jp^k is a nonpositive integer power of q for some integer k and a $j\in\{1,2,\ldots,r+1\}$, then the series in (11.2.7) terminates. Clearly, if z and the a's and b's are independent of p, then

$$\begin{aligned} &\lim_{p\to 0} {}_{r+1}E_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;z) \\ &= {}_{r+1}E_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;q,0;z) \\ &= {}_{r+1}\phi_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;q,z), \end{aligned} \tag{11.2.8}$$

where the limit of the series is a termwise limit.

As is customary, the notation ${}_{r+1}E_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;z)$ is also used to denote the sum of the series in (11.2.7) inside the circle of convergence and its analytic continuation (called a *theta hypergeometric function*) outside the circle of convergence. Unlike nonterminating ${}_{r+1}F_r$ series and nonterminating ${}_{r+1}\phi_r$ series with $0<|q|<1$, which have radius of convergence $R=1$ (as can be easily seen by applying the ratio test to (1.2.25) and (1.2.26)), for $0 < |q|, |p| < 1$ and any $R \in [0, \infty]$ there is a nonterminating ${}_{r+1}E_r$ series with radius of convergence R. In particular, there are nonterminating ${}_{r+1}E_r$ series with $0 < |q|, |p| < 1$ that converge to entire functions of z, which is not the case for nonterminating ${}_{r+1}F_r$ and ${}_{r+1}\phi_r$ series with $r = 0, 1, \ldots$ and $0 < |q| < 1$. For example, consider a nonterminating ${}_{r+1}E_r$ series with $0 < |q|, |p| < 1$ and

$$b_k = a_k p^{m_k}, \quad k = 1, \ldots, r+1, \tag{11.2.9}$$

where $m_1, \ldots, m_{r+1}$ are integers and $b_{r+1} = q$. Since the double product theta function identity $\theta(ap^m; p) = (-a)^{-m} p^{-\binom{m}{2}} \theta(a; p)$ implies that

$$(ap^m; q, p)_n = (a; q, p)_n (-a)^{-mn} p^{-n\binom{m}{2}} q^{-m\binom{n}{2}}$$

for $m, n = 0, \pm 1, \ldots$, we find that

$${}_{r+1}E_r(a_1, \ldots, a_{r+1}; a_1 p^{m_1}, \ldots, a_r p^{m_r}; q, p; z) = \sum_{n=0}^{\infty} (z\rho_r)^n q^{M_r \binom{n}{2}}, \tag{11.2.10}$$

with $a_{r+1} = qp^{-m_{r+1}}$, $\rho_r = \prod_{k=1}^{r+1} (-a_k)^{m_k} p^{\binom{m_k}{2}}$, and $M_r = m_1 + \cdots + m_{r+1}$. From (11.2.10) it is clear that this series converges to an entire function of z when $M_r > 0$, converges only for $z = 0$ when $M_r < 0$, and converges to $1/(1 - z\rho_r)$ when $M_r = 0$ and $|z\rho_r| < 1$.

As in Spiridonov [2002a], we call a (unilateral or bilateral) series $\sum c_n$ an *elliptic hypergeometric series* if $g(n) = c_{n+1}/c_n$ is an elliptic function of n with n considered as a complex variable, i.e., $g(x)$ is a doubly periodic meromorphic function of the complex variable x. For the ${}_{r+1}E_r$ series in (11.2.7)

$$g(x) = z \prod_{k=1}^{r+1} \frac{\theta(a_k q^x; p)}{\theta(b_k q^x; p)} \tag{11.2.11}$$

with $b_{r+1} = q$. Clearly $g(x)$ is a meromorphic function of x. From (11.2.4) it is obvious that $q^{x+\sigma^{-1}} = q^x$ and hence $g(x + \sigma^{-1}) = g(x)$. Since, by (11.2.1),

$$\theta(aq^{x+\tau\sigma^{-1}}; p) = (-aq^x)^{-1} \theta(aq^x; p), \tag{11.2.12}$$

it follows that

$$g(x + \tau\sigma^{-1}) = g(x) \prod_{k=1}^{r+1} \frac{b_k}{a_k}. \tag{11.2.13}$$

Thus $g(x + \tau\sigma^{-1}) = g(x)$ when

$$a_1 a_2 \cdots a_{r+1} = (b_1 b_2 \cdots b_r) q\,, \tag{11.2.14}$$

in which case $g(x)$ is an elliptic (doubly periodic meromorphic) function of x with periods σ^{-1} and $\tau\sigma^{-1}$. Therefore, we call (11.2.14) the *elliptic balancing condition*, and when (11.2.14) holds we say that ${}_{r+1}E_r$ is *elliptically balanced* (*E-balanced*). In Spiridonov [2002a] an ${}_{r+1}E_r$ series is called, simply, "balanced" when (11.2.14) holds, but here we need to distinguish between the different balancing conditions that arise. Notice that, unlike the requirement that $z = q$ in the definition of a balanced ${}_{r+1}\phi_r$ series, no restrictions are placed on the argument z in the above definition of an E-balanced ${}_{r+1}E_r$ series. If $z = q$, then, by the definition of a k-balanced ${}_{r+1}\phi_r$ series in §1.2, the ${}_{r+1}\phi_r$ series in (11.2.8) is (-1)-balanced if and only if the elliptic balancing condition (11.2.14) holds.

Analogous to the basic hypergeometric special case, we call the ${}_{r+1}E_r$ series in (11.2.7) *well-poised* if

$$qa_1 = a_2b_1 = a_3b_2 = \ldots = a_{r+1}b_r, \tag{11.2.15}$$

in which case the elliptic balancing condition (11.2.14) reduces to

$$a_1^2a_2^2 \cdots a_{r+1}^2 = (a_1q)^{r+1}. \tag{11.2.16}$$

Via (11.2.5) and the $n \to \infty$ limit case of Ex. 1.1(iv) with q replaced by p, we find that

$$\frac{\theta(aq^{2n};p)}{\theta(a;p)} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/p^{\frac{1}{2}}, -qa^{\frac{1}{2}}p^{\frac{1}{2}};q,p)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, a^{\frac{1}{2}}p^{\frac{1}{2}}, -a^{\frac{1}{2}}/p^{\frac{1}{2}};q,p)_n}(-q)^{-n} \tag{11.2.17}$$

is an elliptic analogue of the quotient

$$\frac{1-aq^{2n}}{1-a} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}};q)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}};q)_n},$$

which is the very-well-poised part of the ${}_{r+1}W_r$ series in (2.1.11) with $a_1 = a$; see Ex. 11.3. Therefore the ${}_{r+1}E_r$ series in (11.2.7) is called *very-well-poised* if it is well-poised, $r \geq 4$, and

$$a_2 = qa_1^{\frac{1}{2}},\ a_3 = -qa_1^{\frac{1}{2}},\ a_4 = qa_1^{\frac{1}{2}}/p^{\frac{1}{2}},\ a_5 = -qa_1^{\frac{1}{2}}p^{\frac{1}{2}}. \tag{11.2.18}$$

Corresponding to Spiridonov [2002b, (2.15)], we define the ${}_{r+1}V_r$ very-well-poised theta hypergeometric series by

$$\begin{aligned}&{}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p;z)\\&= \sum_{n=0}^{\infty} \frac{\theta(a_1q^{2n};p)}{\theta(a_1;p)} \frac{(a_1,a_6,a_7,\ldots,a_{r+1};q,p)_n}{(q,a_1q/a_6,a_1q/a_7,\ldots,a_1q/a_{r+1};q,p)_n}(qz)^n.\end{aligned} \tag{11.2.19}$$

It follows that if (11.2.15) and (11.2.18) hold, then

$$\begin{aligned}&{}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p;z)\\&= {}_{r+1}E_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;-z),\end{aligned} \tag{11.2.20}$$

and that ${}_{r+1}V_r$ is elliptically balanced if and only if

$$(a_6^2a_7^2\cdots a_{r+1}^2)q^2 = (a_1q)^{r-5}. \tag{11.2.21}$$

As in Warnaar [2003c], when the argument z in the ${}_{r+1}V_r$ series equals 1 we suppress it and denote the series in (11.2.19) by the simpler notation ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$. If $a_1,a_6,a_7,\ldots,a_{r+1}$ are independent of p, then

$$\begin{aligned}&\lim_{p\to 0} {}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)\\&= {}_{r-1}W_{r-2}(a_1;a_6,\ldots,a_{r+1};q,q),\end{aligned} \tag{11.2.22}$$

which shows that there is a shift $r \to r-2$ when taking the $p \to 0$ limit, and that the $p \to 0$ limit of an ${}_{r+1}V_r(a_1; a_6, a_7, \ldots, a_{r+1}; q, p)$ series with $a_1, a_6, a_7, \ldots, a_{r+1}$ independent of p is an ${}_{r-1}W_{r-2}$ series.

As mentioned in the Introduction, Frenkel and Turaev showed that the elliptic $6j$-symbols, which are elliptic solutions of the Yang-Baxter equation (formula (1.2.b) in their paper) found by Baxter [1973], [1982] and Date *et al.* [1986–1988] can be expressed as ${}_{12}V_{11}$ series (in the additive notation discussed in the next section). Then they employed the tetrahedral symmetry of the elliptic $6j$-symbols, which is analogous to the symmetry of the classical, quantum and trigonometric $6j$-symbols (see Frenkel and Turaev [1995, 1997]), and the finite dimensionality of cusp forms (see Eichler and Zagier [1985]) to derive (in their additive form) the following elliptic analogue of Bailey's ${}_{10}\phi_9$ transformation formula (2.9.1)

$$\begin{aligned} &{}_{12}V_{11}(a; b, c, d, e, f, \lambda a q^{n+1}/ef, q^{-n}; q, p) \\ &= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q, p)_n} \\ &\quad \times {}_{12}V_{11}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda a q^{n+1}/ef, q^{-n}; q, p) \end{aligned} \tag{11.2.23}$$

for $n = 0, 1, \ldots,$ provided that the balancing condition

$$bcdef(\lambda a q^{n+1}/ef)q^{-n}q = (aq)^3, \tag{11.2.24}$$

which is equivalent to $\lambda = qa^2/bcd$, holds. Note that both of the series in (11.2.23) are E-balanced when (11.2.24) holds. Setting $\lambda = a/d$ in (11.2.23) yields a summation formula for ${}_{10}V_9$ series that is an elliptic analogue of Jackson's ${}_8\phi_7$ summation formula (2.6.2) and of Dougall's ${}_7F_6$ summation formula (2.1.6), which, after a change in parameters, can be written in the form:

$${}_{10}V_9(a; b, c, d, e, q^{-n}; q, p) = \frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_n} \tag{11.2.25}$$

for $n = 0, 1, \ldots,$ provided that the balancing condition $bcde = a^2 q^{n+1}$, which can be written in the form

$$(bcdeq^{-n})q = (aq)^2, \tag{11.2.26}$$

holds. Clearly, if a, b, c, d, e are independent of p, then (11.2.25) tends to Jackson's ${}_8\phi_7$ summation formula (2.6.2) as $p \to 0$. Unlike in the basic hypergeometric limit cases of (2.6.2) discussed in Chapters 1 and 2, one cannot take termwise limits of (11.2.25) to obtain a ${}_3E_2$ analogue of the q-Saalschütz formula (1.7.2), ${}_2E_1$ analogues of the q-Vandermonde formulas (1.5.2) and (1.5.3), or even a ${}_1E_0$ analogue of the terminating case of the q-binomial theorem (1.3.2) in Ex. 1.3(i). Therefore one cannot derive (11.2.25) by working up from sums at the ${}_1E_0$, ${}_2E_1$, and ${}_3E_2$ levels as was done in Chapters 1 and 2, and so one is forced to employ a different approach, such as in the above-mentioned Frenkel and Turaev derivation, or by some other method. Rather than repeating the Frenkel and Turaev [1997] derivation of (11.2.25), we will present in §11.4 a simpler derivation of (11.2.25) via mathematical induction, which is a modification of those discovered independently by Rosengren, Spiridonov, and Warnaar. We will then show in §11.5 that (11.2.23) follows from

(11.2.25) in the same way as in our derivation of Bailey's ${}_{10}\phi_9$ transformation formula (2.9.1) from Jackson's ${}_8\phi_7$ summation formula (2.6.2).

Observe that if we set $e = \pm(aq)^{\frac{1}{2}}$, then (11.2.25) reduces to

$$ {}_9V_8(a;b,c,d,q^{-n};q,p) = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n} \tag{11.2.27}$$

for $n = 0,1,\ldots$, provided that the balancing condition

$$(bcdq^{-n})q = \big(\pm(aq)^{\frac{1}{2}}\big)^3 \tag{11.2.28}$$

holds, which is equivalent to the elliptic balancing condition $b^2c^2d^2 = a^3q^{2n+1}$.

In view of (11.2.24), (11.2.26), (11.2.28), and of the required balancing conditions for other significant special cases of the Frenkel and Turaev transformation formula (11.2.23) to hold, analogous to the definition of a VWP-balanced series given in §2.1 we call the series ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p;z)$ a *very-well-poised-balanced* (*VWP-balanced*) series when the *very-well-poised balancing condition*

$$(a_6a_7\cdots a_{r+1})qz = \big(\pm(a_1q)^{\frac{1}{2}}\big)^{r-5} \tag{11.2.29}$$

holds. It follows that ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$ is VWP-balanced if and only if

$$(a_6a_7\cdots a_{r+1})q = \big(\pm(a_1q)^{\frac{1}{2}}\big)^{r-5}, \tag{11.2.30}$$

and that the summation formulas (11.2.25), (11.2.27) and the transformation formula (11.2.23) hold for $n = 0,1,\ldots$, when the series are VWP-balanced. Note that (11.2.30) reduces to $(a_6a_7\cdots a_{2j+6})q = (a_1q)^j$ when $r = 2j+5$ is odd. It should also be noted that if either (11.2.30) or $(a_6\cdots a_{r+1})q = -(\pm(a_1q)^{\frac{1}{2}})^{r-5}$ holds, then ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$ is E-balanced, and hence an elliptic hypergeometric series. If ${}_{r+1}V_r(a_1;a_6,a_7,\ldots,a_{r+1};q,p)$ is E-balanced, then it is VWP-balanced when r is even, but not necessarily when r is odd. In particular, the elliptic balancing condition (11.2.21) is *not* a sufficient condition for the Frenkel and Turaev transformation and summation formulas (11.2.23) and (11.2.25) to hold; the series in these formulas need to be VWP-balanced in order for these formulas to hold for $n = 0,1,\ldots$.

Since, if ${}_{r+1}E_r$ is a well-poised series satisfying the relations in (11.2.15),

$$\begin{aligned} &{}_{r+1}E_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;q,p;z) \\ &= {}_{r+9}V_{r+8}\big(a_1;a_1^{\frac{1}{2}},-a_1^{\frac{1}{2}},a_1^{\frac{1}{2}}p^{\frac{1}{2}},-a_1^{\frac{1}{2}}/p^{\frac{1}{2}},a_2,a_3,\ldots,a_{r+1};q,p;-z\big), \end{aligned} \tag{11.2.31}$$

we find that the very-well-poised balancing condition for the above ${}_{r+9}V_{r+8}$ series is equivalent to the *well-poised balancing condition*

$$(a_1a_2\cdots a_{r+1})z = -\big(\pm(a_1q)^{\frac{1}{2}}\big)^{r+1} \tag{11.2.32}$$

for the ${}_{r+1}E_r$ series in (11.2.31). Hence, a well-poised ${}_{r+1}E_r$ series is called *well-poised-balanced* (*WP-balanced*) when (11.2.32) holds. In particular, the well-poised ${}_4E_3$ series in the transformation formula in Ex. 11.6 is WP-balanced. Clearly, every VWP-balanced theta hypergeometric series is WP-balanced and, by the above observations, every WP-balanced theta hypergeometric series can be rewritten to be a VWP-balanced series of the form in (11.2.31).

Analogous to the bilateral ${}_r\psi_r$ series, we follow Spiridonov [2002a] in defining a ${}_rG_r$ bilateral theta hypergeometric series by

$$\begin{aligned}&{}_rG_r(a_1,\ldots,a_r;b_1,\ldots,b_r;q,p;z)\\&=\sum_{n=-\infty}^{\infty}\frac{(a_1,\ldots,a_r;q,p)_n}{(b_1,\ldots,b_r;q,p)_n}z^n.\end{aligned}\tag{11.2.33}$$

Note that

$$\begin{aligned}&{}_rG_r(a_1,\ldots,a_r;q,b_1,b_2,\ldots,b_{r-1};q,p;z)\\&={}_rE_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};q,p;z)\end{aligned}\tag{11.2.34}$$

and, more generally, as in the bilateral basic hypergeometric case in (5.1.5), if the index of summation in an ${}_{r+1}E_r$ series is shifted by an integer amount, then the resulting series is an ${}_{r+1}G_{r+1}$ series multiplied by a quotient of products of q,p-shifted factorials. Also note that corresponding to (11.2.9) we can consider nonterminating ${}_rG_r$ series with $0<|q|,|p|<1$ and

$$b_k=a_kp^{m_k},\quad k=1,\ldots,r,$$

where $m_1,\ldots,m_r$ are integers. As in (11.2.10) we find that

$${}_rG_r(a_1,\ldots,a_r;a_1p^{m_1},\ldots,a_rp^{m_r};q,p;z)=\sum_{n=-\infty}^{\infty}(z\sigma_r)^nq^{N_r\binom{n}{2}}\tag{11.2.35}$$

with $\sigma_r=\prod_{k=1}^r(-a_k)^{m_k}p^{\binom{m_k}{2}}$ and $N_r=m_1+\cdots+m_r$, which clearly converges for any $z\neq 0$ if and only if $N_r>0$.

If, as in Spiridonov [2002a], we replace $r+1$ by r in the upper limit of the product in (11.2.11) and proceed as in the derivation of the condition (11.2.14) for an ${}_{r+1}E_r$ series to be elliptically balanced, we find that the series ${}_rG_r$ is *elliptically balanced* (*E-balanced*) if and only if

$$a_1a_2\cdots a_r=b_1b_2\cdots b_r.\tag{11.2.36}$$

If ${}_rG_r$ is a *well-poised* series with

$$a_1b_1=a_2b_2=\ldots=a_rb_r,\tag{11.2.37}$$

then the elliptic balancing condition (11.2.36) reduces to

$$a_1^2a_2^2\cdots a_r^2=(a_1b_1)^r.\tag{11.2.38}$$

In view of (11.2.32), (11.2.34), and (5.1.7), it is consistent to call a well-poised ${}_rG_r$ series *well-poised-balanced* (*WP-balanced*) when

$$(a_1a_2\cdots a_r)z=-\big(\pm(a_1b_1)^{\frac12}\big)^r.\tag{11.2.39}$$

In our consideration of modular series in the next section we are led to consider additive forms of special cases of the rather general power series

$$\begin{aligned}&{}_rE_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q,p;\mathbf{A},z)\\&=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q,p)_n}{(q,b_1,\ldots,b_s;q,p)_n}A_nz^n,\end{aligned}\tag{11.2.40}$$

and the Laurent series

$$ {}_rG_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,p;\mathbf{A},z) = \sum_{n=-\infty}^{\infty}\frac{(a_1,\ldots,a_r;q,p)_n}{(b_1,\ldots,b_s;q,p)_n}A_nz^n, \tag{11.2.41} $$

where $\mathbf{A}=\{A_n\}$ is an arbitrary sequence of complex numbers. Some special cases of these series with $A_n=q^{\alpha n(n-1)/2}$ or, more generally, with $A_n=\exp(\alpha_1 n+\alpha_2 n^2+\alpha_3 n^3)$ are considered in Spiridonov [2002a, 2003b]. If $A_n=1$ for all n, then we will suppress $\mathbf{A}$ from the left sides of (11.2.40) and (11.2.41). When we encounter series with more than one base or nome, such as in Ex. 11.25(i), Ex. 11.26 and in several of the formulas in §11.6, or multivariable formulas, such as the multivariable extension of the Frenkel and Turaev summation formula in §11.7, we will write the series in terms of elliptic shifted factorials.

To help keep the size of this book down we will not repeat the main elliptic identities and summation and transformation formulas in the appendices. Nevertheless, for the convenience of the readers we collect below some of the most useful identities involving $\theta(a;p)$, the q,p-shifted factorials, and the q,p-binomial coefficients.

$$ \theta(a;p)=\theta(p/a;p)=-a\,\theta(1/a;p)=-a\,\theta(ap;p), \tag{11.2.42} $$

$$ \theta(a^2;p^2)=\theta(a,-a;p), \tag{11.2.43} $$

$$ \theta(a;p)=\theta(a,ap;p^2)=\theta(a^{\frac12},-a^{\frac12},(aq)^{\frac12},-(aq)^{\frac12};p), \tag{11.2.44} $$

$$ \theta(a;p)=\theta(ap^n;p)(-a)^np^{\binom{n}{2}}, \tag{11.2.45} $$

$$ (a;q,p)_n\theta(aq^n;p)=(a;q,p)_{n+1}=\theta(a;p)(aq;q,p)_n, \tag{11.2.46} $$

$$ (a;q,p)_{n+k}=(a;q,p)_n(aq^n;q,p)_k, \tag{11.2.47} $$

$$ (a;q,p)_n=(q^{1-n}/a;q,p)_n(-a)^nq^{\binom{n}{2}}, \tag{11.2.48} $$

$$ (a;q,p)_{n-k}=\frac{(a;q,p)_n}{(q^{1-n}/a;q,p)_k}\Big(-\frac{q}{a}\Big)^kq^{\binom{k}{2}-nk}, \tag{11.2.49} $$

$$ (aq^{-n};q,p)_n=(q/a;q,p)_n\Big(-\frac{a}{q}\Big)^nq^{-\binom{n}{2}}, \tag{11.2.50} $$

$$ (aq^{-n};q,p)_k=\frac{(a;q,p)_k(q/a;q,p)_n}{(q^{1-k}/a;q,p)_n}q^{-nk}, \tag{11.2.51} $$

$$ (a;q^{-1},p)_n=(a^{-1};q,p)_n(-a)^nq^{-\binom{n}{2}}, \tag{11.2.52} $$

$$ (a;q,p)_{-n}=\frac{1}{(aq^{-n};q,p)_n}=\frac{(-q/a)^n}{(q/a;q,p)_n}q^{\binom{n}{2}}, \tag{11.2.53} $$

$$ (aq^n;q,p)_k=\frac{(a;q,p)_k(aq^k;q,p)_n}{(a;q,p)_n}=\frac{(a;q,p)_{n+k}}{(a;q,p)_n}, \tag{11.2.54} $$

$$ (a;q,p)_n=(ap^k;q,p)_n(-a)^{nk}p^{n\binom{k}{2}}q^{k\binom{n}{2}}, \tag{11.2.55} $$

$$ (a^2;q^2,p^2)_n=(a,-a;q,p)_n, \tag{11.2.56} $$

$$ (a;q,p^2)_{2n}=(a^{\frac12},-a^{\frac12},(aq)^{\frac12},-(aq)^{\frac12};q,p)_n, \tag{11.2.57} $$

$$(a;q,p)_{2n} = (a, aq; q^2, p)_n, \tag{11.2.58}$$
$$(a;q,p)_{3n} = (a, aq, aq^2; q^3, p)_n, \tag{11.2.59}$$
$$(a;q,p)_{kn} = (a, aq, \ldots, aq^{k-1}; q^k, p)_n. \tag{11.2.60}$$

Corresponding to the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ defined in Ex. 1.2, we define the q,p-*binomial coefficient* (or *elliptic binomial coefficient*) by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,p} = \frac{(q;q,p)_n}{(q;q,p)_n(q;q,p)_{n-k}} \tag{11.2.61}$$

for $k = 0, 1, \ldots, n$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,p} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{q,p} = \frac{(q^{-n};q,p)_k}{(q;q,p)_k}(-q^n)^k q^{-\binom{k}{2}}. \tag{11.2.62}$$

For complex α we can employ (11.2.62) to define

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_{q,p} = \frac{(q^{-\alpha};q,p)_k}{(q;q,p)_k}(-q^\alpha)^k q^{-\binom{k}{2}} \tag{11.2.63}$$

when $k = 0, 1, \ldots$. As in Ex. 1.2 it follows that $\begin{bmatrix} \alpha \\ k \end{bmatrix}_{q,p}$ satisfies the identities

$$\begin{bmatrix} k+\alpha \\ k \end{bmatrix}_{q,p} = \frac{(q^{\alpha+1};q,p)_k}{(q;q,p)_k}, \tag{11.2.64}$$
$$\begin{bmatrix} -\alpha \\ k \end{bmatrix}_{q,p} = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_{q,p}(-q^{-\alpha})^k q^{-\binom{k}{2}}, \tag{11.2.65}$$
$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^{-1},p} = \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q,p} q^{k^2-\alpha k}. \tag{11.2.66}$$

Similarly, additional elliptic identities can be obtained by replacing each $(a;q)_n$ by $(a;q,p)_n$ and making any necessary changes in the other identities of Appendix I that do *not* contain any infinite products. Elliptic analogues of (I.6), (I.41), and of some other identities containing infinite shifted factorials and/or q-gamma functions can be obtained by using the Jackson [1905d] and Ruijsenaars [1997, 2001] *elliptic gamma function* defined by

$$\Gamma(z;q,p) = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}q^{j+1}p^{k+1}}{1-zq^jp^k}, \tag{11.2.67}$$

where z, q, p are complex numbers and $|q|$, $|p| < 1$, whose main properties are considered in Ex. 11.12. Since

$$(z;q,p)_n = \frac{\Gamma(zq^n;q,p)}{\Gamma(z;q,p)}, \quad n \in \mathbb{Z}, \tag{11.2.68}$$

and

$$\Gamma_q(z) = (1-q)^{1-z}(q;q)_\infty \Gamma(q^z;q,0), \quad 0 < q < 1, \tag{11.2.69}$$

one can extend the definition of $(z;q)_\alpha$ in (I.6) by defining

$$(z;q,p)_\alpha = \frac{\Gamma(zq^\alpha;q,p)}{\Gamma(z;q,p)}, \tag{11.2.70}$$

and extend the definition of $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q$ in (I.41) and of $\begin{bmatrix}\alpha\\ k\end{bmatrix}_{q,p}$ in (11.2.63) by defining

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{q,p} = \frac{\Gamma(q^{\alpha+1};q,p)\Gamma(q;q,p)}{\Gamma(q^{\beta+1};q,p)\Gamma(q^{\alpha-\beta+1};q,p)} \tag{11.2.71}$$

for $|q|$, $|p| < 1$ and complex α and β. The Felder and Varchenko [2003a] elliptic analogue of the Gauss multiplication formula (1.10.10) and of its q-analogue (1.10.11) is considered in Ex. 11.13.

11.3 Additive notations and modular series

In what follows it will be convenient to employ the standard notation $\mathbb{C}$ for the set of all complex numbers, and $\mathbb{Z}$ for the set of all integers. In terms of the elliptic number $[a;\sigma,\tau]$ defined in (1.6.14) the (*additive*) *elliptic shifted factorials* (or *σ,τ-shifted factorials*) are defined by

$$[a;\sigma,\tau]_n = \begin{cases} \prod_{k=0}^{n-1}[a+k;\sigma,\tau], & n=1,2,\ldots,\\ 1, & n=0,\\ 1/\prod_{k=0}^{-n-1}[a+n+k;\sigma,\tau], & n=-1,-2,\ldots, \end{cases} \tag{11.3.1}$$

where a, σ, $\tau \in \mathbb{C}$ and the modular parameters σ, τ are such that that $\mathrm{Im}(\tau) > 0$ and $\sigma \notin \mathbb{Z}+\tau\mathbb{Z}$ with $\mathbb{Z}+\tau\mathbb{Z} = \{m+\tau n : m,n\in\mathbb{Z}\}$. Let $[n;\sigma,\tau]! = [1;\sigma,\tau]_n$ and

$$[a_1,a_2,\ldots,a_m;\sigma,\tau]_n = \prod_{k=1}^{m}[a_k;\sigma,\tau]_n. \tag{11.3.2}$$

Then we can formally define the additive forms of the ${}_{r+1}E_r$ and ${}_{r+1}V_r$ series to be the series

$$\begin{aligned}&{}_{r+1}e_r(a_1,a_2,\ldots,a_{r+1};b_1,\ldots,b_r;\sigma,\tau;z)\\ &\quad= \sum_{n=0}^{\infty}\frac{[a_1,a_2,\ldots,a_{r+1};\sigma,\tau]_n}{[1,b_1,\ldots,b_r;\sigma,\tau]_n}z^n,\end{aligned} \tag{11.3.3}$$

and

$$\begin{aligned}&{}_{r+1}v_r(a_1;a_6,a_7,\ldots,a_{r+1};\sigma,\tau;z)\\ &= \sum_{n=0}^{\infty}\frac{[a_1+2n;\sigma,\tau]}{[a_1;\sigma,\tau]}\frac{[a_1,a_6,a_7,\ldots,a_{r+1};\sigma,\tau]_n}{[1,1+a_1-a_6,1+a_1-a_7,\ldots,1+a_1-a_{r+1};\sigma,\tau]_n}z^n,\end{aligned} \tag{11.3.4}$$

respectively, where, as usual, it is assumed that the parameters are such that each term in these series is well-defined. When the argument z in the ${}_{r+1}v_r$

series equals 1 we suppress it and denote the series in (11.3.4) by the simpler notation ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)$. Since, by (11.2.3) and (11.3.1),

$$\frac{[a;\sigma,\tau]_n}{[b;\sigma,\tau]_n} = \frac{(q^a;q,p)_n}{(q^b;q,p)_n} q^{n(b-a)/2} \tag{11.3.5}$$

and

$$\frac{[a+2n;\sigma,\tau]}{[a;\sigma,\tau]} = \frac{\theta(q^{a+2n};p)}{\theta(q^a;p)} q^{-n}, \tag{11.3.6}$$

we have that

$$\begin{aligned} &{}_{r+1}e_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;\sigma,\tau;z) \\ &= {}_{r+1}E_r\left(q^{a_1},\ldots,q^{a_{r+1}};q^{b_1},\ldots,q^{b_r};q,p;zq^{(1+b_1+\cdots+b_r-(a_1+\cdots+a_{r+1}))/2}\right) \end{aligned} \tag{11.3.7}$$

and

$$\begin{aligned} &{}_{r+1}v_r(a_1;a_6,a_7,\ldots,a_{r+1};\sigma,\tau;z) \\ &= {}_{r+1}V_r\left(q^{a_1};q^{a_6},q^{a_7},\ldots,q^{a_{r+1}};q,p;zq^{(a_1+1)(r-5)/2-(1+a_6+a_7+\cdots+a_{r+1})}\right). \end{aligned} \tag{11.3.8}$$

From (11.2.4) and the multiplicative form of the elliptic balancing condition in (11.2.14), it follows that ${}_{r+1}e_r$ is an elliptic hypergeometric series when

$$\sigma[a_1+a_2+\cdots+a_{r+1}-(1+b_1+b_2+\cdots+b_r)] \in \mathbb{Z}\,, \tag{11.3.9}$$

which is the additive form of the elliptic balancing condition. Hence, ${}_{r+1}e_r$ is called *E-balanced* when (11.3.9) holds, and it follows that

$$\begin{aligned} &{}_{r+1}e_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;\sigma,\tau;z) \\ &= {}_{r+1}E_r\left(q^{a_1},\ldots,q^{a_{r+1}};q^{b_1},\ldots,q^{b_r};q,p;\pm z\right) \end{aligned} \tag{11.3.10}$$

holds with the plus sign of the $\pm$ when $\sigma[a_1+a_2+\cdots+a_{r+1}-(1+b_1+b_2+\cdots+b_r)]$ is an even integer and the minus sign when it is an odd integer. Using (11.2.4) we find that the additive form of the multiplicative well-poised condition (11.2.15) is

$$\sigma[1+a_1-a_{k+1}-b_k] \in \mathbb{Z}\,, \quad k=1,\ldots,r, \tag{11.3.11}$$

in which case ${}_{r+1}e_r$ is called *well-poised*, and the elliptic balancing condition (11.3.9) reduces to

$$2\sigma\left[a_1+a_2+\cdots+a_{r+1}-\frac{r+1}{2}(a_1+1)\right] \in \mathbb{Z}\,. \tag{11.3.12}$$

Note that (11.3.9) and (11.3.11) are not equivalent to the additive forms of the "balanced" and "well-poised" constraints in Spiridonov [2002a, (20) and (26)], respectively, which correspond to requiring that the expressions inside the square brackets in (11.3.9) and (11.3.11) are equal to zero.

If ${}_{r+1}e_r$ is a well-poised series that can be written in the very-well-poised form (11.3.4), e.g., when, via (11.2.17) and (11.3.6),

$$a_2 = 1 + \frac{a_1}{2},\; a_3 = 1 + \frac{a_1}{2} - \frac{1}{2\sigma},\; a_4 = 1 + \frac{a_1}{2} - \frac{\tau}{2\sigma},\; a_5 = 1 + \frac{a_1}{2} + \frac{1+\tau}{2\sigma}, \tag{11.3.13}$$

then the elliptic balancing condition reduces to

$$2\sigma \left[1 + a_6 + a_7 + \cdots + a_{r+1} - \frac{r-5}{2}(a_1+1)\right] \in \mathbb{Z}\,, \tag{11.3.14}$$

which is the condition for the ${}_{r+1}v_r$ series in (11.3.4) to be *elliptically balanced* (*E-balanced*) and hence an elliptic hypergeometric series. Since $e^{2\pi i(1\mp1)/4} = \pm1$, it follows by setting $z = q^w$ that the additive form of the very-well-poised balancing condition in (11.2.29) is

$$\sigma \left[w + 1 + a_6 + a_7 + \cdots + a_{r+1} - \frac{r-5}{2}(a_1+1)\right] + \frac{1\mp1}{4}(r+1) \in \mathbb{Z}\,, \tag{11.3.15}$$

which, via (11.3.8), is the condition for the series

$${}_{r+1}v_r\left(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau; (\pm1)^{r+1}e^{2\pi i\sigma w}\right) \tag{11.3.16}$$

to be *very-well-poised-balanced* (*VWP-balanced*), where, as elsewhere, either both of the upper signs or both of the lower signs in ∓1 and ±1 are used simultaneously. In particular, setting $w = 0$, it follows that if r is odd, then ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)$ is VWP-balanced when

$$\sigma \left[1 + a_6 + a_7 + \cdots + a_{r+1} - \frac{r-5}{2}(a_1+1)\right] \in \mathbb{Z}\,, \tag{11.3.17}$$

and if r is even, then ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau; \pm1)$ is VWP-balanced when

$$\sigma \left[1 + a_6 + a_7 + \cdots + a_{r+1} - \frac{r-5}{2}(a_1+1)\right] + \frac{1\mp1}{4} \in \mathbb{Z}\,. \tag{11.3.18}$$

Thus, by replacing the parameters a, b, c, d, e in (11.2.25) by q^a, q^b, q^c, q^d, q^e, respectively, and applying (11.3.8), we obtain the additive form of the Frenkel and Turaev summation formula:

$$\begin{aligned} &{}_{10}v_9(a; b, c, d, e, -n; \sigma, \tau) \\ &\quad = \frac{[a+1, a+1-b-c, a+1-b-d, a+1-c-d; \sigma, \tau]_n}{[a+1-b, a+1-c, a+1-d, a+1-b-c-d; \sigma, \tau]_n} \end{aligned} \tag{11.3.19}$$

for $n = 0, 1, \ldots,$ provided that the series is VWP-balanced, i.e.,

$$\sigma[b + c + d + e - n - 2a - 1] \in \mathbb{Z}\,. \tag{11.3.20}$$

Similarly, it follows from (11.2.27), (11.2.28), and (11.3.8) that

$$\begin{aligned} &{}_{9}v_8(a; b, c, d, -n; \sigma, \tau; \pm1) \\ &\quad = \frac{[a+1, a+1-b-c, a+1-b-d, a+1-c-d; \sigma, \tau]_n}{[a+1-b, a+1-c, a+1-d, a+1-b-c-d; \sigma, \tau]_n} \end{aligned} \tag{11.3.21}$$

for $n = 0, 1, \ldots$, provided that the series is VWP-balanced, i.e.,

$$\sigma\left[b + c + d - n - \frac{3}{2}a - \frac{1}{2}\right] + \frac{1 \mp 1}{4} \in \mathbb{Z}. \tag{11.3.22}$$

This formula also follows by setting $e = (a+1)/2 + k/2\sigma$ for $k = 0, 1$ in (11.3.19) and using the fact that $[(a+1)/2+k/2\sigma; \sigma, \tau]_n = (-1)^k[(a+1)/2-k/2\sigma; \sigma, \tau]_n$. In addition, the Frenkel and Turaev transformation formula (11.2.23) has the additive form

$$\begin{aligned}
&{}_{12}v_{11}(a; b, c, d, e, f, \lambda + a + n + 1 - e - f, -n; \sigma, \tau) \\
&= \frac{[a + 1, a + 1 - e - f, \lambda + 1 - e, \lambda + 1 - f; \sigma, \tau]_n}{[a + 1 - e, a + 1 - f, \lambda + 1 - e - f, \lambda + 1; \sigma, \tau]_n} \\
&\times {}_{12}v_{11}(\lambda; \lambda + b - a, \lambda + c - a, \lambda + d - a, e, f, \lambda + a + n + 1 - e - f, -n; \sigma, \tau)
\end{aligned} \tag{11.3.23}$$

for $n = 0, 1, \ldots$, provided that the series is VWP-balanced, i.e.,

$$\sigma[b + c + d + \lambda - 2a - 1] \in \mathbb{Z}. \tag{11.3.24}$$

The summation formula (11.3.19) follows from (11.3.23) by setting $\lambda = a - d$.

Frenkel and Turaev [1997] used a slightly different notation for the $z = 1$ case of the series in (11.3.4) (with ${}_{r-1}\omega_{r-2}$ denoting an ${}_{r+1}v_r$ series when $z = 1$) and assumed that the series terminates (to avoid the problem of convergence) and satisfies the balancing condition

$$1 + (a_6 + a_7 + \cdots + a_{r+1}) = \frac{r-5}{2}(a_1 + 1). \tag{11.3.25}$$

They called the function that is the sum of their terminating series a *modular hypergeometric function* and justified the name "modular" by showing that if the series ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)$ terminates and (11.3.25) holds, then these functions are invariant under the natural action of $SL(2, \mathbb{Z})$ on the σ and τ variables, i.e., if $a_1, a_6, a_7, \ldots, a_{r+1} \in \mathbb{C}$, then

$$\begin{aligned}
&{}_{r+1}v_r\left(a_1; a_6, a_7, \ldots, a_{r+1}; \frac{\sigma}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) \\
&= {}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)
\end{aligned} \tag{11.3.26}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, where $SL(2, \mathbb{Z})$ is the modular group of 2×2 matrices such that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Therefore, we call (11.3.25) the *modular balancing condition*, and when it holds we say that the series ${}_{r+1}v_r$ is *M-balanced.* Note that if the series ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)$ is M-balanced, then it is E-balanced and VWP-balanced, but if it is E-balanced or VWP-balanced then it is not necessarily M-balanced. Frenkel and Turaev also showed that if $a_1, a_6, a_7, \ldots, a_{r+1} \in \mathbb{Z}$, (11.3.25) holds, and the series ${}_{r+1}v_r(a_1; a_6, a_7, \ldots, a_{r+1}; \sigma, \tau)$ terminates, then the sum of the series is invariant under the natural action of $\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}$ on the variable σ, i.e.,

$$\begin{aligned}
&{}_{r+1}v_r(a_1; a_6, \ldots, a_{r+1}; \sigma + m + n\tau, \tau) \\
&= {}_{r+1}v_r(a_1; a_6, \ldots, a_{r+1}; \sigma, \tau)
\end{aligned} \tag{11.3.27}$$

for all $(m, n) \in \mathbb{Z}^2$, and hence is elliptic in σ with periods 1 and τ.

A formal unilateral or bilateral series of the form $\sum_n c_n(\sigma, \tau)$ is called a *modular series* (or *modular invariant*) if it is invariant under the natural action of $SL(2, \mathbb{Z})$ on the σ and τ variables, i.e.,

$$\sum_n c_n\left(\frac{\sigma}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \sum_n c_n(\sigma, \tau) \tag{11.3.28}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, where, as usual, two formal series are considered to be equal if their corresponding nth terms are equal. Spiridonov [2002a] extended (11.3.26) by proving that the *unilateral theta hypergeometric series*

$$\begin{aligned} &{}_re_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma, \tau; z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, \ldots, a_r; \sigma, \tau]_n}{[1, b_1, \ldots, b_s; \sigma, \tau]_n} z^n \end{aligned} \tag{11.3.29}$$

is modular if

$$r = s + 1, \tag{11.3.30a}$$

$$a_1 + a_2 + \cdots + a_{s+1} = 1 + b_1 + b_2 + \cdots + b_s, \tag{11.3.30b}$$

and

$$a_1^2 + a_2^2 + \cdots + a_{s+1}^2 = 1 + b_1^2 + b_2^2 + \cdots + b_s^2, \tag{11.3.30c}$$

and, more generally, that the *bilateral theta hypergeometric series*

$$\begin{aligned} &{}_rg_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma, \tau; z) \\ &= \sum_{n=-\infty}^{\infty} \frac{[a_1, \ldots, a_r; \sigma, \tau]_n}{[b_1, \ldots, b_s; \sigma, \tau]_n} z^n \end{aligned} \tag{11.3.31}$$

is modular if

$$r = s, \tag{11.3.32a}$$

$$a_1 + a_2 + \cdots + a_r = b_1 + b_2 + \cdots + b_r, \tag{11.3.32b}$$

and

$$a_1^2 + a_2^2 + \cdots + a_r^2 = b_1^2 + b_2^2 + \cdots + b_r^2, \tag{11.3.32c}$$

where it is assumed that z and the a's and b's are complex numbers such that the series are well-defined. Hence, ${}_re_s$ and ${}_rg_s$ will be called *M-balanced* when (11.3.30a)–(11.3.30c) and (11.3.32a)–(11.3.32c), respectively, hold. Notice that if ${}_rg_r$ is well-poised with

$$a_k + b_k = a + 1, \qquad k = 1, \ldots, r, \tag{11.3.33}$$

then

$$\sum_{k=1}^{r}(a_k^2 - b_k^2) = (a+1)\sum_{k=1}^{r}(a_k - b_k) = 2(a+1)\left(\sum_{k=1}^{r} a_k - \frac{r}{2}(a+1)\right), \tag{11.3.34}$$

and so this well-poised ${}_rg_r$ series is modular when the *modular balancing condition*

$$a_1 + a_2 + \cdots + a_r = \frac{r}{2}(a+1) \tag{11.3.35}$$

holds. Similarly, a well-poised ${}_{r+1}e_r$ series with

$$a_{k+1} + b_k = a_1 + 1, \qquad k = 1, \ldots, r, \tag{11.3.36}$$

is modular when the *modular balancing condition*

$$a_1 + a_2 + \cdots + a_{r+1} = \frac{r+1}{2}(a_1+1) \tag{11.3.37}$$

holds, which reduces to (11.3.25) if (11.3.13) also holds.

In order to prove the Frenkel and Turaev, and the Spiridonov modularity results, (11.3.27), and some more general results, we start by observing that, by (1.6.9) and Whittaker and Watson [1965, §21.11 and §21.5], $\vartheta_1(x, e^{\pi i\tau})$ satisfies the *modular symmetry relations*

$$\vartheta_1(x, e^{\pi i(\tau+1)}) = e^{\pi i/4}\vartheta_1(x, e^{\pi i\tau}), \tag{11.3.38}$$

$$\vartheta_1(x/\tau, e^{-\pi i/\tau}) = -i(-i\tau)^{\frac{1}{2}}e^{ix^2/\pi\tau}\vartheta_1(x, e^{\pi i\tau}), \tag{11.3.39}$$

where the square root of $-i\tau$ is chosen so that its real part is positive, and the *quasi-periodicity relations*

$$\vartheta_1(x+\pi, e^{\pi i\tau}) = -\vartheta_1(x, e^{\pi i\tau}), \tag{11.3.40}$$

$$\vartheta_1(x+\pi\tau, e^{\pi i\tau}) = -e^{-\pi i\tau-2ix}\vartheta_1(x, e^{\pi i\tau}). \tag{11.3.41}$$

Hence, by (1.6.14), the elliptic number $[a;\sigma,\tau]$ has the *modular symmetries*

$$[a;\sigma,\tau+1] = [a;\sigma,\tau], \tag{11.3.42}$$

$$[a;\sigma/\tau,-1/\tau] = e^{\pi i(a^2-1)\sigma^2/\tau}[a;\sigma,\tau], \tag{11.3.43}$$

and the *quasi-periodicity relations*

$$[k;\sigma+1,\tau] = (-1)^{k+1}[k;\sigma,\tau], \tag{11.3.44}$$

$$[k;\sigma+\tau,\tau] = (-1)^{k+1}e^{\pi i(1-k^2)(2\sigma+\tau)}[k;\sigma,\tau], \tag{11.3.45}$$

where $k \in \mathbb{Z}$. See, e.g., van Diejen and Spiridonov [2000, 2001b], but note that (11.3.43) corrects a typo in both of these papers.

Let us first consider when the bilateral series ${}_rg_s$ is modular. Since the modular group $SL(2,\mathbb{Z})$ is generated by the two matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{11.3.46}$$

which, respectively, correspond to the two transformations

$$(\sigma,\tau) \to (\sigma,\tau+1), \qquad (\sigma,\tau) \to (\sigma/\tau,-1/\tau), \tag{11.3.47}$$

it suffices to consider when each term

$$c_n(\sigma,\tau) = \frac{[a_1,\ldots,a_r;\sigma,\tau]_n}{[b_1,\ldots,b_s;\sigma,\tau]_n}\, z^n \tag{11.3.48}$$

of the series in (11.3.31) is invariant under these transformations. From (11.3.1) and (11.3.42) it is clear that $[a;\sigma,\tau+1]_n = [a;\sigma,\tau]_n$ for $n\in\mathbb{Z}$, and hence

$$c_n(\sigma,\tau+1) = c_n(\sigma,\tau), \qquad n\in\mathbb{Z}. \tag{11.3.49}$$

For the second transformation in (11.3.47) observe that from (11.3.1) and (11.3.43) it follows that

$$[a;\sigma/\tau,-1/\tau]_n = e^{\pi i\nu_n(a)\sigma^2/\tau}[a;\sigma,\tau]_n, \qquad n\in\mathbb{Z}, \tag{11.3.50}$$

with $\nu_n(a) = n[a^2+a(n-1)+(n-1)(2n-1)/6-1]$, and thus

$$c_n(\sigma/\tau,-1/\tau) = e^{\pi i\lambda_n\sigma^2/\tau}c_n(\sigma,\tau), \qquad n\in\mathbb{Z}, \tag{11.3.51}$$

with

$$\begin{aligned}\lambda_n =& n[a_1^2+a_2^2+\cdots+a_r^2-(b_1^2+b_2^2+\cdots+b_s^2)]\\ &+n(n-1)[a_1+a_2+\cdots+a_r-(b_1+b_2+\cdots+b_s)]\\ &+n(r-s)[(n-1)(2n-1)/6-1].\end{aligned} \tag{11.3.52}$$

Consequently,

$$c_n(\sigma/\tau,-1/\tau) = c_n(\sigma,\tau) \tag{11.3.53}$$

if and only if

$$e^{\pi i\lambda_n\sigma^2/\tau} = 1 \tag{11.3.54}$$

whenever $c_n(\sigma,\tau)\neq 0$. Obviously, (11.3.54) holds for $n=0$ since we have $\lambda_0=0$, and in order for it to hold for any $n\neq 0$ with (σ,τ) replaced by $(\sigma/(c\tau+d),(a\tau+b)/(c\tau+d))$ for $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$, it is necessary and sufficient that $\lambda_n=0$. Setting $\lambda_1=0$ in (11.3.52) gives

$$a_1^2+a_2^2+\cdots+a_r^2 = b_1^2+b_2^2+\cdots+b_s^2+r-s, \tag{11.3.55}$$

which, with $a_1=-1, b_1=1$, is a necessary and sufficient condition for the series

$${}_rg_s(-1,a_2,\ldots,a_r;1,b_2,\ldots,b_s;\sigma,\tau;z) = {}_re_{s-1}(-1,a_2,\ldots,a_r;b_2,\ldots,b_s;\sigma,\tau;z)$$

to be modular when $a_k\neq 0$ for $k=2,\ldots,r$. Setting $\lambda_1=\lambda_2=0$ gives (11.3.55) and

$$a_1+a_2+\cdots+a_r = b_1+b_2+\cdots+b_s+(s-r)/2, \tag{11.3.56}$$

which, with $a_1=-2, b_1=1$, are necessary and sufficient for the series

$${}_rg_s(-2,a_2,\ldots,a_r;1,b_2,\ldots,b_s;\sigma,\tau;z) = {}_re_{s-1}(-2,a_2,\ldots,a_r;b_2,\ldots,b_s;\sigma,\tau;z)$$

to be modular when $a_k\neq 0,-1$ for $k=2,\ldots,r$. Similarly, if N and M are nonnegative integers, then

$$\begin{aligned}&{}_rg_s(-N,a_2,a_3,\ldots,a_r;M+1,b_2,b_3,\ldots,b_s;\sigma,\tau;z)\\ &=\sum_{n=-M}^{N}\frac{[-N,a_2,a_3,\ldots,a_r;\sigma,\tau]_n}{[M+1,b_2,b_3,\ldots,b_s;\sigma,\tau]_n}z^n\end{aligned} \tag{11.3.57}$$

is modular for arbitrary values of the a's and b's if and only if

$$\lambda_n = 0, \qquad -M \le n \le N, \tag{11.3.58}$$

which, since λ_n is a cubic polynomial in powers of n, is equivalent to (11.3.32a)–(11.3.32c) if and only if $N + M \ge 3$. In particular, when $N + M < 3$ the restriction $r = s$ is not needed for the series in (11.3.57) to be modular. It follows from the above observations that the ${}_rg_s$ series in (11.3.31) is modular for arbitrary values of its parameters if and only if (11.3.32a)–(11.3.32c) hold. Replacing s by $s+1$ in ${}_rg_s$ and then setting $b_{s+1} = 1$ yields that the ${}_re_s$ series in (11.3.29) is modular for arbitrary values of its parameters if and only if (11.3.30a)–(11.3.30c) hold.

It should be noted that, since the modularity of the above ${}_re_s$ and ${}_rg_s$ series hold termwise under the stated conditions, when we extend ${}_re_s$ and ${}_rg_s$ to

$$\begin{aligned} &{}_re_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma, \tau; \mathbf{A}, z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, \ldots, a_r; \sigma, \tau]_n}{[1, b_1, \ldots, b_s; \sigma, \tau]_n} A_n z^n, \end{aligned} \tag{11.3.59}$$

and

$$\begin{aligned} &{}_rg_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma, \tau; \mathbf{A}, z) \\ &= \sum_{n=-\infty}^{\infty} \frac{[a_1, \ldots, a_r; \sigma, \tau]_n}{[b_1, \ldots, b_s; \sigma, \tau]_n} A_n z^n, \end{aligned} \tag{11.3.60}$$

respectively, where $\mathbf{A} = \{A_n\}$ is an arbitrary sequence of complex numbers and $a_1, a_2, \ldots, a_{r+1}, z \in \mathbb{C}$, we obtain that the series in (11.3.59) is modular when (11.3.30a)–(11.3.30c) hold, and series in (11.3.60) is modular when (11.3.32a)–(11.3.32c) hold. This is also true when $\mathbf{A} = \{A_n(\sigma, \tau)\}$ is a sequence of complex numbers such that the series $\sum_n A_n(\sigma, \tau)$ is modular.

Rather than just proving (11.3.27), we will now consider when a bilateral series ${}_rg_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma, \tau)$ with $a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{Z}$ is invariant under the natural action of $\mathbb{Z}^2$ on the variable σ, i.e.,

$$\begin{aligned} &{}_rg_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \sigma + m + n\tau, \tau) \\ &= {}_rg_s(a_1, \ldots, a_r; b_s, \ldots, b_s; \sigma, \tau) \end{aligned} \tag{11.3.61}$$

for all $(m, n) \in \mathbb{Z}^2$, and hence is doubly periodic in σ with periods 1 and τ. Let $a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{Z}$. Using (11.3.1) and (11.3.44), we find that

$$c_n(\sigma + 1, \tau) = (-1)^{\gamma_n} c_n(\sigma, \tau) \qquad n \in \mathbb{Z}, \tag{11.3.62}$$

with

$$\gamma_n = n[a_1 + \cdots + a_r - (b_1 + \cdots + b_s) + (n+1)(r-s)/2] \tag{11.3.63}$$

and $c_n(\sigma, \tau)$ defined as in (11.3.48). So,

$$c_n(\sigma + 1, \tau) = c_n(\sigma, \tau) \tag{11.3.64}$$

if and only if

$$\gamma_n \in 2\mathbb{Z} \tag{11.3.65}$$

whenever $c_n(\sigma,\tau) \neq 0$, where $2\mathbb{Z} = \{2m\colon\ m \in \mathbb{Z}\}$ is the set of all even integers. Via (11.3.1) and (11.3.45),

$$[k;\sigma+\tau,\tau]_n = (-1)^{kn+\binom{n+1}{2}} e^{-\pi i\mu_n(k)(2\sigma+\tau)}[k;\sigma,\tau]_n, \qquad k,n \in \mathbb{Z}, \quad (11.3.66)$$

with $\mu_n(k) = -n[k^2 + k(n-1) + (n-1)(2n-1)/6 - 1]$, and hence

$$c_n(\sigma+\tau,\tau) = (-1)^{\gamma_n} e^{-\pi i\lambda_n(2\sigma+\tau)} c_n(\sigma,\tau) \quad (11.3.67)$$

for $n \in \mathbb{Z}$ with λ_n as defined in (11.3.52). Thus, if (11.3.65) holds, then

$$c_n(\sigma+\tau,\tau) = c_n(\sigma,\tau) \quad (11.3.68)$$

if and only if

$$e^{\pi i\lambda_n(2\sigma+\tau)} = 1 \quad (11.3.69)$$

when $c_n(\sigma,\tau) \neq 0$. Clearly, (11.3.69) holds for $n=0$ since $\lambda_0 = 0$, and in order for it to hold for any $n \neq 0$ with σ replaced by $\sigma + m + m\tau$ for $(m,n) \in \mathbb{Z}^2$, it is necessary and sufficient that $\lambda_n = 0$.

As in the modular case, it follows that if N and M are nonnegative integers, then

$$\begin{aligned} &{}_rg_s(-N, a_2, a_3, \ldots, a_r; M+1, b_2, b_3, \ldots, b_s; \sigma, \tau; z) \\ &= \sum_{n=-M}^{N} \frac{[-N, a_2, a_3, \ldots, a_r; \sigma, \tau]_n}{[M+1, b_2, b_3, \ldots, b_s; \sigma, \tau]_n} z^n \end{aligned} \quad (11.3.70)$$

is invariant under the natural action of $\mathbb{Z}^2$ on the variable σ for arbitrary values of the a's and b's if and only if

$$\lambda_n = 0, \quad \gamma_n \in 2\mathbb{Z}, \qquad -M \leq n \leq N, \quad (11.3.71)$$

which is equivalent to (11.3.32a)–(11.3.32c) when $N + M \geq 3$. Consequently, the ${}_rg_s$ series in (11.3.31) is invariant under the natural action of $\mathbb{Z}^2$ on the variable σ for arbitrary values of its parameters if and only if (11.3.32a)–(11.3.32c) hold. Replacing s by $s+1$ in ${}_rg_s$ and then setting $b_{s+1} = 1$ yields that the ${}_re_s$ series in (11.3.29) is invariant under the natural action of $\mathbb{Z}^2$ on the variable σ for arbitrary values of its parameters if and only if (11.3.30a)–(11.3.30c) hold. In addition, since the invariance under the natural action of $\mathbb{Z}^2$ on the variable σ in the above ${}_re_s$ and ${}_rg_s$ series hold termwise under the stated conditions, it follows that the series in (11.3.59) and (11.3.60) are invariant under the natural action of $\mathbb{Z}^2$ on the variable σ when (11.3.30a)–(11.3.30c) and (11.3.32a)–(11.3.32c), respectively, hold.

Spiridonov [2002a] called a (unilateral or bilateral) theta hypergeometric series $\sum c_n$ *totally elliptic* if $g(n) = c_{n+1}/c_n$ is an elliptic function of all *free parameters* entering it (except for z) with equal periods of double periodicity. He proved that the most general (in the sense of a maximal number of independent free parameters) totally elliptic theta hypergeometric series coincide with the terminating well-poised M-balanced ${}_re_{r-1}$ (in the unilateral case) and ${}_rg_r$ (in the bilateral case) for $r > 2$.

11.4 Elliptic analogue of Jackson's ${}_8\phi_7$ summation formula

Recall from (11.2.25) that the multiplicative form of the Frenkel and Turaev summation formula (11.3.19) is

$$ {}_{10}V_9(a;b,c,d,e,q^{-n};q,p) = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n} \tag{11.4.1} $$

with $a^2q^{n+1} = bcde$ and $|p| < 1$. When $p = 0$ it clearly reduces to Jackson's summation formula (2.6.2). We will give a simple inductive proof of (11.4.1), which is a modification of those found independently by Rosengren, Spiridonov, and Warnaar. When $n = 0$ both sides of (11.4.1) are equal to 1, and for $n = 1$ it becomes

$$ \begin{aligned} &1 + \frac{\theta(aq^2;p)(a,b,c,d,a^2q^2/bcd,q^{-1};q,p)_1}{\theta(a;p)(q,aq/b,aq/c,aq/d,bcd/aq,aq^2;q,p)_1} q \\ &= \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_1}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_1}. \end{aligned} \tag{11.4.2} $$

By using (11.2.5) and (11.2.42), (11.4.2) can be rewritten as

$$ \begin{aligned} &\theta(aq/b,aq/c,aq/d,aq/bcd;p) + \theta(b,c,d,a^2q^2/bcd;p)aq/bcd \\ &= \theta(aq,aq/bc,aq/bd,aq/cd;p). \end{aligned} \tag{11.4.3} $$

However, (11.4.3) is equivalent to the identity in Ex. 2.16(i) if we replace x, λ, μ and ν by $aq/(bc)^{\frac12}$, $(bc)^{\frac12}$, $aq/d(bc)^{\frac12}$ and $(c/b)^{\frac12}$, respectively. So (11.4.1) is true for $n = 1$. Let us now assume that (11.4.1) is true when $n = m$. We need to prove that

$$ \begin{aligned} &\sum_{k=0}^{m+1} \frac{\theta(aq^{2k};p)(a,b,c,d,a^2q^{m+2}/bcd,q^{-m-1};q,p)_k}{\theta(a;p)(q,aq/b,aq/c,aq/d,bcdq^{-m-1}/a,aq^{m+2};q,p)_k} q^k \\ &= \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_{m+1}}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_{m+1}}. \end{aligned} \tag{11.4.4} $$

Observe that, by (11.2.43),

$$ \begin{aligned} &\frac{(a^2q^{m+2}/bcd,q^{-m-1};q,p)_k}{(bcdq^{-m-1}/a,aq^{m+2};q,p)_k} \\ &= \frac{(a^2q^{m+1}/bcd,q^{-m};q,p)_k}{(bcdq^{-m}/a,aq^{m+1};q,p)_k} \\ &\times \left[\frac{\theta(a^2q^{m+k+1}/bcd,bcdq^{k-m-1}/a,aq^{m+1},q^{-m-1};p)}{\theta(aq^{m+k+1},q^{k-m-1},a^2q^{m+1}/bcd,bcdq^{-m-1}/a;p)}\right]. \end{aligned} \tag{11.4.5} $$

However, by Ex. 2.16(i) the ratio of the θ-terms inside the brackets in (11.4.5) is equal to

$$ 1 + \frac{bcdq^{-m-1}}{a} \frac{\theta(aq^k,q^k,a/bcd,a^2q^{2m+2}/bcd;p)}{\theta(aq^{m+k+1},q^{k-m-1},a^2q^{m+1}/bcd,bcdq^{-m-1}/a;p)}. \tag{11.4.6} $$

When we multiply (11.4.6) by

$$ \frac{\theta(aq^{2k};p)(a,b,c,d,a^2q^{m+1}/bcd,q^{-m};q,p)_k}{\theta(a;p)(q,aq/b,aq/c,aq/d,bcdq^{-m}/a,aq^{m+1};q,p)_k} q^k $$

and sum over k from 0 to $m+1$ we find that the first term in (11.4.6) contributes a zero term at $k=m+1$ because $(q^{-m};q,p)_{m+1}=0$, while the second term contributes a zero term at $k=0$ because of the factor $\theta(q^k;p)$. After some simplifications it follows that

$$\begin{aligned}
&{}_{10}V_9(a;b,c,d,a^2q^{m+2}/bcd,q^{-m-1};q,p)\\
&= {}_{10}V_9(a;b,c,d,a^2q^{m+1}/bcd,q^{-m};q,p)\\
&\quad+\frac{bcdq^{-m}}{a}\frac{(aq;q,p)_2\theta(b,c,d,a/bcd,a^2q^{2m+2}/bcd;p)}{(aq^{m+1},bcdq^{-m-1}/a;q,p)_2\theta(aq/b,aq/c,aq/d;p)}\\
&\quad\times {}_{10}V_9(aq^2;bq,cq,dq,a^2q^{m+2}/bcd,q^{-m};q,p).
\end{aligned}\tag{11.4.7}$$

However, by the induction assumption both ${}_{10}V_9$ series on the right side of (11.4.7) can be summed. Thus, from (11.4.7) and the $n=m$ case of (11.4.1),

$$\begin{aligned}
&{}_{10}V_9(a;b,c,d,a^2q^{m+2}/bcd,q^{-m-1};q,p)\\
&=\frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_m}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_m}\\
&\quad+\frac{bcdq^{-m}}{a}\frac{(aq;q,p)_2\theta(b,c,d,a/bcd,a^2q^{2m+2}/bcd;p)}{(aq^{m+1},bcdq^{-m-1}/a;q,p)_2\theta(aq/b,aq/c,aq/d;p)}\\
&\quad\times\frac{(aq^3,aq/bc,aq/bd,aq/cd;q,p)_m}{(aq^2/b,aq^2/c,aq^2/d,a/bcd;q,p)_m}\\
&=\frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_m}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_m}\\
&\quad\times\left[1-\frac{\theta(b,c,d,a^2q^{2m+2}/bcd;p)}{\theta(aq^{m+1}/b,aq^{m+1}/c,aq^{m+1}/d,bcdq^{-m-1}/a;p)}\right],
\end{aligned}\tag{11.4.8}$$

by use of the identities (11.2.42) and (11.2.43). Applying Ex. 2.16(i) once again we find that

$$\begin{aligned}
&1-\frac{\theta(b,c,d,a^2q^{2m+2}/bcd;p)}{\theta(aq^{m+1}/b,aq^{m+1}/c,aq^{m+1}/d,bcdq^{-m-1}/a;p)}\\
&=\frac{\theta(aq^{m+1},aq^{m+1}/bc,aq^{m+1}/bd,aq^{m+1}/cd;p)}{\theta(aq^{m+1}/b,aq^{m+1}/c,aq^{m+1}/d,aq^{m+1}/bcd;p)}.
\end{aligned}\tag{11.4.9}$$

Substituting (11.4.9) into (11.4.8) and using (11.2.43) shows that (11.4.1) is true for $n=m+1$ whenever it is assumed to be true for $n=m$. This completes the proof of (11.4.1) by induction. Another proof of (11.4.1) by induction is indicated in Ex. 11.4.

Note that by letting $e\to aq^{n+1}$ in (11.4.1) we obtain the summation formula for a truncated ${}_8V_7$ series

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{\theta(aq^{2k};p)}{\theta(a;p)}\frac{(a,b,c,a/bc;q,p)_k}{(q,aq/b,aq/c,bcq;q,p)_k}\,q^k\\
&=\frac{(aq,bq,cq,aq/bc;q,p)_n}{(aq/b,aq/c,bcq,q;q,p)_n},
\end{aligned}\tag{11.4.10}$$

which is an elliptic analogue of Ex. 2.5.

As alluded to in §11.2 the limit of the summation formula (11.4.1) as one of three parameters b, c, d tends to infinity does not exist when $p \neq 0$, even though these limits do exist when $p = 0$ yielding formula (2.4.2). Similarly, one cannot take the limit $a \to 0$ after replacing d by aq/d in (11.4.1) to get an elliptic analogue of the q-Saalschütz formula (1.7.2), as was done in the basic hypergeometric case in §2.7. The root cause of the nonexistence of these limits is the fact that $\lim_{a\to 0} \theta(a;p)$ and $\lim_{a\to\infty} \theta(a;p)$ do not exist in the extended complex plane when $p \neq 0$. Nevertheless, one can obtain an elliptic analogue of the q-Saalschütz formula (1.7.2) at the ${}_{10}V_9$ level by employing Schlosser's observation (see Warnaar [2003e]) that by making the simultaneous substitutions $\{a, c, d, e, p\} \to \{dp, dqp/c, a, ep, p^2\}$ in (11.4.1) we obtain the summation formula

$$ {}_{10}V_9(dp; a, b, dqp/c, ep, q^{-n}; q, p^2) = \frac{(c/a, c/b, dqp, dqp/ab; q, p^2)_n}{(c, c/ab, dqp/a, dqp/b; q, p^2)_n} \qquad (11.4.11) $$

with $cdq^n = abe$, $|p| < 1$, and $n = 0, 1, \ldots$, which tends to (1.7.2) as $p \to 0$.

11.5 Elliptic analogue of Bailey's transformation formula for a terminating ${}_{10}\phi_9$ series

Recall from (11.2.23) that the multiplicative form of Frenkel and Turaev's transformation formula (11.3.23) is

$$ \begin{aligned} &{}_{12}V_{11}(a; b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p) \\ &= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q)_n} \\ &\quad \times {}_{12}V_{11}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p) \end{aligned} \qquad (11.5.1) $$

with $\lambda = qa^2/bcd$. This formula reduces to the Frenkel and Turaev summation formula (11.4.1) when $aq = bc, bd$, or cd, and it reduces to Bailey's transformation formula (2.9.1) when $p = 0$. It can be proved by proceeding as in (2.9.2) and (2.9.3) with the q-shifted factorials and the ${}_8\phi_7$ and ${}_{10}\phi_9$ series replaced by q, p-shifted factorials and ${}_{10}V_9$ and ${}_{12}V_{11}$ series, respectively. In view of the importance of this transformation formula and for the sake of completeness, we decided to present its proof here. First observe that by (11.4.1)

$$ \begin{aligned} &{}_{10}V_9(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, aq^m, q^{-m}; q, p) \\ &= \frac{(\lambda q, b, c, d; q, p)_m}{(a/\lambda, aq/b, aq/c, aq/d; q, p)_m}, \end{aligned} \qquad (11.5.2) $$

and hence the ${}_{12}V_{11}$ series on the left side of (11.5.1) equals

$$ \begin{aligned} &\sum_{m=0}^{n} \frac{\theta(aq^{2m}; p)}{\theta(a; p)} \frac{(a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p)_m}{(q, aq/e, aq/f, efq^{-n}/a, aq^{n+1}; q, p)_m} \frac{(a/\lambda; q, p)_m}{(\lambda q; q, p)_m} q^m \\ &\times \sum_{j=0}^{m} \frac{\theta(\lambda q^{2j}; p)}{\theta(\lambda; p)} \frac{(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, aq^m, q^{-m}; q, p)_j}{(q, aq/b, aq/c, aq/d, \lambda q^{1-m}/a, \lambda q^{m+1}; q, p)_j} q^j \end{aligned} $$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \sum_{j=0}^{m} \frac{\theta(aq^{2m};p)\theta(\lambda q^{2j};p)}{\theta(a;p)\theta(\lambda;p)} \frac{(a;q,p)_{m+j}(a/\lambda;q,p)_{m-j}}{(\lambda q;q,p)_{m+j}(q;q,p)_{m-j}} q^m \\
&\quad \times \frac{(e,f,\lambda aq^{n+1}/ef,q^{-n};q,p)_m(\lambda,\lambda b/a,\lambda c/a,\lambda d/a;q,p)_j}{(aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1};q,p)_m(q,aq/b,aq/c,aq/d;q,p)_j} \left(\frac{a}{\lambda}\right)^j \\
&= \sum_{j=0}^{n} \frac{\theta(\lambda q^{2j};p)}{\theta(\lambda;p)} \frac{(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{n+1}/ef,q^{-n};q,p)_j}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1};q,p)_j} \\
&\quad \times \frac{(aq;q,p)_{2j}}{(\lambda q;q,p)_{2j}} \left(\frac{aq}{\lambda}\right)^j {}_{10}V_9(aq^{2j};a/\lambda,eq^j,fq^j,\lambda aq^{n+j+1}/ef,q^{j-n};q,p).
\end{aligned}
\tag{11.5.3}
$$

However, by (11.4.1)

$$
\begin{aligned}
&{}_{10}V_9(aq^{2j};a/\lambda,eq^j,fq^j,\lambda aq^{n+j+1}/ef,q^{j-n};q,p) \\
&= \frac{(aq^{2j+1},\lambda q^{j+1}/e,\lambda q^{j+1}/f,aq/ef;q,p)_{n-j}}{(\lambda q^{2j+1},aq^{j+1}/e,aq^{j+1}/f,\lambda q/ef;q,p)_{n-j}} \\
&= \frac{(aq,\lambda q/e,\lambda q/f,aq/ef;q,p)_n}{(\lambda q,aq/e,aq/f,\lambda q/ef;q,p)_n} \\
&\quad \times \frac{(\lambda q;q)_{2j}(aq/e,aq/f,efq^{-n}/\lambda;q)_j}{(aq;q)_{2j}(\lambda q/e,\lambda q/f,efq^{-n}/a;q)_j} \left(\frac{\lambda}{a}\right)^j.
\end{aligned}
\tag{11.5.4}
$$

Substitution of (11.5.4) into (11.5.3) gives (11.5.1).

Note that by applying the transformation formula (11.5.1) to the ${}_{12}V_{11}$ series on the right side of (11.5.1), keeping e, $\lambda c/a$, $\lambda d/a$ and q^{-n} unchanged, we obtain that

$$
\begin{aligned}
&{}_{12}V_{11}(a;b,c,d,e,f,\lambda aq^{n+1}/ef,q^{-n};q,p) \\
&= \frac{(aq,aq/ce,aq/de,aq/ef,\lambda q/f,b;q,p)_n}{(aq/c,aq/d,aq/e,aq/f,\lambda q/ef,b/e;q,p)_n} \\
&\quad \times {}_{12}V_{11}(eq^{-n}/b;e,\lambda c/a,\lambda d/a,aq/bf,eq^{-n}/a,efq^{-n}/\lambda b,q^{-n};q,p),
\end{aligned}
\tag{11.5.5}
$$

which is the elliptic analogue of Ex. 2.19. Setting $f = qa^2/bcde$ in (11.5.1) gives the transformation formula for a truncated ${}_{10}V_9$ series

$$
\begin{aligned}
&\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)(a,b,c,d,e,qa^2/bcde;q,p)_k}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,bcde/a;q,p)_k} q^k \\
&= \frac{(aq,bcd/a,a^2q^2/bcde,eq;q,p)_n}{(aq/e,bcde/a,a^2q^2/bcd,q;q,p)_n} \\
&\quad \times {}_{12}V_{11}(qa^2/bcd;aq/bc,aq/bd,aq/cd,e,qa^2/bcde,aq^{n+1},q^{-n};q,p)
\end{aligned}
\tag{11.5.6}
$$

in which the ${}_{12}V_{11}$ can be transformed via (11.5.1) and (11.5.5) yielding other ${}_{12}V_{11}$ series representations for the above truncated series.

11.6 Multibasic summation and transformation formulas for theta hypergeometric series

First observe that by replacing a in (11.4.1) by a/q it follows that the $n = 1$ case of (11.4.1) is equivalent to the identity

$$1 - \frac{\theta(b, c, d, a^2/bcd; p)}{\theta(a/b, a/c, a/d, bcd/a; p)} = \frac{\theta(a, a/bc, a/bd, a/cd; p)}{\theta(a/bcd, a/d, a/c, a/b; p)}. \tag{11.6.1}$$

Next, corresponding to (3.6.12), define

$$\prod_{k=m}^{n} a_k = \begin{cases} a_m a_{m+1} \cdots a_n, & m \le n, \\ 1, & m = n+1, \\ (a_{n+1} a_{n+2} \cdots a_{m-1})^{-1}, & m \ge n+2, \end{cases} \tag{11.6.2}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$, and let

$$u_n = \prod_{k=0}^{n-1} \frac{\theta(b_k, c_k, d_k, a_k^2/b_k c_k d_k; p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, b_k c_k d_k/a_k; p)}, \tag{11.6.3}$$

for integer n, where it is assumed that the a's, b's, c's, d's are complex numbers such that u_n is well defined for $n = 0, \pm 1, \pm 2, \ldots$. Then, by using (11.6.1) with a, b, c, d replaced by a_k, b_k, c_k, d_k, respectively, we obtain the indefinite summation formula

$$\begin{aligned} u_{-m} - u_{n+1} &= \sum_{k=-m}^{n} (u_k - u_{k+1}) \\ &= \sum_{k=-m}^{n} \frac{\theta(a_k, a_k/b_k c_k, a_k/b_k d_k, a_k/c_k d_k; p)}{\theta(a_k/b_k c_k d_k, a_k/d_k, a_k/c_k, a_k/b_k; p)} u_k \end{aligned} \tag{11.6.4}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$.

Since $u_0 = 1$ by (11.6.2), setting $m = 0$ in (11.6.4) gives the summation formula

$$\begin{aligned} &\sum_{k=0}^{n} \frac{\theta(a_k, a_k/b_k c_k, a_k/b_k d_k, a_k/c_k d_k; p)}{\theta(a_k/b_k c_k d_k, a_k/d_k, a_k/c_k, a_k/b_k; p)} \\ &\quad \times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, a_j^2/b_j c_j d_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_j c_j d_j/a_j; p)} \\ &= 1 - \prod_{j=0}^{n} \frac{\theta(b_j, c_j, d_j, a_j^2/b_j c_j d_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_j c_j d_j/a_j; p)} \end{aligned} \tag{11.6.5}$$

for $n = 0, 1, \ldots$, which is equivalent to formula (3.2) in Warnaar [2002b]. When $p = 0$ this formula reduces to a summation formula of Macdonald that was first published in Bhatnagar and Milne [1997, Theorem 2.27], and it contains the summation formulas in W. Chu [1993, Theorems A, B, C] as special cases.

Notice that in (11.6.1), (11.6.3), (11.6.4) and (11.6.5) we have arranged the components of each quotient of products of theta functions so that the well-poised property of these quotients is clearly displayed; e.g., in (11.6.5) the quotients of the theta functions that depend on k are arranged so that each product of corresponding numerator and denominator parameters equals $a_k^2/b_kc_kd_k$, and each of the corresponding products that depend on j equals a_j.

If we set

$$a_k = ad(rst/q)^k,\ b_k = br^k,\ c_k = cs^k,\ d_k = ad^2t^k/bc,$$

then u_n reduces to

$$\tilde{u}_n = \frac{(a;rst/q^2,p)_n(b;r,p)_n(c;s,p)_n(ad^2/bc;t,p)_n}{(dq;q,p)_n(adst/bq;st/q,p)_n(adrt/cq;rt/q,p)_n(bcrs/dq;rs/q,p)_n}$$

and it follows from (11.6.4) by applying (11.2.42), (11.2.43) and (11.2.49) that we have the Gasper and Schlosser [2003] indefinite multibasic theta hypergeometric summation formula

$$\begin{aligned}&\sum_{k=-m}^{n} \frac{\theta(ad(rst/q)^k, br^k/dq^k, cs^k/dq^k, adt^k/bcq^k;p)}{\theta(ad,b/d,c/d,ad/bc;p)}\\
&\quad\times \frac{(a;rst/q^2,p)_k(b;r,p)_k(c;s,p)_k(ad^2/bc;t,p)_k}{(dq;q,p)_k(adst/bq;st/q,p)_k(adrt/cq;rt/q,p)_k(bcrs/dq;rs/q,p)_k}q^k\\
&= \frac{\theta(a,b,c,ad^2/bc;p)}{d\,\theta(ad,b/d,c/d,ad/bc;p)}\\
&\quad\times\left\{\frac{(arst/q^2;rst/q^2,p)_n(br;r,p)_n(cs;s,p)_n(ad^2t/bc;t,p)_n}{(dq;q,p)_n(adst/bq;st/q,p)_n(adrt/cq;rt/q,p)_n(bcrs/dq;rs/q,p)_n}\right.\\
&\quad\left.-\frac{(c/ad;rt/q,p)_{m+1}(d/bc;rs/q,p)_{m+1}(1/d;q,p)_{m+1}(b/ad;st/q,p)_{m+1}}{(1/c;s,p)_{m+1}(bc/ad^2;t,p)_{m+1}(1/a;rst/q^2,p)_{m+1}(1/b;r,p)_{m+1}}\right\}\end{aligned}\tag{11.6.6}$$

for $n,m = 0,\pm1,\pm2,\ldots$. Formula (3.6.13) follows from (11.6.6) by setting $p=0$ and then setting $r=p$ and $s=t=q$.

If $p=0$ and

$$\max(|q|,|r|,|s|,|t|,|rs/q|,|rt/q|,|st/q|,|rst/q^2|) < 1,$$

then we can let n or m in (11.6.6) tend to infinity to obtain that this special case of (11.6.6) also holds with n and/or m replaced by ∞, just as in the special case (3.6.14). Even though one cannot let $n\to\infty$ or $m\to\infty$ in (11.6.6) when $p\neq 0$ to obtain summation formulas for nonterminating theta hypergeometric series since $\lim_{a\to 0}\theta(a;p)$ does not exist when $p\neq 0$, it is possible to let $n\to\infty$ or $m\to\infty$ in (11.6.4) to obtain summation formulas for nonterminating series containing products of certain theta functions. For example, if we let

$$z_k = \frac{\theta(b_k,c_k,d_k,a_k^2/b_kc_kd_k;p)}{\theta(a_k/b_k,a_k/c_k,a_k/d_k,b_kc_kd_k/a_k;p)}$$

denote the kth factor in the product representation (11.6.3) for u_n and observe that when a is not an integer power of p

$$\lim_{b\to a^{\frac{1}{2}}} \frac{\theta(b;p)}{\theta(a/b;p)} = 1, \quad |p|<1,$$

then it is clear that there exist bilateral sequences of the a's, b's, c's, and d's in (11.6.4) such that Re $z_k > 0$ for integer k and the series

$$\sum_{k=-\infty}^{\infty} \log z_k \quad \text{converges,} \tag{11.6.7}$$

where $\log z_k$ is the principal branch of the logarithm (take, e.g., b_k, c_k, and d_k so close to $a_k^{\frac{1}{2}}$ that $|\log z_k| < 1/k^2$ for $k = \pm 1, \pm 2, \ldots$). Then $\lim_{n\to\infty} u_n$ and $\lim_{m\to\infty} u_{-m}$ exist, and we have the Gasper and Schlosser [2003] bilateral summation formula

$$\sum_{k=-\infty}^{\infty} \frac{\theta(a_k, a_k/b_kc_k, a_k/b_kd_k, a_k/c_kd_k; p)}{\theta(a_k/b_kc_kd_k, a_k/d_k, a_k/c_k, a_k/b_k; p)}$$
$$\times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, a_j^2/b_jc_jd_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_jc_jd_j/a_j; p)}$$
$$= \prod_{k=-\infty}^{-1} \frac{\theta\left(\frac{a_k}{b_k}, \frac{a_k}{c_k}, \frac{a_k}{d_k}, \frac{b_kc_kd_k}{a_k}; p\right)}{\theta\left(b_k, c_k, d_k, \frac{a_k^2}{b_kc_kd_k}; p\right)} - \prod_{k=0}^{\infty} \frac{\theta\left(b_k, c_k, d_k, \frac{a_k^2}{b_kc_kd_k}; p\right)}{\theta\left(\frac{a_k}{b_k}, \frac{a_k}{c_k}, \frac{a_k}{d_k}, \frac{b_kc_kd_k}{a_k}; p\right)} \tag{11.6.8}$$

provided that (11.6.7) holds. However, such bilateral sums do not appear to be particularly useful.

It is more useful to use the $m = 0$ case of (11.6.6) in the form

$$\sum_{k=0}^{n} \frac{\theta(ad(rst/q)^k, br^k/dq^k, cs^k/dq^k, adt^k/bcq^k; p)}{\theta(ad, b/d, c/d, ad/bc; p)}$$
$$\times \frac{(a; rst/q^2, p)_k (b; r, p)_k (c; s, p)_k (ad^2/bc; t, p)_k}{(dq; q, p)_k (adst/bq; st/q, p)_k (adrt/cq; rt/q, p)_k (bcrs/dq; rs/q, p)_k} q^k$$
$$= \frac{\theta(a, b, c, ad^2/bc; p)}{d\,\theta(ad, b/d, c/d, ad/bc; p)}$$
$$\times \frac{(arst/q^2; rst/q^2, p)_n (br; r, p)_n (cs; s, p)_n (ad^2t/bc; t, p)_n}{(dq; q, p)_n (adst/bq; st/q, p)_n (adrt/cq; rt/q, p)_n (bcrs/dq; rs/q, p)_n}$$
$$- \frac{\theta(d, ad/b, ad/c, bc/d; p)}{d\,\theta(ad, b/d, c/d, ad/bc; p)}, \tag{11.6.9}$$

its $c = s^{-n}$ special case

$$\sum_{k=0}^{n} \frac{\theta(ad(rst/q)^k, br^k/dq^k, s^{k-n}/dq^k, ads^nt^k/bq^k; p)}{\theta(ad, b/d, s^{-n}/d, ads^n/b; p)}$$
$$\times \frac{(a; rst/q^2, p)_k (b; r, p)_k (s^{-n}; s, p)_k (ad^2s^n/b; t, p)_k}{(dq; q, p)_k (adst/bq; st/q, p)_k (ads^nrt/q; rt/q, p)_k (brs^{1-n}/dq; rs/q, p)_k} q^k$$
$$= \frac{\theta(d, ad/b, ads^n, ds^n/b; p)}{\theta(ad, d/b, ds^n, ads^n/b; p)}, \tag{11.6.10}$$

and the $d \to 1$ limit case of (11.6.10)

$$\sum_{k=0}^{n} \frac{\theta(a(rst/q)^k, br^k/q^k, s^{k-n}/q^k, as^n t^k/bq^k; p)}{\theta(a, b, s^{-n}, as^n/b; p)} \times \frac{(a; rst/q^2, p)_k (b; r, p)_k (s^{-n}; s, p)_k (as^n/b; t, p)_k}{(q; q, p)_k (ast/bq; st/q, p)_k (as^n rt/q; rt/q, p)_k (brs^{1-n}/q; rs/q, p)_k} q^k = \delta_{n,0}, \tag{11.6.11}$$

where $n = 0, 1, \ldots$, which are generalizations of (3.6.15), (3.6.16), and (3.6.17), respectively. In particular, replacing n, a, b, and k in the $s = t = q$ case of (11.6.11) by $n - m$, $ar^m q^m$, $br^m q^{-m}$, and $j - m$, respectively, gives the orthogonality relation

$$\sum_{j=m}^{n} a_{nj} b_{jm} = \delta_{n,m} \tag{11.6.12}$$

with

$$a_{nj} = \frac{(-1)^{n+j} \theta(ar^j q^j, br^j q^{-j}; p)(arq^n, brq^{-n}; r, p)_{n-1}}{(q; q, p)_{n-j} (arq^n, brq^{-n}; r, p)_j (bq^{1-2n}/a; q, p)_{n-j}}, \tag{11.6.13}$$

$$b_{jm} = \frac{(ar^m q^m, br^m q^{-m}; r, p)_{j-m}}{(q, aq^{1+2m}/b; q, p)_{j-m}} \Big(-\frac{a}{b} q^{1+2m} \Big)^{j-m} q^{2\binom{j-m}{2}}, \tag{11.6.14}$$

which shows that the triangular matrix $A = (a_{nj})$ is inverse to the triangular matrix $B = (b_{jm})$, and gives a theta hypergeometric analogue of (3.6.18)–(3.6.20). Proceeding as in the derivation of (3.6.22), it follows that (3.6.22) extends to the bibasic theta hypergeometric summation formula

$$\theta(a/r, b/r; p) \sum_{k=0}^{n} \frac{(aq^k, bq^{-k}; r, p)_{n-1} \theta(aq^{2k}/b; p)}{(q; q, p)_k (q; q, p)_{n-k} (aq^k/b; q, p)_{n+1}} (-1)^k q^{\binom{k}{2}} = \delta_{n,0} \tag{11.6.15}$$

for $n = 0, 1, \ldots$, which reduces to

$$_8V_7(a/b; q/b, aq^{n-1}, q^{-n}; q, p) = \delta_{n,0}$$

when $r = q$.

The summation formula (11.6.10) and the argument in §3.8 can be employed to extend (3.8.14) and (3.8.15) to the quadratic theta hypergeometric transformation formulas

$$\sum_{k=0}^{n} \frac{\theta(acq^{3k}; p)}{\theta(ac; p)} \frac{(a, b, cq/b; q, p)_k (f, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, p)_k}{(cq^2, acq^2/b, abq; q^2, p)_k (acq/f, f/acq^{2n}, acq^{2n+1}; q, p)_k} q^k = \frac{(acq; q, p)_{2n} (ac^2q^2/bf, abq/f; q^2, p)_n}{(acq/f; q, p)_{2n} (abq, ac^2q^2/b; q^2, p)_n} \times {}_{12}V_{11}(ac^2/b; f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, p) \tag{11.6.16}$$

and

$$\sum_{k=0}^{2n} \frac{\theta(acq^{3k}; p)}{\theta(ac; p)} \frac{(d, f, a^2c^2q/df; q^2, p)_k (a, cq^{2n+1}, q^{-2n}; q, p)_k}{(acq/d, acq/f, df/ac; q, p)_k (cq^2, aq^{1-2n}, acq^{2n+2}; q^2, p)_k} q^k$$

$$= \frac{(acq, acq/df; q, p)_n (acq^{1-n}/d, acq^{1-n}/f; q^2, p)_n}{(acq/d, acq/f; q, p)_n (acq^{1-n}, acq^{1-n}/df; q^2, p)_n} \times {}_{12}V_{11}(acq^{-2n-1}; c, d, f, a^2c^2q/df, aq^{-2n-1}, q^{1-2n}, q^{-2n}; q^2, p) \tag{11.6.17}$$

for $n = 0, 1, \ldots$; see Warnaar [2002b, Theorems 4.2 and 4.7].

As in the derivation in Gasper [1989a] of the quadbasic transformation formula in Ex. 3.21, indefinite summation formulas such as in (11.6.5) and (11.6.9) can be extended to transformation formulas by using the identity

$$\sum_{k=0}^{n} \lambda_k \sum_{j=0}^{n-k} \Lambda_j = \sum_{k=0}^{n} \Lambda_k \sum_{j=0}^{n-k} \lambda_j, \tag{11.6.18}$$

which follows by a change in order of summation. In particular taking λ_k to be the kth term in the series in (11.6.5) and Λ_k to be this term with a_k, b_k, c_k, d_k, and p replaced by A_k, B_k, C_k, D_k, and P, respectively, yields the rather general transformation formula

$$\begin{aligned} &\sum_{k=0}^{n} \frac{\theta(a_k, a_k/b_kc_k, a_k/b_kd_k, a_k/c_kd_k; p)}{\theta(a_k/b_kc_kd_k, a_k/d_k, a_k/c_k, a_k/b_k; p)} \\ &\quad \times \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, a_j^2/b_jc_jd_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_jc_jd_j/a_j; p)} \\ &\quad \times \left\{1 - \prod_{j=0}^{n-k} \frac{\theta(B_j, C_j, D_j, A_j^2/B_jC_jD_j; P)}{\theta(A_j/B_j, A_j/C_j, A_j/D_j, B_jC_jD_j/A_j; P)}\right\} \\ &= \sum_{k=0}^{n} \frac{\theta(A_k, A_k/B_kC_k, A_k/B_kD_k, A_k/C_kD_k; P)}{\theta(A_k/B_kC_kD_k, A_k/D_k, A_k/C_k, A_k/B_k; P)} \\ &\quad \times \prod_{j=0}^{k-1} \frac{\theta(B_j, C_j, D_j, A_j^2/B_jC_jD_j; P)}{\theta(A_j/B_j, A_j/C_j, A_j/D_j, B_jC_jD_j/A_j; P)} \\ &\quad \times \left\{1 - \prod_{j=0}^{n-k} \frac{\theta(b_j, c_j, d_j, a_j^2/b_jc_jd_j; p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, b_jc_jd_j/a_j; p)}\right\}. \end{aligned} \tag{11.6.19}$$

The special case of (11.6.19) corresponding to using (11.6.9) instead of (11.6.5) contains the parameters a, b, c, d, A, B, C, D, and the bases r, s, t, R, S, T, the nomes p and P, and it contains the quadbasic transformation formula in Ex. 3.21 as a special case (see Ex. 11.26).

By proceeding as in Gasper and Schlosser [2003] we can use (11.6.11) to derive multibasic extensions of the Fields and Wimp, Verma, and Gasper expansion formulas in (3.7.1)–(3.7.3), (3.7.6)–(3.7.9), and multibasic theta hypergeometric extensions of (3.7.6)–(3.7.8). Set $a = \gamma(rst/q)^j$ and $b = \sigma(r/q)^j$ in (11.6.11). For $j, n = 0, 1, \ldots$, let $B_n(p)$ and $C_{j,n}$ be complex numbers such that $C_{j,0} = 1$ and the sequence $\{B_n(p)\}$ has finite support when $p \neq 0$. Then, as in (3.7.5), it follows that

$$
\begin{aligned}
&B_j(p)x^j \\
&= \sum_{k=0}^{n}\sum_{n=j}^{\infty} \frac{\theta(\gamma(rst/q)^n, \sigma(r/q)^n, \gamma\sigma^{-1}s^{n+k}t^j, \gamma\sigma^{-1}(st)^{n+k}, s^{-k}q^{j-n};p)}{\theta(s^{j-n-k};p)(q;q,p)_n(\gamma rts^{n+k}/q;rt/q,p)_n(\sigma rs^{1-n-k}/q;rs/q,p)_n} \\
&\quad\times (\gamma\sigma^{-1}(st)^{n+1}q^{j-n-1};st/q,p)_{k-1}(\gamma\sigma^{-1}s^{n+k}t^{j+1};t,p)_{n-j-1} \\
&\quad\times (\sigma rq^{-j};r,p)_{n-1}(\gamma rstq^{j-2};rst/q^2,p)_{n-1}(q^{-n};q,p)_j \\
&\quad\times (-1)^n B_{n+k}(p)C_{j,n+k-j}x^{n+k}q^{n(1+j-n-k)+\binom{n}{2}} \qquad (11.6.20)
\end{aligned}
$$

for $j = 0, 1, \ldots$. Now multiply both sides of (11.6.20) by $A_j w^j/(q;q,p)_j$ and sum from $n = 0$ to ∞ to obtain the Gasper and Schlosser multibasic expansion formula

$$
\begin{aligned}
&\sum_{n=0}^{\infty} A_n B_n(p)\frac{(xw)^n}{(q;q,p)_n} \\
&= \sum_{n=0}^{\infty} \frac{\theta(\gamma(rst/q)^n, \sigma(r/q)^n;p)}{(q;q,p)_n}(-x)^n q^{n+\binom{n}{2}} \\
&\quad\times \sum_{k=0}^{\infty} \frac{\theta(\gamma\sigma^{-1}(st)^{n+k};p)}{(q;q,p)_k(\gamma rts^{n+k}/q;rt/q,p)_n(\sigma rs^{1-n-k}/q;rs/q,p)_k} B_{n+k}(p)x^k \\
&\quad\times \sum_{j=0}^{n} \frac{\theta(s^{-k}q^{j-n};p)(\gamma rstq^{j-2};rst/q^2,p)_{n-1}(\sigma rq^{-j};r,p)_{n-1}}{\theta(s^{j-n-k};p)(q;q,p)_j} \\
&\quad\times \theta(\gamma\sigma^{-1}s^{n+k}t^j;p)(\gamma\sigma^{-1}(st)^{n+1}q^{j-n-1};st/q,p)_{k-1} \\
&\quad\times (\gamma\sigma^{-1}s^{n+k}t^{j+1};t,p)_{n-j-1}A_jC_{j,n+k-j}w^jq^{n(j-n-k)}, \qquad (11.6.21)
\end{aligned}
$$

which reduces to (3.7.6) by setting $p = 0$ and then letting $r = p$ and $s = t = q$.

Setting $r = s = t = q$ in (11.6.21) yields an expansion formula that is equivalent to the following extension of (3.7.7)

$$
\begin{aligned}
\sum_{n=0}^{\infty} A_n B_n(p)\frac{(xw)^n}{(q;q,p)_n} &= \sum_{n=0}^{\infty} \frac{(\sigma, \gamma q^{n+1}/\sigma, \alpha, \beta;q,p)_n}{(q, \gamma q^n;q,p)_n}\left(\frac{x}{\sigma}\right)^n \\
&\quad\times \sum_{k=0}^{\infty} \frac{\theta(\gamma q^{2n+2k}/\sigma;p)(\gamma q^n/\sigma, \sigma^{-1}, \alpha q^n, \beta q^n;q,p)_k}{\theta(\gamma q^{2n}/\sigma;p)(q, \gamma q^{2n+1};q,p)_k} B_{n+k}(p)x^k \\
&\quad\times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n;q,p)_j}{(q, \gamma q^{n+1}/\sigma, q^{1-n}/\sigma, \alpha, \beta;q,p)_j} A_j(wq)^j, \qquad (11.6.22)
\end{aligned}
$$

where, as above, $\{B_n(p)\}$ has finite support when $p \neq 0$. Of course, one cannot let $\sigma \to \infty$ in (11.6.22) to get an extension of (3.7.3) that holds for any $p \neq 0$.

Analogous to the q-extension of the Fields and Wimp expansion formula (3.7.1) displayed in (3.7.8), from (11.6.22) one easily obtains the rather general theta hypergeometric expansion formula

$$
\sum_{n=0}^{\infty} \frac{(a_R, c_T;q,p)_n}{(q, b_S, d_U;q,p)_n} A_n B_n(p)(xw)^n
$$

$$= \sum_{n=0}^{\infty} \frac{(c_T, e_K, \sigma, \gamma q^{n+1}/\sigma; q, p)_n}{(q, d_U, f_M, \gamma q^n; q, p)_n} \left(\frac{x}{\sigma}\right)^n$$
$$\times \sum_{k=0}^{\infty} \frac{\theta(\gamma q^{2n+2k}/\sigma; p)(\gamma q^{2n}/\sigma, \sigma^{-1}, c_T q^n, e_K q^n; q, p)_k}{\theta(\gamma q^{2n}/\sigma; p)(q, \gamma q^{2n+1}, d_U q^n, f_M q^n; q, p)_k} B_{n+k}(p) x^k$$
$$\times \sum_{j=0}^{n} \frac{(q^{-n}, \gamma q^n, a_R, f_M; q, p)_j}{(q, \gamma q^{n+1}/\sigma, q^{1-n}/\sigma, b_S, e_K; q, p)_j} A_j(wq)^j, \tag{11.6.23}$$

where we used a contracted notation analogous to that used in (3.7.1) and (3.7.8), and to avoid convergence problems it is assumed that $\{B_n\}$ has finite support when $p \neq 0$. Additional formulas are given in the exercises.

11.7 Rosengren's elliptic extension of Milne's fundamental theorem

Milne's [1985a] fundamental theorem states that

$$\sum_{\substack{k_1,\ldots,k_n \geq 0 \\ k_1+\cdots+k_n=N}} \frac{\Delta(\mathbf{z}q^{\mathbf{k}})}{\Delta(\mathbf{z})} \prod_{r,s=1}^{n} \frac{(a_r z_s/z_r; q)_{k_s}}{(qz_s/z_r; q)_{k_s}} = \frac{(a_1 \cdots a_n; q)_N}{(q; q)_N}, \tag{11.7.1}$$

where $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{z}q^{\mathbf{k}} = (z_1 q^{k_1}, \ldots, z_n q^{k_n})$,

$$\Delta(\mathbf{z}) = \prod_{1 \leq r < s \leq n} (z_r - z_s), \quad \Delta(\mathbf{z}q^{\mathbf{k}}) = \prod_{1 \leq r < s \leq n} (z_r q^{k_r} - z_s q^{k_s}), \tag{11.7.2}$$

and the a's and z's are fixed parameters. This is the identity that played a fundamental role in Milne's derivation of the Macdonald identities for the affine Lie algebra $A_n^{(1)}$, as well as his general approach to hypergeometric series on A_n or $U(n)$. An important tool for proving (11.7.1) is an easily verifiable identity

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{n} (b_j - a_k)}{a_k \prod_{j \neq k} (a_j - a_k)} = \frac{b_1 \cdots b_n}{a_1 \cdots a_n} - 1. \tag{11.7.3}$$

To derive an elliptic extension of Milne's identity, Rosengren [2003c] used the following elliptic extension of (11.7.3)

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{n} \theta(a_k/b_j; p)}{\prod_{j \neq k} \theta(a_k/a_j; p)} = 0, \tag{11.7.4}$$

where it is assumed that the balancing condition $a_1 \cdots a_n = b_1 \cdots b_n$ holds. Note that (11.7.4) is the same identity as in Ex. 5.23. Slater [1966] gave a proof of it by using special relationships between the parameters in the general transformation formula (5.4.3). Also see Tannery and Molk [1898] for a simple proof via residues. However, since Rosengren [2003c] gave a rather elegant yet elementary proof of (11.7.4) that is similar in spirit to the way Milne proved (11.7.3), we will present his proof.

First, it is easy to see that Ex. 2.16(i) is equivalent to the $n = 3$ case of (11.7.4). Assume that (11.7.4) is true for $n = m$. Then, by separating the $(m+1)$-th term from the series on the left side of (11.7.4), we can rewrite it in the form

$$\sum_{k=1}^{m-1} \frac{\theta(a_k/b_m;p)\prod_{j=1}^{m-1}\theta(a_k/b_j;p)}{\theta(a_k/a_m;p)\prod_{j\neq k}\theta(a_k/a_j;p)} = -\frac{\theta(a_m/b_m;p)\prod_{j=1}^{m-1}\theta(a_m/b_j;p)}{\prod_{j=1}^{m-1}\theta(a_m/a_j;p)}, \quad (11.7.5)$$

where the a's and b's are always assumed to satisfy the balancing condition. Considered as a function of a_m the expression on the left side of (11.7.5) resembles a partial fraction expansion of the product on the right side. When $n = m+1$, we write $a_{m+1} = t$, say, and seek an expansion of the form

$$\prod_{j=1}^{m} \frac{\theta(t/b_j;p)}{\theta(t/a_j;p)} = \sum_{k=1}^{m} C_k \frac{\theta(b_1\cdots b_m a_k/a_1\cdots a_m t;p)}{\theta(a_k/t;p)}. \quad (11.7.6)$$

That such an expansion exists follows by using induction on m and the fact that the $m = 2$ case is equivalent to (11.4.3). Multiplying both sides of (11.7.6) by $\theta(t/a_k;p) = -\theta(a_k/t;p)t/a_k$ and setting $t = a_k$, we find that

$$\begin{aligned} C_k &= -\frac{\prod_{j=1}^{m}\theta(a_k/b_j;p)}{\theta(b_1\cdots b_m/a_1\cdots a_m;p)\prod_{j\neq k}\theta(a_k/a_j;p)} \\ &= -\frac{\prod_{j=1}^{m}\theta(a_k/b_j;p)}{\theta(a_{m+1}/b_{m+1};p)\prod_{j\neq k}\theta(a_k/a_j;p)} \end{aligned} \quad (11.7.7)$$

with $a_1\cdots a_{m+1} = b_1\cdots b_{m+1}$. Now substitute (11.7.7) into (11.7.6) and set $t = a_{m+1}$ to get

$$\sum_{k=1}^{m} \frac{\theta(a_k/b_{m+1};p)\prod_{j=1}^{m}\theta(a_k/b_j;p)}{\theta(a_k/a_{m+1};p)\prod_{j\neq k}\theta(a_k/a_j;p)} = -\frac{\theta(a_{m+1}/b_{m+1};p)\prod_{j=1}^{m}\theta(a_{m+1}/b_j;p)}{\prod_{j=1}^{m}\theta(a_{m+1}/a_j;p)},$$

which is the same as (11.7.5) with m replaced by $m+1$. This completes the proof of (11.7.4).

An elliptic extension of (11.7.1) given in Rosengren [2003c, Theorem 5.1] states that

$$\begin{aligned} &\sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=N}} \frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)} \prod_{r=1}^{n} \frac{\prod_{s=1}^{n+1}(a_s z_r;q,p)_{k_r}}{(bz_r;q,p)_{k_r}\prod_{s=1}^{n}(qz_r/z_s;q,p)_{k_r}} \\ &= \frac{(b/a_1,\ldots,b/a_{n+1};q,p)_N}{(q,bz_1,\ldots,bz_n;q,p)_N}, \end{aligned} \quad (11.7.8)$$

where

$$\Delta(\mathbf{z};p) = \Delta_n(\mathbf{z};p) = \prod_{1\le r<s\le n} z_r\theta(z_s/z_r;p) \tag{11.7.9}$$

is the elliptic analogue of the Vandermonde determinant in (11.7.2) and the parameters satisfy the balancing condition

$$b = (a_1\cdots a_{n+1})(z_1\cdots z_n). \tag{11.7.10}$$

If we set $p=0$ and take the limits $b\to 0$, $a_{n+1}\to 0$ such that $b/a_{n+1}\to a_1\cdots a_n\, z_1\cdots z_n$, then it is easy to see that (11.7.8) approaches Milne's limit (11.7.1). When $p=0$, (11.7.8) reduces to Milne's [1988a, Theorem 6.17] A_n Jackson summation formula, which is a multivariable extension of Jackson's sum (2.6.2).

To prove (11.7.8) by induction, observe that when $N=1$ we have that $k_i=\delta_{ij}$ for some j, and thus j can be used as the summation index. Then, after a bit of manipulation,

$$\frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)\prod\limits_{r,s=1}^{n}(qz_r/z_s;q,p)_{k_r}} = \frac{1}{\theta(q;p)\prod\limits_{r\neq j}\theta(z_j/z_r;p)}$$

and formula (11.7.8) becomes

$$\sum_{j=1}^{n}\frac{\prod\limits_{s=1}^{n+1}\theta(a_sz_j;p)}{\theta(bz_j;p)\prod\limits_{s\neq j}\theta(z_j/z_s;p)} = \frac{\prod\limits_{s=1}^{n+1}\theta(b/a_s;p)}{\prod\limits_{s=1}^{n}\theta(bz_s;p)}, \tag{11.7.11}$$

which is the $m=n+1$ case of (11.7.5) with a different set of parameters. So (11.7.8) is true when $N=1$. Assume that it is true for a fixed N. We will show that it is also true for $N+1$. Denoting the right side of (11.7.8) by R_N, we have that

$$\begin{aligned} R_{N+1} &= \frac{(b/a_1,\ldots,b/a_{n+1};q,p)_{N+1}}{(q,bz_1,\ldots,bz_n;q,p)_{N+1}} \\ &= \frac{\theta(q;p)}{\theta(q^{N+1};p)}\frac{\theta(bq^N/a_1,\ldots,bq^N/a_{n+1};p)}{\theta(q,bq^Nz_1,\ldots,bq^Nz_n;p)}R_N. \end{aligned} \tag{11.7.12}$$

However, by the induction hypothesis we can replace R_N by the series on the left side of (11.7.8) and replace the ratio $\theta(bq^N/a_1,\ldots,bq^N/a_{n+1};p)$ $/\theta(q,bq^Nz_1,\ldots, bq^Nz_n;p)$ by the $N=1$ case of the series in (11.7.8) with z_m and b replaced by $z_mq^{k_m}$ and bq^N, respectively. Thus,

$$\begin{aligned} R_{N+1} = \frac{\theta(q;p)}{\theta(q^{N+1};p)} \sum_{\substack{k_1,\ldots,k_n\ge 0\\ k_1+\cdots+k_n=N}} &\Bigg[\Bigg\{\frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)}\prod_{r=1}^{n}\frac{\theta(bz_rq^{N+k_r};p)}{\theta(bz_rq^N;p)(bz_r;q,p)_{k_r}} \\ &\times\frac{\prod\limits_{s=1}^{n+1}(a_sz_r;q,p)_{k_r}}{\prod\limits_{s=1}^{n}(qz_r/z_s;q,p)_{k_r}}\Bigg\}\sum_{\substack{j_1,\ldots,j_n\ge 0\\ j_1+\cdots+j_n=1}}\frac{\Delta(\mathbf{z}q^{\mathbf{j}+\mathbf{k}};p)}{\Delta(\mathbf{z}q^{\mathbf{k}};p)} \end{aligned}$$

$$\times \prod_{r=1}^{n} \frac{\prod_{s=1}^{n+1} (a_s z_r q^{k_r}; q, p)_{j_r}}{(bz_r q^{N+k_r}; q, p)_{j_r} \prod_{s=1}^{n} (z_r q^{1+k_r-k_s}/z_s; q, p)_{j_r}} \Bigg]. \tag{11.7.13}$$

In the above summation, replace $\mathbf{k}$ by $\mathbf{k}-\mathbf{j}$. Since $j_r \in \{0,1\}$, we have

$$\frac{\theta(bz_r q^{N+k_r-j_r}; p)}{(bz_r q^{N+k_r-j_r}; q, p)_{j_r}} = \frac{\theta(bz_r q^{N+k_r}; p)}{(bz_r q^{N+k_r}; q, p)_{j_r}},$$

$$(bz_r; q, p)_{k_r-j_r} = \frac{(bz_r; q, p)_{k_r}}{(bz_r q^{k_r-1}; q, p)_{j_r}},$$

and

$$\prod_{r,s=1}^{n} (z_r q^{1+k_r-k_s-j_r+j_s}/z_s; q, p)_{j_r} = \theta(q; p) \prod_{r\neq s} (z_r q^{k_r-k_s}/z_s; q, p)_{j_r}.$$

Thus

$$R_{N+1} = \frac{1}{\theta(q^{N+1}; p)} \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=N+1}} \Bigg[\frac{\Delta(\mathbf{z}q^{\mathbf{k}}; p)}{\Delta(\mathbf{z}; p)}$$

$$\times \prod_{r=1}^{n} \frac{\theta(bz_r q^{N+k_r}; p) \prod_{s=1}^{n} (a_s z_r; q, p)_{k_r}}{\theta(bz_r q^N; p)(bz_r; q, p)_{k_r} \prod_{s=1}^{n} (z_r q/z_s; q, p)_{k_r}}$$

$$\times \sum_{\substack{j_1,\ldots,j_n\geq 0\\ j_1+\cdots+j_n=1}} \prod_{r=1}^{n} \frac{(bz_r q^{k_r-1}; q, p)_{j_r} \prod_{s=1}^{n} (z_r q^{k_r}/z_s; q, p)_{j_r}}{(bz_r q^{N+k_r}; q, p)_{j_r} \prod_{r\neq s} (z_r q^{k_r-k_s}/z_s; q, p)_{j_r}} \Bigg]. \tag{11.7.14}$$

The sum over $\mathbf{j}$ can be rewritten in the form

$$\sum_{m=1}^{n} \frac{\theta(bz_m q^{k_m-1}; p) \prod_{s=1}^{n} \theta(z_m q^{k_m}/z_s; p)}{\theta(bz_m q^{N+k_m}; p) \prod_{s\neq m} \theta(z_m q^{k_m-k_s}/z_s; p)}$$

$$= \frac{\theta(q^{N+1}; p) \prod_{r=1}^{n} \theta(bz_r q^N; p)}{\prod_{r=1}^{n} \theta(bz_r q^{N+k_r}; p)}, \tag{11.7.15}$$

by (11.7.11). From (11.7.14) and (11.7.15) it follows that

$$R_{N+1} = \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=N+1}} \frac{\Delta(\mathbf{z}q^{\mathbf{k}}; p)}{\Delta(\mathbf{z}; p)} \prod_{r=1}^{n} \frac{\prod_{s=1}^{n+1} (a_s z_r; q, p)_{k_r}}{(bz_r; q, p)_{k_r} \prod_{s=1}^{n} (qz_r/z_s; q, p)_{k_r}}, \tag{11.7.16}$$

which completes the proof of (11.7.8).

Formula (11.7.8) may be regarded as a multivariable extension of Frenkel and Turaev's summation formula (11.4.1) which, as we have seen before, is an elliptic analogue of Jackson's summation formula (2.6.2) for a terminating very-well-poised ${}_8\phi_7$ series. However, the multiple series on the left side of (11.7.8) does not look like well-poised at all even though it is perhaps easier to remember in this form because of the symmetries. To restore its well-poised character, following Milne [1987, pp. 237–238] and Rosengren [2003c], we apply the following procedure.

First we replace n by $n+1$, set $k_{n+1} = N - |\mathbf{k}|$, $|\mathbf{k}| = k_1 + \cdots + k_n$, $z_{n+1} = a^{-1}q^{-N}$, take $a_j \to b_j$, $j = 1, \ldots, n+2$, and replace b by aq/c. Then the balancing condition (11.7.10) takes the form

$$a^2 q^{N+1} = c(b_1 \cdots b_{n+2})(z_1 \cdots z_n), \tag{11.7.17}$$

which resembles the corresponding condition (2.6.1) for the one-dimensional Jackson sum. The transformations of the various terms of the series in (11.7.8) are straightforward, albeit somewhat tedious. First note that

$$\begin{aligned}
\frac{\Delta_{n+1}(\mathbf{z}_{n+1}q^{\mathbf{k}_{n+1}};p)}{\Delta_{n+1}(\mathbf{z}_{n+1};p)} &= \prod_{1\le r<s\le n+1} \frac{q^{k_r}\theta(z_s q^{k_s-k_r}/z_r;p)}{\theta(z_s/z_r;p)} \\
&= \prod_{r=1}^{n} \frac{q^{k_r}\theta(a^{-1}q^{-|\mathbf{k}|-k_r}/z_r;p)}{\theta(a^{-1}q^{-N}/z_r;p)} \prod_{1\le r<s\le n} \frac{q^{k_r}\theta(z_s q^{k_s-k_r}/z_r;p)}{\theta(z_s/z_r;p)} \\
&= q^{n(N-|\mathbf{k}|)} \frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)} \prod_{r=1}^{n} \frac{\theta(az_r q^{|\mathbf{k}|+k_r};p)}{\theta(az_r q^N;p)},
\end{aligned} \tag{11.7.18}$$

where $\mathbf{z}_{n+1} = (z_1, \ldots, z_{n+1})$ and $\mathbf{k}_{n+1} = (k_1, \ldots, k_{n+1})$. Now,

$$\begin{aligned}
&\prod_{r=1}^{n+1}\prod_{s=1}^{n+2}(a_s z_r;q,p)_{k_r} = \prod_{r=1}^{n}\prod_{s=1}^{n+2}(b_s z_r;q,p)_{k_r} \prod_{s=1}^{n+2}(a^{-1}b_s q^{-N};q,p)_{N-|\mathbf{k}|}, \\
&\prod_{r=1}^{n+1}(bz_r;q,p)_{k_r} = (q^{1-N}/c;q,p)_{N-|\mathbf{k}|}\prod_{r=1}^{n}(aqz_r/c;q,p)_{k_r}, \\
&\prod_{r,s=1}^{n+1}(qz_r/z_s;q,p)_{k_r} = (q;q,p)_{N-|\mathbf{k}|}\prod_{r=1}^{n}(az_r q^{N+1};q,p)_{k_r}\prod_{r,s=1}^{n}(qz_r/z_s;q,p)_{k_r}.
\end{aligned}$$

So,

$$\prod_{r=1}^{n+1} \frac{\prod_{s=1}^{n+2}(a_s z_r;q,p)_{k_r}}{(bz_r;q,p)_{k_r}\prod_{s=1}^{n+1}(qz_r/z_s;q,p)_{k_r}}$$

$$= \left[\prod_{r=1}^{n} \frac{\prod_{s=1}^{n+2} (b_s z_r; q, p)_{k_r}}{(az_r q^{N+1}, aqz_r/c; q, p)_{k_r} \prod_{r,s=1}^{n} (qz_r/z_s; q, p)_{k_r}} \right]$$
$$\times \left[\frac{\prod_{s=1}^{n+2} (b_s a^{-1} q^{-N}; q, p)_{N-|\mathbf{k}|}}{(q, q^{1-N}/c; q, p)_{N-|\mathbf{k}|} \prod_{s=1}^{n} (a^{-1} q^{1-N}/z_s; q, p)_{N-|\mathbf{k}|}} \right].$$

Using the identities (11.2.49) and (11.2.50), we can simplify the last expression above and obtain the following well-poised form of (11.7.8)

$$\sum_{\substack{k_1,\ldots,k_n \geq 0 \\ |\mathbf{k}| \leq N}} \left[\frac{\Delta(\mathbf{z}q^{\mathbf{k}}; p)}{\Delta(\mathbf{z}; p)} \prod_{r=1}^{n} \frac{\theta(az_r q^{|\mathbf{k}|+k_r}; p)}{\theta(az_r; p)} \frac{\prod_{s=1}^{n} (az_s; q, p)_{|\mathbf{k}|}}{\prod_{s=1}^{n} (qz_r/z_s; q, p)_{k_r}} \right.$$
$$\left. \times \frac{\prod_{s=1}^{n+2} (b_s z_r; q, p)_{k_r} (c, q^{-N}; q, p)_{|\mathbf{k}|}}{\prod_{s=1}^{n+2} (aq/b_s; q, p)_{|\mathbf{k}|} (aqz_r/c, az_r q^{N+1}; q, p)_{k_r}} q^{|\mathbf{k}|} \right]$$
$$= c^N \prod_{r=1}^{n} \frac{(aqz_r; q, p)_N}{(aqz_r/c; q, p)_N} \prod_{s=1}^{n+2} \frac{(aq/cb_s; q, p)_N}{(aq/b_s; q, p)_N}. \tag{11.7.19}$$

For some elliptic multivariable transformation formulas and additional summation formulas, see the exercises and notes.

Exercises

11.1 Verify the $\theta(a;p)$ and q,p-shifted factorial identities in (11.2.42)–(11.2.60).

11.2 Verify the q,p-binomial coefficient identities in (11.2.62) and (11.2.64)–(11.2.66).

11.3 Show that

$$\frac{\theta(aq^{2n}; p)}{\theta(a; p)} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q, p^{\frac{1}{2}})_n}{(a^{\frac{1}{2}}, -a^{-\frac{1}{2}}; q, p^{\frac{1}{2}})_n}$$
$$= \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q(ap)^{\frac{1}{2}}, -q(ap)^{\frac{1}{2}}; q, p)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (ap)^{\frac{1}{2}}, -(ap)^{\frac{1}{2}}; q, p)_n}$$
$$= \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, q(a/p)^{\frac{1}{2}}, -q(ap)^{\frac{1}{2}}; q, p)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (ap)^{\frac{1}{2}}, -(a/p)^{\frac{1}{2}}; q, p)_n} (-q)^{-n}$$

and convert these identities to the $[a; \sigma, \tau]_n$ notation.

11.4 Prove (11.4.1) via induction by starting with the identity

$$q^k \theta(aq^{2k}, q^{-n-1}, e, aq^{n+1}/e; p)$$
$$= \theta(aq^k, q^{k-n-1}, eq^k, aq^{n+k+1}/e; p) - \theta(aq^{n+k+1}, q^k, eq^{k-n-1}, aq^k/e; p),$$

which is equivalent to (11.4.3).
(Rosengren: June 13, 2002, e-mail message)

11.5 Verify that

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,a/bc;q,p)_k}{(q,aq/b,aq/c,bcq;q,p)_k} q^k = \frac{(aq,bq,cq,aq/bc;q,p)_n}{(q,aq/b,aq/c,bcq;q,p)_n}$$

for $n = 0, 1, \ldots$, which is an elliptic analogue of Ex. 2.5.

11.6 Derive the following transformation formula for a well-poised ${}_4E_3$ series

$$\begin{aligned} &{}_4E_3(a,b,aq^{n+2}/b,q^{-n};aq/b,bq^{-n-1},aq^{n+1};q,p;-1) \\ &= \frac{(aq,q^2(ap)^{\frac12}/b,-q^2(a/p)^{\frac12}/b,-q;q,p)_n}{(q^2/b,q(a/p)^{\frac12},-q(ap)^{\frac12},-aq^2/b;q,p)_n} \\ &\quad \times {}_{12}V_{11}(-aq/b;(ap)^{\frac12},-(a/p)^{\frac12},qa^{\frac12}/b,-qa^{\frac12}/b,-q,aq^{n+2}/b,q^{-n};q,p), \end{aligned}$$

where $n = 0, 1, \ldots$.

11.7 Extend the terminating case of the expansion formula in (2.8.2) to

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a,b,c;q,p)_n}{(q,aq/b,aq/c;q,p)_n} A_n \\ &= \sum_{n=0}^{\infty} \frac{\theta(\lambda q^{2n};p)(\lambda,\lambda b/a,\lambda c/a,aq/bc;q,p)_n(a;q,p)_{2n}}{\theta(\lambda;p)(q,aq/b,aq/c,a^2q/\lambda bc;q,p)_n(\lambda q;q,p)_{2n}} \left(\frac{a}{\lambda}\right)^n \\ &\quad \times \sum_{k=0}^{\infty} \frac{(aq^{2n},a/\lambda,\lambda bcq^n/a;q,p)_k}{(q,\lambda q^{2n+1},a^2q^{n+1}/\lambda bc;q,p)_k} A_{n+k}, \end{aligned}$$

where the sequence $\{A_n\}$ has finite support and λ is an arbitrary parameter. Use this formula to derive (11.5.1).

11.8 Show that

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(a,qa^{\frac12},-qa^{\frac12},b,c,q^{-n};q,p)_k}{(q,a^{\frac12},-a^{\frac12},aq/b,aq/c,aq^{n+1};q,p)_k} (-1)^k \\ &= \frac{(aq,aq/bc,qb^{-1}(a/p)^{\frac12},qc^{-1}(a/p)^{\frac12};q,p)_n}{(aq/b,aq/c,q(a/p)^{\frac12},ab^{-1}c^{-1}(a/p)^{\frac12};q,p)_n}, \end{aligned}$$

where $bc = -aq^{n+1}$, $n = 0, 1, \ldots$.

11.9 (i) Extend Ex. 1.4(i) to the inversion formula

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q,p)_n}{(q,b_1,\ldots,b_s;q,p)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} A_n z^n \\ &= \sum_{n=0}^{\infty} \frac{(a_1^{-1},\ldots,a_r^{-1};q^{-1},p)_n}{(q^{-1},b_1^{-1},\ldots,b_s^{-1};q^{-1},p)_n} A_n \left(\frac{a_1\cdots a_r z}{qb_1\cdots b_s}\right)^n \end{aligned}$$

when the sequence $\{A_n\}$ has finite support.

(ii) Extend Ex. 1.4(ii) to the reverse in order of summation formula

$$\sum_{k=0}^{n} \frac{(a_1,\ldots,a_r,q^{-n};q,p)_k}{(q,b_1,\ldots,b_s;q,p)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} A_k z^k$$
$$= \frac{(a_1,\ldots,a_r;q,p)_n}{(b_1,\ldots,b_s;q,p)_n}\left(\frac{z}{q}\right)^n\left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}$$
$$\times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1,\ldots,q^{1-n}/b_s,q^{-n};q,p)_n}{(q,q^{1-n}/a_1,\ldots,q^{1-n}/a_r;q,p)_n} A_{n-k}\left(\frac{b_1\cdots b_s}{a_1\cdots a_r}\frac{q^{n+1}}{z}\right)^k$$

when $n = 0, 1, \ldots$, and $\{A_n\}$ is an arbitrary sequence of complex numbers.

11.10 Prove that

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,d,e,a^2q/bcde;q,p)_k}{(q,aq/b,aq/c,aq/d,aq/e,bcde/a;q,p)_k} q^k$$
$$= \frac{(aq,bcd/a,a^2q^2/bcde,eq;q,p)_n}{(q,a^2q^2/bcd,bcde/a,aq/e;q,p)_n}$$
$$\times\ {}_{12}V_{11}(a^2q/bcd;aq/bc,aq/bd,aq/cd,e,a^2q/bcde,aq^{n+1},q^{-n};q,p),$$

where $n = 0, 1, \ldots$.

11.11 Verify the transformation formulas (11.6.16) and (11.6.17).

11.12 As in (11.2.67), the $\Gamma(z;q,p)$ *elliptic gamma function* is defined by

$$\Gamma(z;q,p) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}q^{j+1}p^{k+1}}{1 - zq^j p^k},$$

where z, q, p are complex numbers and $|q|$, $|p| < 1$. Show that

(i) $\Gamma(z;q,0) = e_q(z)$,
(ii) $\Gamma(zq;q,p) = \theta(z;p)\Gamma(z;q,p)$,
(iii) $\Gamma(zp;q,p) = \theta(z;q)\Gamma(z;q,p)$,
(iv) $\Gamma(zq^n;q,p) = (z;q,p)_n\Gamma(z;q,p)$,
(v) $\Gamma_q(z) = (1-q)^{1-z}(q;q)_\infty\Gamma(q^z;q,0), \quad 0 < q < 1$,

and that $\Gamma(z;q,p)$ and $\Gamma(q^z;q,p)$ are meromorphic functions of z, which are *not* doubly periodic.
(See Jackson [1905d], Ruijsenaars [1997, 2001], Felder and Varchenko [2000], and Spiridonov [2003b].)

11.13 Define the $\tilde{\Gamma}(z;q,p)$ *elliptic gamma function* by

$$\tilde{\Gamma}(z;q,p) = \frac{(q;q)_\infty}{(p;p)_\infty}(\theta(q;p))^{1-z} \prod_{j,k=0}^{\infty} \frac{1 - q^{j+1-z}p^{k+1}}{1 - q^{j+z}p^k},$$

where z, q, p are complex numbers and $|q|$, $|p| < 1$. Extend the Gauss multiplication formula (1.10.10) and its q-analogue (1.10.11) to

$$\begin{aligned}&\tilde{\Gamma}(nz;q,p)\tilde{\Gamma}\Big(\frac{1}{n};r,p\Big)\tilde{\Gamma}\Big(\frac{2}{n};r,p\Big)\cdots\tilde{\Gamma}\Big(\frac{n-1}{n};r,p\Big)\\&=\left[\frac{\theta(r;p)}{\theta(q;p)}\right]^{nz-1}\tilde{\Gamma}(z;r,p)\tilde{\Gamma}\Big(z+\frac{1}{n};r,p\Big)\cdots\tilde{\Gamma}\Big(z+\frac{n-1}{n};r,p\Big)\end{aligned}$$

with $r = q^n$, and show that

$$\tilde{\Gamma}(z;q,0) = \Gamma_q(z).$$

(Felder and Varchenko [2003a])

11.14 (i) Show that if n is a nonnegative integer and k is a positive integer, then

$$\begin{aligned}&{}_{2k+8}V_{2k+7}(ab;c,ab/c,bq,bq^2,\ldots,bq^k,aq^n,aq^{n+1},\ldots,aq^{n+k-1},q^{-kn};q^k,p)\\&=\frac{(a/c,c/b;q,p)_n(q^k,abq^k;q^k,p)_n}{(cq^k,abq^k/c;q^k,p)_n(a,1/b;q,p)_n}.\end{aligned}$$

(ii) Show that

$$\begin{aligned}&{}_{12}V_{11}(acp;c,-c,cp,-cp,aqp/c,a^2q^{n+1},q^{-n};q,p^2)\\&=\begin{cases}0, & \text{if } n \text{ is odd},\\ \dfrac{c^2(q,a^2q^2/c^2;q^2,p^2)_{n/2}(acqp;q,p^2)_n}{(a^2q^2,c^2q;q^2,p^2)_{n/2}(aqp/c;q,p^2)_n}, & \text{if } n \text{ is even},\end{cases}\end{aligned}$$

and that this formula tends to (II.17) as $p \to 0$.
(iii) Show that

$$\begin{aligned}&{}_{12}V_{11}(cp;c,-c,-cp,ep/c,cqp/e,q^{n+1},q^{-n};q,p^2)\\&=\frac{(cqp,e/c^2;q,p^2)_n(eq^{-n};q^2,p^2)_n}{(qp/c,e;q,p^2)_n(eq^{-n}/c^2;q^2,p^2)_n}\end{aligned}$$

and that this formula tends to (II.19) as $p \to 0$.
(See Warnaar [2002b] for part (i) and Warnaar [2003f] for parts (ii) and (iii).)

11.15 Prove that if a, b_k, c_k are complex numbers such that $c_j \neq c_k$ and $ac_jc_k \neq 1$ for integer j and k, then

$$\sum_{k=m}^{n} f_{nk}g_{km} = \delta_{n,m}$$

with

$$f_{nk} = \frac{\theta(b_kc_k, ac_k/b_k;p)}{\theta(b_nc_n, ac_n/b_n;p)}\prod_{j=k}^{n-1}\frac{\theta(c_nb_{j+1}, ac_n/b_{j+1};p)}{c_j\theta(ac_nc_j, c_n/c_j;p)}$$

and

$$g_{km} = \prod_{j=m}^{k-1}\frac{\theta(c_mb_j, ac_m/k_j;p)}{c_{j+1}\theta(ac_mc_{j+1}, c_m/c_{j+1};p)}.$$

(Warnaar [2002b])

11.16 Prove that

$$\sum_{k=0}^{n} \frac{\theta(a^2q^{5k};p)}{\theta(a^2;p)} \frac{(a^2;q^4,p)_k(a,aq,aq^2;q^3,p)_k(aq^{n+1},q^{-n};q,p)_k}{(q;q,p)_k(a,aq,aq^2;q^2,p)_k(aq^{3-n},a^2q^{n+4};q^4,p)_k} q^k$$
$$= \begin{cases} \dfrac{(q,q^2,q^3,a^2q^4;q^4,p)_{n/4}}{(aq^2,aq^3,aq^4,q/a;q^4,p)_{n/4}}, & n \equiv 0 \pmod 4, \\ 0, & n \not\equiv 0 \pmod 4. \end{cases}$$

(Warnaar [2002b])

11.17 Show that if $bc = a^2q^{n+1}$ and $d = q^{n+1}$, then

$$\sum_{k=0}^{[n/2]} \frac{\theta(aq^{4k};p)(b,c;q^3,p)_k(a,d;q,p)_k(q^{-n};q,p)_{2k}}{\theta(a;p)(aq/b,aq/c;q,p)_k(aq^3/d,q^3;q^3,p)_k(aq^{n+1};q,p)_{2k}} q^k$$
$$= \frac{(aq;q,p)_n(aq^{2-n}/b;q^3,p)_n}{(aq/b;q,p)_n(aq^{2-n};q^3,p)_n}.$$

(Warnaar [2002b])

11.18 For $bc = aq$ and $cd = aq^{n+1}$, show that

$$\sum_{k=0}^{n} \frac{\theta(aq^{4k};p)(a,b;q^3,p)_k(d,q^{-n};q,p)_k(c;q,p)_{2k}}{\theta(a;p)(q,aq/b;q,p)_k(aq^3/d,aq^{n+3};q^3,p)_k(aq/c;q,p)_{2k}} q^k$$
$$= \begin{cases} \dfrac{(q,q^2,aq^3,b^2/a;q^3,p)_{n/3}}{(bq,bq^2,b/a,aq^3/b;q^3,p)_{n/3}}, & n \equiv 0 \pmod 3, \\ 0, & n \not\equiv 0 \pmod 3. \end{cases}$$

(Warnaar [2002b])

11.19 Extend the $c = bq^{-n-1}$ case of (3.8.19) to

$$\sum_{k=0}^{[n/2]} \frac{\theta(aq^{4k};p)}{\theta(a;p)} \frac{(b,c;q^3,p)_k(d,e;q,p)_k(q^{-n};q,p)_{2k}}{(aq/b,aq/c;q,p)_k(aq^3/d,aq^3/e;q^3,p)_k(aq^{n+1};q,p)_{2k}} q^k$$
$$= \frac{(aq;q,p)_n(aq^{2-n}/b;q^3,p)_n}{(aq/b;q,p)_n(aq^{2-n};q^3,p)_n}$$
$$\times\, {}_{12}V_{11}(a^2/de;b,c,a/d,a/e,q^{2-n},q^{1-n},q^{-n};q^3,p),$$

where $bc = a^2q^{n+1}$ and $de = aq^{n+1}$.
(Warnaar [2002b])

11.20 Show that if $bc = aq$ and $de = aq^{n+1}$, then

$$\sum_{k=0}^{n} \frac{\theta(aq^{3k};p)}{\theta(a;p)} \frac{(a,b,c;q^2,p)_k(d,e,q^{-n};q,p)_k}{(q,aq/b,aq/c;q,p)_k(aq^2/d,aq^2/e,aq^{n+2};q^2,p)_k} q^k$$
$$= \begin{cases} \dfrac{(aq^2,aq^2/bc,aq^2/bd,aq^2/cd;q^2,p)_{n/2}}{(aq^2/b,aq^2/c,aq^2/d,aq^2/bcd;q^2,p)_{n/2}}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

(Warnaar [2002b])

11.21 Show that if $bcd = aq$, $ef = a^2q^{2n+1}$, and either $b = a$, or $e = a$, then

$$\sum_{k=0}^{n} \frac{\theta(aq^{3k};p)}{\theta(a;p)} \frac{(b,c,d;q,p)_k(e,f,q^{-2n};q^2,p)_k}{(aq^2/b,aq^2/c,aq^2/d;q^2,p)_k(aq/e,aq/f,aq^{2n+1};q,p)_n} q^k$$
$$= \frac{(aq^2,a^2q^2/bce,a^2q^2/bde,aq^2/cd;q^2,p)_n}{(a^2q^2/be,aq^2/c,aq^2/d,a^2q^2/bcde;q^2,p)_n}.$$

(Warnaar [2002b])

11.22 Show that if $bcd = aq$, $de = a^2q^{3n+1}$, and either $b = a$ or $e = a$, then

$$\sum_{k=0}^{n} \frac{\theta(aq^{4k};p)}{\theta(a;p)} \frac{(b,c;q,p)_k(d;q,p)_{2k}}{(aq^3/b,aq^3/c;q^3,p)_k(aq/d;q,p)_{2k}}$$
$$\times \frac{(e,q^{-3n};q^3,p)_k}{(aq/e,aq^{3n+1};q,p)_k} q^k$$
$$= \frac{(aq^3,a^2q^3/bce,a^2q^3/bde,aq^3/cd;q^3,p)_n}{(a^2q^3/be,aq^3/c,aq^3/d,a^2q^3/bcde;q^3,p)_n}.$$

(Warnaar [2002b])

11.23 (i) Verify the quadratic elliptic transformation formula

$$_{14}V_{13}(a;b,-b,c,-c,dq^n,-dq^n,q^{-n},-q^{-n},qa^2/\lambda^2;q,p)$$
$$= \frac{(a^2q^2,d^2/\lambda^2,b^2d^2/a^2,c^2d^2/a^2;q^2,p^2)_n}{(d^2/a^2,q^2\lambda^2,a^2q^2/b^2,a^2q^2/c^2;q^2,p^2)_n}$$
$$\times {}_{14}V_{13}(\lambda^2;b^2,c^2,d^2q^{2n},a^2q^2/\lambda^2,q^{-2n},-\lambda^2/a,-\lambda^2q/a,$$
$$-\lambda^2/ap,-\lambda^2qp/a;q^2,p^2),$$

where $\lambda = bcd/aq$ and $n = 0, 1, \ldots$.

(ii) Prove that the $p \to 0$ limit of the above transformation is

$$_{12}W_{11}(a;b,-b,c,-c,dq^n,-dq^n,q^{-n},-q^{-n},qa^2/\lambda^2;q,q)$$
$$= \frac{(a^2q^2,d^2/\lambda^2,b^2d^2/a^2,c^2d^2/a^2;q^2)_n}{(d^2/a^2,q^2\lambda^2,a^2q^2/b^2,a^2q^2/c^2;q^2)_n}$$
$$\times {}_{10}W_9(\lambda^2;b^2,c^2,d^2q^{2n},a^2q^2/\lambda^2,q^{-2n},-\lambda^2/a,-\lambda^2q/a;q^2,\lambda^2q/a^2)$$

and that the $_{12}W_{11}$ series is VWP-balanced and balanced, while the $_{10}W_9$ series is VWP-balanced but not balanced.

(See (Spiridonov [2002a]) for part (i), and Nassrallah and Rahman [1981] for part (ii). Also see (3.10.15), Andrews and Berkovich [2002] and Warnaar [2003c,e].)

11.24 (i) Verify the quadratic elliptic transformation formula

$$_{14}V_{13}(a;a^2/\lambda^2,b,bq,c,cq,dq^n,dq^{n+1},q^{-n},q^{1-n};q^2,p)$$
$$= \frac{(aq,d/\lambda,\lambda q/b,\lambda q/c;q,p)_n}{(\lambda q,d/a,aq/b,aq/c;q,p)_n}$$
$$\times {}_{14}V_{13}(\lambda;a/\lambda,\mu,-\mu,\mu p^{\frac{1}{2}},-\mu p^{-\frac{1}{2}},b,c,dq^n,dq^{-n};q,p),$$

where $\lambda = bcd/aq$, $\mu = \pm\lambda(q/a)^{\frac{1}{2}}$, and $n = 0, 1, \ldots$.

(ii) Prove that the $p \to 0$ limit of the above identity is

$$\begin{aligned}&{}_{12}W_{11}(a; a^2/\lambda^2, b, bq, c, cq, dq^n, dq^{n+1}, q^{1-n}, q^{-n}; q^2, q^2)\\ &= \frac{(aq, d/\lambda, \lambda q/b, \lambda q/c; q)_n}{(\lambda q, d/a, aq/b, aq/c; q)_n}\\ &\quad \times {}_{10}W_9(\lambda; a/\lambda, \mu, -\mu, b, c, dq^n, q^{-n}; q, -q\lambda^2/a)\end{aligned}$$

and that the ${}_{12}W_{11}$ series is VWP-balanced and balanced, while the ${}_{10}W_9$ series is VWP-balanced but not balanced.

(See Warnaar [2002b] for part (i), and Rahman and Verma [1993] for the transformation in part (ii). Also see Andrews and Berkovich [2002] and Warnaar [2003c,e].)

11.25 (i) Extend the quadbasic transformation formula in Ex. 3.21 to

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\theta(ar^kq^k, br^kq^{-k}; p)}{\theta(a, b; p)} \frac{(a, b; r, p)_k (c, a/bc; q, p)_k}{(q, aq/b; q, p)_k (ar/c, bcr; r, p)_k}\\ &\quad \times \frac{(CR^{-n}/A, R^{-n}/BC; R, P)_k (Q^{-n}, BQ^{-n}/A; Q, P)_k}{(Q^{-n}/C, BCQ^{-n}/A; Q, P)_k (R^{-n}/A, R^{-n}/B; R, P)_k} q^k\\ &= \frac{(ar, br; r, p)_n (cq, aq/bc; q, p)_n (Q, AQ/B; Q, P)_n (AR/C, BCR; R, P)_n}{(q, aq/b; q, p)_n (arc, bc/r; r, p)_n (AR, BR; R, P)_n (CQ, AQ/BC; Q, P)_n}\\ &\quad \times \sum_{k=0}^{n} \frac{\theta(AR^kQ^k, BR^kQ^{-k}; P)}{\theta(A, B; P)} \frac{(A, B; R, P)_k (C, A/BC; Q, P)_k}{(Q, AQ/B; Q, P)_k (AR/C, BCR; R, P)_k}\\ &\quad \times \frac{(cr^{-n}/a, r^{-n}/bc; r, p)_k (q^{-n}, bq^{-n}/a; q, p)_k}{(q^{-n}/c, bcq^{-n}/a; q, p)_k (r^{-n}/a, r^{-n}/b; r, p)_k} Q^k\end{aligned}$$

for $n = 0, 1, \ldots$.

(ii) Deduce the following transformation formula for a "split-poised" theta hypergeometric series

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\theta(aq^{2k}; p)}{\theta(a; p)} \frac{(a, b, c, a/bc; q, p)_k}{(q, aq/b, aq/c, bcq; q, p)_k}\\ &\quad \times \frac{(q^{-n}, B/Aq^n, C/Aq^n, 1/BCq^n; q, p)_k}{(1/Aq^n, 1/Bq^n, 1/Cq^n, BC/Aq^n; q, p)_k} q^k\\ &= \frac{(aq, bq, cq, aq/bc, Aq/B, Aq/C, BCq; q, p)_n}{(Aq, Bq, Cq, Aq/BC, aq/b, aq/c, bcq; q, p)_n}\\ &\quad \times \sum_{k=0}^{n} \frac{\theta(Aq^{2k}; p)}{\theta(A; p)} \frac{(A, B, C, A/BC; q, p)_k}{(q, Aq/B, Aq/C, BCq; q, p)_k}\\ &\quad \times \frac{(q^{-n}, b/aq^n, c/aq^n, 1/bcq^n; q, p)_k}{(1/aq^n, 1/bq^n, 1/cq^n, bc/aq^n; q, p)_k} q^k\end{aligned}$$

for $n = 0, 1, \ldots$, which is an extension of the transformation formula for a split-poised ${}_{10}\phi_9$ series given in Ex. 3.21. Write this formula as a transformation formula for a split-poised ${}_{12}E_{11}$ series.

(Gasper and Schlosser [2003])

11.26 Extend Ex. 11.25(i) to

$$\sum_{k=0}^{n} \frac{\theta(a(rst/q)^k, br^kq^{-k}, cs^kq^{-k}, at^k/bcq^k;p)}{\theta(a,b,c,a/bc;p)}$$
$$\times \frac{(a;rst/q^2,p)_k(b;r,p)_k(c;s,p)_k(a/bc;t,p)_k}{(q;q,p)_k(ast/bq;st/q,p)_k(art/cq;rt/q,p)_k(bcrs/q;rs/q,p)_k}$$
$$\times \frac{(Q^{-n};Q,P)_k(B(Q/ST)^n/A;ST/Q,P)_k(C(Q/RT)^n/A;RT/Q,P)_k}{((Q^2/RST)^n/A;RST/Q^2,P)_k(R^{-n}/B;R,P)_k(S^{-n}/C;S,P)_k}$$
$$\times \frac{((Q/RS)^n/BC;RS/Q,P)_k}{(BC/AT^n;T,P)_k} q^k$$
$$= \frac{(arst/q^2;rst/q^2,p)_n(br;r,p)_n(cs;s,p)_n(at/bc;t,p)_n}{(q;q,p)_n(ast/bq;st/q,p)_n(art/cq;rt/q,p)_n(bcrs/q;rs/q,p)_n}$$
$$\times \frac{(Q;Q,P)_n(AST/BQ;ST/Q,P)_n}{(ARST/Q^2;RST/Q^2,P)_n(BR;R,P)_n}$$
$$\times \frac{(ART/CQ;RT/Q,P)_n(BCRS/Q;RS/Q,P)_n}{(CS;S,P)_n(AT/BC;T,P)_n}$$
$$\times \sum_{k=0}^{n} \frac{\theta(A(RST/Q)^k, BR^kQ^{-k}, CS^kQ^{-k}, AT^k/BCQ^k;P)}{\theta(A,B,C,A/BC;P)}$$
$$\times \frac{(A;RST/Q^2,P)_k(B;R,P)_k}{(Q;Q,P)_k(AST/BQ;ST/Q,P)_k}$$
$$\times \frac{(C;S,P)_k(A/BC;T,P)_k}{(ART/CQ;RT/Q,P)_k(BCRS/Q;RS/Q,P)_k}$$
$$\times \frac{(q^{-n};q,p)_k(b(q/st)^n/a;st/q,p)_k}{((q^2/rst)^n/a;rst/q^2,p)_k}$$
$$\times \frac{(c(q/rt)^n/a;rt/q,p)_k((q/rs)^n/bc;rs/q,p)_k}{(r^{-n}/b;r,p)_k(s^{-n}/c;s,p)_k(bc/at^n;t,p)_k} Q^k$$

for $n = 0, 1, \ldots$. Use (11.6.9) and (11.6.18) to extend this formula to a transformation formula containing the two additional parameters d and D.

(Gasper and Schlosser [2003])

11.27 Show that if

$$U_k = \prod_{j=0}^{k-1} \frac{\theta(b_j, c_j, d_j, e_j, f_j, g_j;p)}{\theta(a_j/b_j, a_j/c_j, a_j/d_j, a_j/e_j, a_j/f_j, a_j/g_j;p)}$$

and $a_k^3 = b_k c_k d_k e_k f_k g_k$ for $k = 0, \pm 1, \pm 2, \ldots$, then

$$\sum_{k=m}^{n} \frac{\theta(a_k^2/b_kc_kd_kf_k, a_k^2/b_kc_kd_ke_k, a_k, a_k/e_kf_k;p)}{\theta(a_k/e_k, a_k/f_k, a_k^2/b_kc_kd_ke_kf_k, a_k^2/b_kc_kd_k;p)} U_k$$
$$\times \left[1 - \frac{\theta(a_k/c_kd_k, a_k/b_kd_k, a_k/b_kc_k, e_k, f_k, g_k;p)}{\theta(a_k/b_k, a_k/c_k, a_k/d_k, a_k^2/b_kc_kd_ke_k, a_k^2/b_kc_kd_kf_k, a_k^2/b_kc_kd_kg_k;p)}\right]$$
$$= U_m - U_{n+1}$$

for $n, m = 0, \pm 1, \pm 2, \ldots$.

(Gasper and Schlosser [2003])

11.28 Extend the indefinite multibasic theta hypergeometric summation formula in (11.6.9) to

$$\sum_{k=0}^{n} \frac{\theta\big(av^k, \frac{a}{fg}\big(\frac{v}{ta}\big)^k, \frac{a^2}{bcdf}\big(\frac{v^2}{qrst}\big)^k, \frac{a^2}{vbcdg}\big(\frac{v^2}{qrsu}\big)^k, \frac{a^2}{bcd};p\big)}{\theta\big(a, \frac{a}{fg}, \frac{a^2}{bcdf}, \frac{a^2}{bcdg}, \frac{a^2}{bcd}\big(\frac{v^2}{qrs}\big)^k;p\big)}$$
$$\times \frac{(b;q,p)_k(c;r,p)_k(d;s,p)_k(f;t,p)_k(g;u,p)_k\big(\frac{a^3}{bcdfg}; \frac{v^3}{qrstu},p\big)_k (qrstu/v^2)^k}{\big(\frac{a}{b}; \frac{v}{q},p\big)_k\big(\frac{a}{c}; \frac{v}{r},p\big)_k\big(\frac{a}{d}; \frac{v}{s},p\big)_k\big(\frac{av}{ft}; \frac{v}{t},p\big)_k\big(\frac{av}{gu}; \frac{v}{u},p\big)_k\big(\frac{bcdfgqrstu}{a^2v^2}; \frac{qrstu}{v^2},p\big)_k}$$
$$\times \left[1 - \frac{\theta\big(\frac{a}{cd}\big(\frac{v}{rs}\big)^k, \frac{a}{bd}\big(\frac{v}{qs}\big)^k, \frac{a}{bc}\big(\frac{v}{qr}\big)^k, ft^k, gu^k, \frac{a^3}{bcdfg}\big(\frac{v^3}{qrstu}\big)^k;p\big)}{\theta\big(\frac{a}{b}\big(\frac{v}{q}\big)^k, \frac{a}{c}\big(\frac{v}{r}\big)^k, \frac{a}{d}\big(\frac{v}{s}\big)^k, \frac{a^2}{bcdf}\big(\frac{v^2}{qrst}\big)^k, \frac{a^2}{bcdg}\big(\frac{v^2}{qrsu}\big)^k, \frac{fg}{a}\big(\frac{tu}{v}\big)^k;p\big)}\right]$$
$$= \frac{\theta(a/f, a/g, a^2/bcdfg, a^2/bcd;p)}{\theta(a^2/bcdg, a^2/bcdf, a, a/fg;p)}$$
$$\times \left[1 - \frac{(b;q,p)_{n+1}(c;r,p)_{n+1}(d;s,p)_{n+1}(f;t,p)_{n+1}(g;u,p)_{n+1}}{\big(\frac{a}{b}; \frac{v}{q},p\big)_{k+1}\big(\frac{a}{c}; \frac{v}{r},p\big)_{n+1}\big(\frac{a}{d}; \frac{v}{s},p\big)_{n+1}\big(\frac{a}{f}; \frac{v}{t},p\big)_{n+1}\big(\frac{a}{g}; \frac{v}{u},p\big)_{n+1}}\right.$$
$$\left.\times \frac{\big(\frac{a^3}{bcdfg}; \frac{v^3}{qrstu},p\big)_{n+1}}{\big(\frac{bcdfg}{a^2}; \frac{qrstu}{v^2},p\big)_{n+1}}\right],$$

where $n = 0, 1, \ldots$.
(Gasper and Schlosser [2003])

11.29 Define the *elliptic beta function* $B_E(\mathbf{t};q,p)$ via its *elliptic beta contour integral* representation

$$B_E(\mathbf{t};q,p) = \int_{\mathbb{T}} \Delta_E(z;\mathbf{t};q,p)\frac{dz}{z},$$

where $\mathbb{T}$ is the positively oriented unit circle,

$$\Delta_E(z;\mathbf{t};q,p) = \frac{1}{2\pi i} \frac{\prod_{k=0}^{4} \Gamma(zt_k, t_k/z;q,p)}{\Gamma(z^2, 1/z^2, zA, A/z;q,p)},$$

$\mathbf{t} = (t_0, t_1, t_2, t_3, t_4)$, $\max(|t_0|, |t_1|, |t_2|, |t_3|, |t_4|, |q|, |p|) < 1$, $A = \prod_{k=1}^{4} t_k$, $|qp| < A$, and

$$\Gamma(z_1, \ldots, z_n; q, p) = \prod_{k=1}^{n} \Gamma(z_k; q, p), \quad n = 1, 2, \ldots,$$

with $\Gamma(z; q, p)$ as defined in Ex. 11.12. Prove that

$$B_E(\mathbf{t}; q, p) = \frac{2 \prod_{0 \le j < k \le 4} \Gamma(t_j t_k; q, p)}{(q;q)_\infty (p;p)_\infty \prod_{k=0}^{4} \Gamma(A/t_k; q, p)}.$$

(Spiridonov [2001a])

11.30 Let $\Delta_E(z; \mathbf{t}) = \Delta_E(z; \mathbf{t}; q, p)$ and $B_E(\mathbf{t}) = B_E(\mathbf{t}; q, p)$, with $\Delta_E(z; \mathbf{t}; q, p)$, $B_E(\mathbf{t}; q, p)$ and A defined as in the previous exercise, and let

$$R_{m,j}(z) = {}_{12}V_{11}\Big(\frac{t_3}{t_4}; \frac{q}{t_0 t_4}, \frac{q}{t_1 t_4}, \frac{q}{t_2 t_4}, t_3 z, \frac{t_3}{z}, q^{-m}, \frac{Aq^{m-1}}{t_4}; q, p\Big)$$
$$\times\, {}_{12}V_{11}\Big(\frac{t_3}{t_4}; \frac{p}{t_0 t_4}, \frac{p}{t_1 t_4}, \frac{p}{t_2 t_4}, t_3 z, \frac{t_3}{z}, p^{-j}, \frac{Ap^{j-1}}{t_4}; p, q\Big)$$

$$T_{n,k}(z) = {}_{12}V_{11}\Big(\frac{At_3}{q}; \frac{A}{t_0}, \frac{A}{t_1}, \frac{A}{t_2}, t_3 z, \frac{t_3}{z}, q^{-n}, \frac{Aq^{n-1}}{t_4}; q, p\Big)$$
$$\times\, {}_{12}V_{11}\Big(\frac{At_3}{p}; \frac{A}{t_0}, \frac{A}{t_1}, \frac{A}{t_2}, t_3 z, \frac{t_3}{z}, p^{-k}, \frac{Ap^{k-1}}{t_4}; p, q\Big)$$

for $j, k, m, n = 0, 1, \ldots$. Note that the base and nome in the second ${}_{12}V_{11}$ factors are p and q, respectively. Prove that $R_{m,j}(z)$ and $T_{n,k}(z)$ satisfy the biorthogonality relation

$$\int_{C_{j,k,m,n}} R_{m,j}(z) T_{n,k}(z) \Delta_E(z; \mathbf{t}) \frac{dz}{z} = h_{n,k} B_E(\mathbf{t}) \delta_{m,n} \delta_{j,k},$$

where

$$h_{n,k} = \frac{\theta(A/qt_4; p)(q, qt_3/t_4, t_0t_1, t_0t_2, t_1t_2, At_3; q, p)_n q^{-n}}{\theta(Aq^{2n}/qt_4; p)(1/t_3t_4, t_0t_3, t_1t_3, t_2t_3, A/qt_3, A/qt_4; q, p)_n}$$
$$\times \frac{\theta(A/pt_4; q)(p, pt_3/t_4, t_0t_1, t_0t_2, t_1t_2, At_3; p, q)_n p^{-k}}{\theta(Ap^{2k}/pt_4; q)(1/t_3t_4, t_0t_3, t_1t_3, t_2t_3, A/pt_3, A/pt_4; p, q)_k}$$

and $C_{j,k,m,n}$ is a closed positively oriented contour separating the points $z = t_{0,1,2,3}\, p^r q^s, t_4\, p^{r-j} q^{s-m}$, $A^{-1} p^{r+1-k} q^{s+1-n}$ with $r, s = 0, 1, \ldots$, from their inverses.
(Spiridonov [2003b])

11.31 Show that if

$$w_1 \cdots w_m = (z_1 \cdots z_n)(a_1 \cdots a_{n+m}),$$

then

$$\sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=N}} \frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)} \prod_{r=1}^{n} \frac{\prod_{s=1}^{m+n}(a_s z_r;q,p)_{k_r}}{\prod_{s=1}^{m}(w_s z_r;q,p)_{k_r} \prod_{s=1}^{n}(q z_r/z_s;q,p)_{k_r}}$$

$$= \sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=N}} \frac{\Delta(\mathbf{w}q^{\mathbf{k}};p)}{\Delta(\mathbf{w};p)} \prod_{r=1}^{m} \frac{\prod_{s=1}^{m+n}(w_r/a_s;q,p)_{k_r}}{\prod_{s=1}^{n}(w_r z_s;q,p)_{k_r} \prod_{s=1}^{m}(q w_r/w_s;q,p)_{k_r}},$$

which is an extension of (11.5.5), (11.7.7), and (11.7.10).
(Kajihara and Noumi [2003] and Rosengren [2003c])

11.32 Let $X_1,\ldots,X_n$, $A_2,\ldots,A_n$ and C be indeterminates. Prove that if, for $m=0,\ldots,n-1$, P_m is a Laurent polynomial of degree less than or equal to m such that $P_m(C/X)=P_m(X)$, then

$$\det_{1\leq i,j\leq n}\Big(P_{j-1}(X_i)\prod_{k=j+1}^{n}(1-A_kX_i)(1-CA_k/X_i)\Big)$$

$$= \prod_{1\leq i<j\leq n} A_jX_j(1-X_i/X_j)(1-C/X_iX_j)\prod_{i=1}^{n}P_{i-1}(1/A_i),$$

where the degree of the Laurent polynomial $P(x)=\sum_{i=M}^{N} a_i x^i$, $a_N\neq 0$, is defined to be N.

(Krattenthaler [1995a])

11.33 Prove the following elliptic extension of the identity in the above exercise

$$\det_{1\leq i,j\leq n}\Big(P_{j-1}(X_i)\prod_{k=j+1}^{n}\theta(A_kX_i;p)\theta(CA_k/X_i;p)\Big)$$

$$= \prod_{1\leq i<j\leq n} A_jX_j\theta(X_i/X_j;p)\theta(C/X_iX_j;p)\prod_{i=1}^{n}P_{i-1}(1/A_i),$$

where $X_1,\ldots,X_m$, $A_2,\ldots,A_n$, and C are indeterminates, and $P_j(x)$ is analytic in $0<|x|<\infty$ for $j=0,\ldots,(n-1)$ with periodicity $P_j(px)=\big(C/px^2\big)^j P_j(x)$ and symmetry $P_j(C/x)=P_j(x)$.
(Warnaar [2002b])

11.34 Deduce from the above exercise that

$$\det_{1\leq i,j\leq n}\frac{(AX_i,AC/X_i;q,p)_{n-j}}{(BX_i,BC/X_i;q,p)_{n-j}}$$

$$= A^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{1\leq i<j\leq n}X_j\theta(X_i/X_j;p)\theta(C/X_iX_j;p)$$

$$\times\prod_{i=1}^{n}\frac{(B/A,ABCq^{2n-2i};q,p)_{i-1}}{(BX_i,BC/X_i;q,p)_{n-1}}.$$

(Warnaar [2002b])

11.35 Let $z_1, \ldots, z_n$, a, b, c, d, and e be indeterminates and N a nonnegative integer such that $a^2q^{N-n+2} = bcde$. Show that

$$\sum_{k_1,\ldots,k_n=0}^{N} \frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)} \prod_{1\le r<s\le n} \frac{\theta(az_rz_sq^{k_r+k_s};p)}{\theta(az_rz_s;p)}$$

$$\times \prod_{r=1}^{n} \frac{\theta(az_r^2q^{2k_r};p)}{\theta(az_r^2;p)} \frac{(az_r^2, bz_r, cz_r, dz_r, ez_r, q^{-N};q,p)_{k_s}}{(q, aqz_r/b, aqz_r/e, aqz_r/d, aqz_r/e, aq^{N+1}z_r^2;q,p)_{k_s}} q^{|\mathbf{k}|}$$

$$= \prod_{r=1}^{n} \frac{(aqz_r^2, aq^{2-r}/bc, aq^{2-r}/bd, aq^{2-r}/cd;q,p)_N}{(aq^{2-n}/bcdz_r, aqz_r/b, aqz_r/c, aqz_r/d;q,p)_N},$$

which is an elliptic extension of Schlosser's [2000a] C_n Jackson sum.

(Warnaar [2002b])

11.36 Prove the following elliptic extension of Gasper's summation formula in Ex. 2.33(i):

$$\sum_{k=0}^{N} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a, b, a/b, q^{-N};q,p)_k}{(q, aq/b, bq, aq^{N+1};q,p)_k} q^k \prod_{j=1}^{r} \frac{(c_jq^{m_j}, aq/c_j;q,p)_k}{(aq^{1-m_j}/c_j, c_j;q,p)_k}$$

$$= \frac{(q, aq;q,p)_N}{(bq, aq/b;q,p)_N} \prod_{j=1}^{r} \frac{(c_j/b, c_jb/a;q,p)_{m_j}}{(c_j, c_j/a;q,p)_{m_j}}$$

with $m_1 + \ldots + m_r = N$, where $m_1, \ldots, m_r$ are nonnegative integers.

(Rosengren and Schlosser [2003b])

11.37 Prove the following multidimensional extension of Ex. 11.29:

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \prod_{1\le j<k\le n} \frac{\Gamma(tz_jz_k, tz_jz_k^{-1}, tz_j^{-1}z_k, tz_j^{-1}z_k^{-1};q,p)}{\Gamma(z_jz_k, z_jz_k^{-1}, z_j^{-1}z_k, z_j^{-1}z_k^{-1};q,p)}$$

$$\times \prod_{j=1}^{n} \frac{\prod_{r=0}^{n} \Gamma(t_rz_j, t_rz_j^{-1};q,p)}{\Gamma(z_j, Bz_j, z_j^{-1}, Bz_j^{-1};q,p)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

$$= \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{j=1}^{n} \frac{\Gamma(t^j;q,p)}{\Gamma(t;q,p)} \frac{\prod_{0\le r<s\le 4} \Gamma(t^{s-1}t_rt_s;q,p)}{\prod_{0\le r\le 4} \Gamma(t^{1-j}t_r^{-1}B;q,p)},$$

where $B = t^{2n-2} \prod_{0\le k\le 4} t_k$, $\max(|t|, |t_0|, |t_1|, |t_2|, |t_3|, |t_4|, |p|, |q|) < 1$, $|pq| < B$, and $\mathbb{T}^n$ is the n-dimensional unit torus.

(van Diejen and Spiridonov [2003, (29)] and Rains [2003b])

11.38 Prove Rosengren's [2003c] elliptic D_n Jackson summation formula

$$\sum_{\substack{k_1+\cdots+k_n=N\\ k_1,\ldots,k_n\geq 0}} \frac{\Delta(\mathbf{z}q^{\mathbf{k}},p)}{\Delta(\mathbf{z};p)} \prod_{1\leq r<s\leq n} \frac{1}{(z_rz_s;q,p)_{k_r+k_s}}$$

$$\times \prod_{s=1}^{n} \frac{z_s^{k_s} q^{\binom{k_s}{2}} \prod_{r=1}^{n-1} (z_sa_r, z_s/a_r;q,p)_{k_s}}{(bz_s, z_sq^{1-N}/b;q,p)_{k_s} \prod_{r=1}^{n} (qz_s/z_r;q,p)_{k_s}}$$

$$= (-q^{N-1}b)^N \frac{\prod_{s=1}^{n-1} (ba_s, b/a_s;q,p)_N}{(q;q,p)_N \prod_{s=1}^{n} (bz_s, b/z_s;q,p)_N}.$$

11.39 From the previous exercise deduce that

$$\sum_{\substack{k_1+\cdots+k_n\leq N\\ k_1,\ldots,k_n\geq 0}} \left\{ \frac{\Delta(\mathbf{z}q^{\mathbf{k}},p)}{\Delta(\mathbf{z};p)} \prod_{s=1}^{n} \frac{\theta(az_sq^{k_s+|\mathbf{k}|};p)}{\theta(az_s;p)} \right.$$

$$\times \prod_{1\leq r<s\leq n} \frac{1}{(z_rz_s;q,p)_{k_r+k_s}} \prod_{r,s=1}^{n} \frac{(z_sb_r, z_s/b_r;q,p)_{k_s}}{(qz_s/z_r;q,p)_{k_s}}$$

$$\times \prod_{r=1}^{n} \frac{(az_r;q,p)_{|\mathbf{k}|}(aq/z_r;q,p)_{|\mathbf{k}|-k_r}}{(aqb_r, aq/b_r;q,p)_{|\mathbf{k}|}}$$

$$\left. \times \frac{(q^{-N}, c, a^2q^{N+1}/c;q,p)_{|\mathbf{k}|}}{\prod_{s=1}^{n} (aqz_s/c, cz_s/aq^N, az_sq^{N+1};q,p)_{k_s}} q^{|\mathbf{k}|} \right\}$$

$$= \prod_{s=1}^{n} \frac{(aqz_s, aq/z_s, aqb_s/c, aqb_s/c;q,p)_N}{(aqz_s/c, aq/z_sc, aqb_s, aq/b_s;q,p)_N},$$

where $|\mathbf{k}| = k_1 + \cdots + k_n$.
(Rosengren [2003c])

11.40 Show that

$$\sum_{k_1,\ldots,k_n=0}^{m_1,\ldots,m_n} \left\{ \frac{\Delta(\mathbf{z}q^{\mathbf{k}};p)}{\Delta(\mathbf{z};p)} q^{|\mathbf{k}|} \left[\prod_{s=1}^{n} \frac{\theta(az_sq^{k_s+|\mathbf{k}|};p)}{\theta(az_s;p)} \right] \right.$$

$$\times \frac{(b,c,d;q,p)_{|\mathbf{k}|}}{(aq/e, aq/f, aq/g;q,p)_{|\mathbf{k}|}} \prod_{r=1}^{n} \frac{(az_r;q,p)_{|\mathbf{k}|}}{(azq^{m_r+1};q,p)_{|\mathbf{k}|}}$$

$$\left. \times \prod_{s=1}^{n} \frac{(ez_s, fz_s, gz_s;q,p)_{k_s}}{(aqz_s/b, aqz_s/c, aqz_s/d;q,p)_{k_s}} \prod_{r,s=1}^{n} \frac{(q^{-m_r}z_s/z_r;q,p)_{k_s}}{(qz_s/z_r;q,p)_{k_s}} \right\}$$

$$= \left(\frac{a}{\lambda}\right)^{|\mathbf{m}|} \frac{(\lambda q/f, \lambda q/g; q, p)_{|\mathbf{m}|}}{(aq/f, aq/g; q, p)_{|\mathbf{m}|}} \prod_{s=1}^{n} \frac{(aqz_s, \lambda qz_s/d; q, p)_{m_s}}{(\lambda qz_s, aqz_s/d; q, p)_{m_s}}$$

$$\times \sum_{k_1,\ldots,k_n=0}^{m_1,\ldots,m_n} \left\{ \frac{\Delta(\mathbf{z}q^{\mathbf{k}}; p)}{\Delta(\mathbf{z}; p)} q^{|\mathbf{k}|} \left[\prod_{s=1}^{n} \frac{\theta(\lambda z_s q^{k_s+|\mathbf{k}|}; p)}{\theta(\lambda z_s; p)} \right] \right.$$

$$\times \frac{(\lambda b/a, \lambda c/a, d; q, p)_{|\mathbf{k}|}}{(aq/e, \lambda q/f, \lambda q/g; q, p)_{|\mathbf{k}|}} \prod_{r=1}^{n} \frac{(\lambda z_r; q, p)_{|\mathbf{k}|}}{(\lambda q^{1+m_r} z_r; q, p)_{|\mathbf{k}|}}$$

$$\left. \times \prod_{s=1}^{n} \frac{(\lambda e z_s/a, f z_s, g z_s; q, p)_{k_s}}{(aqz_s/b, aqz_s/c, \lambda qz_s/d; q, p)_{k_s}} \prod_{r,s=1}^{n} \frac{(q^{-m_r} z_s/z_r; q, p)_{k_s}}{(qz_s/z_r; q, p)_{k_s}} \right\},$$

where $\lambda = a^2q/bcd$ and $a^2q^{2+|\mathbf{m}|} = bcdefg$ with $|\mathbf{k}| = k_1 + \cdots + k_n$ and $|\mathbf{m}| = m_1 + \cdots + m_n$.
(Rosengren [2003c])

Notes

§11.2 and 11.3 Totally elliptic multiple hypergeometric series are defined and considered in Spiridonov [2002a, 2003a]. Spiridonov [2003a] showed that the elliptic Milne A_n, Jackson C_n and Bhatnagar-Schlosser D_n theta hypergeometric series in the left sides of his equations (21), (18), and (22), respectively, are totally elliptic and modular invariant. He used these results and other observations to motivate his conjecture that, as in the one-variable series case, every totally elliptic multiple hypergeometric series is modular invariant.

§11.6 The $p = 0, s = q$ special case of (11.6.21) is equivalent to a corrected version of the generalization of (3.7.6) given in Subbarao and Verma [1999].

§11.7 Rosengren's inductive proof of (11.7.8) is similar to Milne's [1985a, pp. 49–50] inductive proof of (11.7.1) based on the analysis in Milne [1980b, pp. 179–182, and 1985c, pp. 17–19]. For additional material on basic and elliptic summation and transformation formulas, see Bhatnagar [1998, 1999], Bhatnagar and Milne [1997], Bhatnagar and Schlosser [1998], van Diejen [1997b], van Diejen and Spiridonov [2000–2003], Gustafson [1987a–1994b], Gustafson and Krattenthaler [1997], Gustafson and Rakha [2000], Ito [2002], Leininger and Milne [1999a,b], Lilly and Milne [1993], Milne [1980a–2002], Milne and Bhatnagar [1998], Milne and Lilly [1992, 1995], Milne and Schlosser [2002], Rains [2003a,b], Rosengren [1999–2003f], Rosengren and Schlosser [2003a,b], Schlosser [1997–2003e], Spiridonov [1999–2003b], and Warnaar [1999–2003e].

Ex. 11.15 This orthogonality relation is an elliptic analogue of Krattenthaler [1996, (1.5)].

Ex. 11.17 This summation formula is an elliptic analogue of Gasper [1989a, (5.22) with $b = q^{n+1}$].

Ex. 11.18 This identity is an elliptic extension of W. Chu [1995, (4.6d)].

Ex. 11.20 This formula is a generalization of Gessel and Stanton [1983, (6.14)].

Ex. 11.21 When $p = 0$ and $b = a$ this formula reduces to Gessel and Stanton [1983, (1.4)]. When $p = 0$ and $e = a$ it reduces to W. Chu [1995, (5.1d)] and to Rahman [1993, (1.9) with $b = q^{-2n}$].

Ex. 11.22 When $p = 0$ and $b = a$ this is an elliptic analogue of Gasper [1989a, (5.22) with $c = q^{-3n}$].

Ex. 11.29 Also see the material on elliptic integrals and elliptic gamma functions in Felder, Stevens and Varchenko [2003a,b], Felder and Varchenko [2000-2003b], Narukawa [2003], and Nishizawa [2002].

Ex. 11.36 Also see the elliptic summation formula in Rosengren and Schlosser [2003b, Corollary 5.3].

Ex. 11.38 The $p = 0$ case was independently discovered by Bhatnagar [1999] and Schlosser [1997].

Ex. 11.39 When $p = 0$ this reduces to an identity in Schlosser [1997].

Ex. 11.40 Equivalent forms of the $p = 0$ case were independently discovered by Denis and Gustafson [1992, Theorem 3.1], who derived it from a multivariable integral transformations via residues, and by Milne and Newcomb [1996, Theorem 3.1] via series manipulations. Also see Rosengren [2003a].

Appendix I

IDENTITIES INVOLVING q-SHIFTED FACTORIALS, q-GAMMA FUNCTIONS AND q-BINOMIAL COEFFICIENTS

q-Shifted factorials:

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots, \\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})\right]^{-1}, & n=-1,-2,\ldots . \end{cases} \tag{I.1}$$

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{\binom{n}{2}} \tag{I.2}$$

and

$$(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\binom{n}{2}}, \tag{I.3}$$

where $\binom{n}{2} = n(n-1)/2$.

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \tag{I.4}$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{I.5}$$

and, for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{I.6}$$

where the principal value of q^α is taken and it is assumed that $|q|<1$.

$$(a;q)_n = \left(q^{1-n}/a;q\right)_n (-a)^n q^{\binom{n}{2}}. \tag{I.7}$$

$$\left(aq^{-n};q\right)_n = (q/a;q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}. \tag{I.8}$$

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\left(\frac{a}{b}\right)^n. \tag{I.9}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}. \tag{I.10}$$

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n}{(b;q)_n}\frac{\left(q^{1-n}/b;q\right)_k}{\left(q^{1-n}/a;q\right)_k}\left(\frac{b}{a}\right)^k. \tag{I.11}$$

$$\left(q^{-n};q\right)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{I.12}$$

$$\left(aq^{-n};q\right)_k = \frac{(a;q)_k(q/a;q)_n}{\left(q^{1-k}/a;q\right)_n} q^{-nk}. \tag{I.13}$$

$$\left(aq^{-n};q\right)_{n-k} = \frac{(q/a;q)_n}{(q/a;q)_k}\left(-\frac{a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}}. \tag{I.14}$$

$$\left(aq^{-2n};q\right)_n = \frac{(q/a;q)_{2n}}{(q/a;q)_n}\left(-\frac{a}{q^2}\right)^n q^{-3\binom{n}{2}}. \tag{I.15}$$

$$\left(aq^{-kn};q\right)_n = \frac{(q/a;q)_{kn}}{(q/a;q)_{(k-1)n}}(-a)^n q^{\binom{n}{2}-kn^2}. \tag{I.16}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k. \tag{I.17}$$

$$(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}. \tag{I.18}$$

$$(aq^{kn};q)_n = \frac{(a;q)_{(k+1)n}}{(a;q)_{kn}}. \tag{I.19}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}. \tag{I.20}$$

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}. \tag{I.21}$$

$$\left(aq^{jk};q\right)_{n-k} = \frac{(a;q)_n(aq^n;q)_{(j-1)k}}{(a;q)_{jk}}. \tag{I.22}$$

$$(a_1,a_2,\ldots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n. \tag{I.23}$$

$$(a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty. \tag{I.24}$$

$$(a;q)_{2n} = (a,aq;q^2)_n, \tag{I.25}$$

$$(a;q)_{3n} = (a,aq,aq^2;q^3)_n, \tag{I.26}$$

and, in general,

$$(a;q)_{kn} = (a,aq,\ldots,aq^{k-1};q^k)_n. \tag{I.27}$$

$$(a^2;q^2)_n = (a,-a;q)_n, \tag{I.28}$$

$$(a^3;q^3)_n = (a,a\omega,a\omega^2;q)_n, \quad \omega = e^{2\pi i/3}, \tag{I.29}$$

and, in general,

$$(a^k;q^k)_n = (a,a\omega_k,\ldots,a\omega_k^{k-1};q)_n, \quad \omega_k = e^{2\pi i/k}. \tag{I.30}$$

$$\frac{(qa^{\frac{1}{2}},-qa^{\frac{1}{2}};q)_n}{(a^{\frac{1}{2}},-a^{\frac{1}{2}};q)_n} = \frac{(aq^2;q^2)_n}{(a;q^2)_n} = \frac{1-aq^{2n}}{1-a}, \tag{I.31}$$

$$\frac{\left(qa^{\frac{1}{3}},q\omega a^{\frac{1}{3}},q\omega^2a^{\frac{1}{3}};q\right)_n}{\left(a^{\frac{1}{3}},\omega a^{\frac{1}{3}},\omega^2a^{\frac{1}{3}};q\right)_n} = \frac{(aq^3;q^3)}{(a;q^3)_n} = \frac{1-aq^{3n}}{1-a}, \tag{I.32}$$

and, in general,

$$\frac{\left(qa^{\frac{1}{k}},q\omega_k a^{\frac{1}{k}},\ldots q\omega_k^{k-1}a^{\frac{1}{k}};q\right)_n}{\left(a^{\frac{1}{k}},\omega_k a^{\frac{1}{k}},\ldots,\omega_k^{k-1}a^{\frac{1}{k}};q\right)_n} = \frac{(aq^k;q^k)_n}{(a;q^k)_n} = \frac{1-aq^{kn}}{1-a}, \tag{I.33}$$

where $\omega = e^{2\pi i/3}$ and $\omega_k = e^{2\pi i/k}$.

$$\lim_{q\to 1^-}\frac{(zq^\alpha;q)_\infty}{(z;q)_\infty} = (1-z)^{-\alpha}, \quad |z|<1. \tag{I.34}$$

q-Gamma function:

$$\Gamma_q(x) = \begin{cases} \dfrac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, & 0<q<1, \\ \dfrac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, & q>1. \end{cases} \tag{I.35}$$

$$\lim_{q\to 1}\Gamma_q(x) = \Gamma(x). \tag{I.36}$$

$$\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right) = \Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)(1+q)^{2x-1}. \tag{I.37}$$

$$\Gamma_q(nx)\Gamma_r\left(\frac{1}{n}\right)\Gamma_r\left(\frac{2}{n}\right)\cdots\Gamma_r\left(\frac{n-1}{n}\right)$$
$$= \left(1+q+\ldots+q^{n-1}\right)^{nx-1}\Gamma_r(x)\Gamma_r\left(x+\frac{1}{n}\right)\cdots\Gamma_r\left(x+\frac{n-1}{n}\right), \tag{I.38}$$

with $r=q^n$.

q-Binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \tag{I.39}$$

and, for $|q|<1$ and complex α and β,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\left(q^{\beta+1},q^{\alpha-\beta+1};q\right)_\infty}{(q,q^{\alpha+1};q)_\infty}, \tag{I.40}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}, \tag{I.41}$$

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\left(-q^\alpha\right)^k q^{-\binom{k}{2}}, \tag{I.42}$$

$$\begin{bmatrix} k+\alpha \\ k \end{bmatrix}_q = \frac{\left(q^{\alpha+1};q\right)_k}{(q;q)_k}, \tag{I.43}$$

$$\begin{bmatrix} -\alpha \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q \left(-q^{-\alpha}\right)^k q^{-\binom{k}{2}}, \tag{I.44}$$

$$\begin{bmatrix} \alpha+1 \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^k + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q q^{\alpha+1-k}, \tag{I.45}$$

$$\begin{bmatrix} n+\alpha \\ n-k \end{bmatrix}_q = \frac{(q^{\alpha+1};q)_n}{(q;q)_{n-k}(q^{\alpha+1};q)_k}, \tag{I.46}$$

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{k^2-\alpha k}, \tag{I.47}$$

where n, k are nonnegative integers. For elliptic analogues, see Chapter 11.

Appendix II

SELECTED SUMMATION FORMULAS

Sums of basic hypergeometric series:

The two q-exponential functions,

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1, \tag{II.1}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-z;q)_\infty. \tag{II.2}$$

The q-binomial theorem,

$${}_1\phi_0(a;—;q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \tag{II.3}$$

or, when $a = q^{-n}$, where, as elsewhere in this appendix, n denotes a nonnegative integer,

$${}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n. \tag{II.4}$$

The sum of a ${}_1\phi_1$ series,

$${}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{II.5}$$

The q-Vandermonde (q-Chu-Vandermonde) sums,

$${}_2\phi_1(a,q^{-n};c;q,q) = \frac{(c/a;q)_n}{(c;q)_n} a^n \tag{II.6}$$

and, reversing the order of summation,

$${}_2\phi_1(a,q^{-n};c;q,cq^n/a) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{II.7}$$

The q-Gauss sum,

$${}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \tag{II.8}$$

The q-Kummer (Bailey-Daum) sum,

$${}_2\phi_1(a,b;aq/b;q,-q/b) = \frac{(-q;q)_\infty (aq,aq^2/b^2;q^2)_\infty}{(-q/b,aq/b;q)_\infty}. \tag{II.9}$$

A q-analogue of Bailey's ${}_2F_1(-1)$ sum,

$${}_2\phi_2(a,q/a;-q,b;q,-b) = \frac{(ab,bq/a;q^2)_\infty}{(b;q)_\infty}. \tag{II.10}$$

A q-analogue of Gauss' ${}_2F_1(-1)$ sum,

$$ {}_2\phi_2\left(a^2, b^2; abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}; q, -q\right) = \frac{(a^2q, b^2q; q^2)_\infty}{(q, a^2b^2q; q^2)_\infty}. \tag{II.11} $$

The q-Saalschütz (q-Pfaff-Saalschütz) sum,

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{II.12} $$

The q-Dixon sum,

$$ {}_4\phi_3\left[\begin{matrix} a, -qa^{\frac{1}{2}}, b, c \\ -a^{\frac{1}{2}}, aq/b, aq/c \end{matrix}; q, \frac{qa^{\frac{1}{2}}}{bc}\right] = \frac{\left(aq, qb^{-1}a^{\frac{1}{2}}, qc^{-1}a^{\frac{1}{2}}, aq/bc; q\right)_\infty}{\left(aq/b, aq/c, qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/bc; q\right)_\infty}, \tag{II.13} $$

or, when $c = q^{-n}$,

$$ {}_4\phi_3\left[\begin{matrix} a, -qa^{\frac{1}{2}}, b, q^{-n} \\ -a^{\frac{1}{2}}, aq/b, aq^{1+n} \end{matrix}; q, \frac{q^{1+n}a^{\frac{1}{2}}}{b}\right] = \frac{\left(aq, qa^{\frac{1}{2}}/b; q\right)_n}{\left(qa^{\frac{1}{2}}, aq/b; q\right)_n}. \tag{II.14} $$

Jackson's terminating q-analogue of Dixon's sum,

$$ {}_3\phi_2\left[\begin{matrix} q^{-2n}, & b, & c \\ & q^{1-2n}/b, & q^{1-2n}/c \end{matrix}; q, \frac{q^{2-n}}{bc}\right] = \frac{(b, c; q)_n(q, bc; q)_{2n}}{(q, bc; q)_n(b, c; q)_{2n}}. \tag{II.15} $$

A q-analogue of Watson's ${}_3F_2$ sum,

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} \lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, a, b, c, -c, \lambda q/c^2 \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, \lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, c^2 \end{matrix}; q, -\frac{\lambda q}{ab}\right] \\ &= \frac{(\lambda q, c^2/\lambda; q)_\infty(aq, bq, c^2q/a, c^2q/b; q^2)_\infty}{(\lambda q/a, \lambda q/b; q)_\infty(q, abq, c^2q, c^2q/ab; q^2)_\infty}, \end{aligned} \tag{II.16} $$

where $\lambda = -c(ab/q)^{\frac{1}{2}}$; and Andrews' terminating q-analogue,

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, a^2q^{n+1}, c, -c \\ aq, -aq, c^2 \end{matrix}; q, q\right] \\ &= \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{c^n(q, a^2q^2/c^2; q^2)_{n/2}}{(a^2q^2, c^2q; q^2)_{n/2}}, & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{II.17} $$

A q-analogue of Whipple's ${}_3F_2$ sum,

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} -c, & q(-c)^{\frac{1}{2}}, & -q(-c)^{\frac{1}{2}}, & a, & q/a, & c, & -d, & -q/d \\ & (-c)^{\frac{1}{2}}, & -(-c)^{\frac{1}{2}}, & -cq/a, & -ac, & -q, & cq/d, & cd \end{matrix}; q, c\right] \\ &= \frac{(-c, -cq; q)_\infty\left(acd, acq/d, cdq/a, cq^2/ad; q^2\right)_\infty}{(cd, cq/d, -ac, -cq/a; q)_\infty}, \end{aligned} \tag{II.18} $$

and a terminating q-analogue,

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+1}, c, -c \\ e, c^2q/e, -q \end{matrix}; q, q\right] \\ &= \frac{\left(eq^{-n}, eq^{n+1}, c^2q^{1-n}/e, c^2q^{n+2}/e; q^2\right)_\infty}{(e, c^2q/e; q)_\infty} q^{n(n+1)/2}. \end{aligned} \tag{II.19} $$

The sum of a very-well-poised ${}_6\phi_5$ series,

$$
{}_6\phi_5\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{aq}{bcd}\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty} \tag{II.20}
$$

or, when $d = q^{-n}$,

$$
{}_6\phi_5\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \tag{II.21}
$$

Jackson's q-analogue of Dougall's ${}_7F_6$ sum,

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, q\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{II.22}
$$

where $a^2 q = bcdeq^{-n}$.

A nonterminating form of the q-Vandermonde sum,

$$
{}_2\phi_1(a, b; c; q, q) + \frac{(q/c, a, b; q)_\infty}{(c/q, aq/c, bq/c; q)_\infty}
\times {}_2\phi_1(aq/c, bq/c; q^2/c; q, q) = \frac{(q/c, abq/c; q)_\infty}{(aq/c, bq/c; q)_\infty}. \tag{II.23}
$$

A nonterminating form of the q-Saalschütz sum,

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; q, q\right] + \frac{(q/e, a, b, c, qf/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, f; q)_\infty}
\times {}_3\phi_2\left[\begin{matrix} aq/e, & bq/e, & cq/e \\ q^2/e, & qf/e & \end{matrix} ; q, q\right] = \frac{(q/e, f/a, f/b, f/c; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty}, \tag{II.24}
$$

where $ef = abcq$.

Bailey's nonterminating extension of Jackson's ${}_8\phi_7$ sum,

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & f \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} ; q, q\right]
- \frac{b}{a}\,\frac{(aq, c, d, e, f, bq/a, bq/c, bq/d, bq/e, bq/f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a; q)_\infty}
\times {}_8\phi_7\left[\begin{matrix} b^2/a, qba^{-\frac12}, -qba^{-\frac12}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac12}, -ba^{-\frac12}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix} ; q, q\right]
= \frac{(aq, b/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a; q)_\infty}, \tag{II.25}
$$

where $qa^2 = bcdef$.

q-Analogues of the Karlsson-Minton sums,

$$
{}_{r+2}\phi_{r+1}\left[\begin{matrix} a, b, b_1q^{m_1}, \ldots, b_rq^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right]
= \frac{(q, bq/a; q)_\infty (b_1/b; q)_{m_1} \cdots (b_r/b; q)_{m_r}}{(bq, q/a; q)_\infty (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}} b^{m_1+\cdots+m_r} \tag{II.26}
$$

and

$$
{}_{r+1}\phi_r\left[\begin{matrix} a, b_1q^{m_1}, \ldots, b_rq^{m_r} \\ b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{-(m_1+\cdots+m_r)}\right] = 0, \tag{II.27}
$$

where $m_1, \ldots, m_r$ are arbitrary nonnegative integers.

Sums of bilateral basic series:

Jacobi's triple product,

$$
\sum_{k=-\infty}^{\infty} q^{k^2} z^k = \left(q^2, -qz, -q/z; q^2\right)_\infty . \tag{II.28}
$$

Ramanujan's sum,

$$
{}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}. \tag{II.29}
$$

The sum of a well-poised ${}_2\psi_2$ series,

$$
{}_2\psi_2(b, c; aq/b, aq/c; q, -aq/bc)
= \frac{(aq/bc; q)_\infty (aq^2/b^2, aq^2/c^2, q^2, aq, q/a; q^2)_\infty}{(aq/b, aq/c, q/b, q/c, -aq/bc; q)_\infty}. \tag{II.30}
$$

Bailey's sum of a well-poised ${}_3\psi_3$,

$$
{}_3\psi_3\left[\begin{matrix} b, c, d \\ q/b, q/c, q/d \end{matrix}; q, \frac{q}{bcd}\right]
= \frac{(q, q/bc, q/bd, q/cd; q)_\infty}{(q/b, q/c, q/d, q/bcd; q)_\infty}. \tag{II.31}
$$

A basic bilateral analogue of Dixon's sum,

$$
{}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right]
= \frac{(aq, aq/bc, aq/bd, aq/cd, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c, qa^{\frac{1}{2}}/d, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, qa^{\frac{3}{2}}/bcd; q)_\infty}. \tag{II.32}
$$

The sum of a very-well-poised ${}_6\psi_6$ series,

$$
{}_6\psi_6\left[\begin{matrix} qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e \\ a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix}; q, \frac{qa^2}{bcde}\right]
= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}. \tag{II.33}
$$

Bibasic sums:

Gosper's indefinite bibasic sum,

$$\sum_{k=0}^{n} \frac{1-ap^kq^k}{1-a} \frac{(a;p)_k(c;q)_k}{(q;q)_k(ap/c;p)_k} q^k = \frac{(ap;p)_n(cq;q)_n}{(q;q)_n(ap/c;p)_n} c^{-n}. \tag{II.34}$$

An extension of (II.34),

$$\begin{aligned}&\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k \\ &= \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}\end{aligned} \tag{II.35}$$

and, more generally,

$$\begin{aligned}&\sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k \\ &= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \left\{ \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} \right. \\ &\quad \left. - \frac{(c/ad,d/bc;p)_{m+1}(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}(1/a,1/b;p)_{m+1}} \right\},\end{aligned} \tag{II.36}$$

where m is an integer or $+\infty$.

An extension of the formula for the n-th q-difference of $(ap^k;q)_{n-1}$,

$$\left(1-\frac{a}{q}\right)\left(1-\frac{b}{q}\right) \sum_{k=0}^{n} \frac{(ap^k,bp^{-k};q)_{n-1}(1-ap^{2k}/b)}{(p;p)_k(p;p)_{n-k}(ap^k/b;p)_{n+1}} (-1)^k p^{\binom{k}{2}} = \delta_{n,0}. \tag{II.37}$$

Appendix III

SELECTED TRANSFORMATION FORMULAS

Heine's transformations of $_2\phi_1$ series:

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b) \tag{III.1}
$$

$$
= \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(abz/c,b;bz;q,c/b) \tag{III.2}
$$

$$
= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{III.3}
$$

Jackson's transformations of $_2\phi_1$, $_2\phi_2$ and $_3\phi_2$ series:

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2(a,c/b;c,az;q,bz) \tag{III.4}
$$

$$
\begin{aligned}
&= \frac{(abz/c;q)_\infty}{(bz/c;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} a, c/b, 0 \\ c, cq/bz \end{matrix}; q, q\right] \\
&\quad + \frac{(a,bz,c/b;q)_\infty}{(c,z,c/bz;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} z, abz/c, 0 \\ bz, bzq/c \end{matrix}; q, q\right].
\end{aligned} \tag{III.5}
$$

Transformations of terminating $_2\phi_1$ series:

$$
\begin{aligned}
{}_2\phi_1(q^{-n},b;c;q,z) &= \frac{(c/b;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n \\
&\quad \times {}_3\phi_2(q^{-n},q/z,c^{-1}q^{1-n};bc^{-1}q^{1-n},0;q,q)
\end{aligned} \tag{III.6}
$$

$$
= \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{matrix}; q, q\right] \tag{III.7}
$$

$$
= \frac{(c/b;q)_n}{(c;q)_n} b^n\, {}_3\phi_1\left[\begin{matrix} q^{-n}, b, q/z \\ bq^{1-n}/c \end{matrix}; q, \frac{z}{c}\right], \tag{III.8}
$$

where, as elsewhere in this appendix, n denotes a non-negative integer.

Transformations of $_3\phi_2$ series:

$$
{}_3\phi_2\left[\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}; q, \frac{de}{abc}\right]
$$

$$
= \frac{(e/a,de/bc;q)_\infty}{(e,de/abc;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} a,\ d/b,\ d/c \\ d,\ de/bc \end{matrix}; q, \frac{e}{a}\right] \tag{III.9}
$$

$$
= \frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b\right], \tag{III.10}
$$

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n},\ b,\ c \\ d,\ e \end{matrix} ; q, q\right] \\
&= \frac{(de/bc;q)_n}{(e;q)_n}\left(\frac{bc}{d}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n},\ d/b,\ d/c \\ d,\ de/bc \end{matrix} ; q, q\right] \qquad \text{(III.11)} \\
&= \frac{(e/c;q)_n}{(e;q)_n} c^n\, {}_3\phi_2\left[\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix} ; q, \frac{bq}{e}\right], \qquad \text{(III.12)}
\end{aligned}
$$

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, \frac{deq^n}{bc}\right] = \frac{(e/c;q)_n}{(e;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n},\ c,\ d/b \\ d,\ cq^{1-n}/e \end{matrix} ; q, q\right]. \qquad \text{(III.13)}
$$

The Sears-Carlitz transformation of a terminating well-poised ${}_3\phi_2$ series,

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} a, & b, & c \\ & aq/b, & aq/c \end{matrix} ; q, \frac{aqz}{bc}\right] \\
&= \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_5\phi_4\left[\begin{matrix} a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, aq/bc \\ aq/b, aq/c, az, q/z \end{matrix} ; q, q\right] \qquad \text{(III.14)}
\end{aligned}
$$

provided that $a = q^{-n}$. See (III.35) for a nonterminating case.

Sears' transformations of terminating balanced ${}_4\phi_3$ series:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix} ; q, q\right] \\
&= \frac{(e/a, f/a;q)_n}{(e, f;q)_n} a^n\, {}_4\phi_3\left[\begin{matrix} q^{-n},\ a,\ d/b,\ d/c \\ d,\ aq^{1-n}/e,\ aq^{1-n}/f \end{matrix} ; q, q\right] \qquad \text{(III.15)} \\
&= \frac{(a, ef/ab, ef/ac;q)_n}{(e, f, ef/abc;q)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, e/a, f/a, ef/abc \\ ef/ab, ef/ac, q^{1-n}/a \end{matrix} ; q, q\right], \qquad \text{(III.16)}
\end{aligned}
$$

where $def = abcq^{1-n}$.

Watson's transformation formulas:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & f \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef}\right] \\
&= \frac{(aq, aq/de, aq/df, aq/ef;q)_\infty}{(aq/d, aq/e, aq/f, aq/def;q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix} ; q, q\right]
\end{aligned}
$$

(III.17)

whenever the ${}_8\phi_7$ series converges and the ${}_4\phi_3$ series terminates, and, when $f = q^{-n}$,

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, \frac{a^2q^{n+2}}{bcde}\right] \\
&= \frac{(aq, aq/de;q)_n}{(aq/d, aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & q^{-n} \\ & aq/b, & aq/c, & deq^{-n}/a \end{matrix} ; q, q\right]
\end{aligned}
$$

(III.18)

or, equivalently,

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right] = \frac{(d/b, d/c; q)_n}{(d, d/bc; q)_n}
$$
$$
\times\ {}_8\phi_7\left[\begin{matrix} \sigma, & q\sigma^{\frac{1}{2}}, & -q\sigma^{\frac{1}{2}}, & f/a, & e/a, & b, & c, & q^{-n} \\ & \sigma^{\frac{1}{2}}, & -\sigma^{\frac{1}{2}}, & e, & f, & ef/ab, & ef/ac, & efq^n/a \end{matrix}; q, \frac{efq^n}{bc}\right], \tag{III.19}
$$

where $def = abcq^{1-n}$ and $\sigma = ef/aq$.

Another transformation of a terminating balanced ${}_4\phi_3$ series to a very-well-poised ${}_8\phi_7$ series,

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right] = \frac{(abq/f, acq/f, bcq/f, q/f; q)_\infty}{(aq/f, bq/f, cq/f, abcq/f; q)_\infty}
$$
$$
\times\ {}_8\phi_7\left[\begin{matrix} \mu, & q\mu^{\frac{1}{2}}, & -q\mu^{\frac{1}{2}}, & a, & b, & c, & dq^n, & eq^n \\ & \mu^{\frac{1}{2}}, & -\mu^{\frac{1}{2}}, & \mu q/a, & \mu q/b, & \mu q/c, & e, & d \end{matrix}; q, \frac{de}{abc}\right], \tag{III.20}
$$

where $def = abcq^{1-n}$ and $\mu = abc/f$.

Singh's quadratic transformation:

$$
{}_4\phi_3\left[\begin{matrix} a^2, b^2, c, d \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -cd \end{matrix}; q, q\right]
= {}_4\phi_3\left[\begin{matrix} a^2, b^2, c^2, d^2 \\ a^2b^2q, -cd, -cdq \end{matrix}; q^2, q^2\right], \tag{III.21}
$$

provided the series terminate.

A q-analogue of Clausen's formula:

$$
\left\{ {}_4\phi_3\left[\begin{matrix} a, b, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix}; q, q\right]\right\}^2
= {}_5\phi_4\left[\begin{matrix} a^2, b^2, ab, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab, a^2b^2 \end{matrix}; q, q\right] \tag{III.22}
$$

provided both series terminate. A non-terminating q-analogue of Clausen's formula is given in (8.8.17).

Transformations of very-well-poised ${}_8\phi_7$ series:

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & f \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right]
$$
$$
= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef; q)_\infty}
$$
$$
\times\ {}_8\phi_7\left[\begin{matrix} \lambda, & q\lambda^{\frac{1}{2}}, & -q\lambda^{\frac{1}{2}}, & \lambda b/a, & \lambda c/a, & \lambda d/a, & e, & f \\ & \lambda^{\frac{1}{2}}, & -\lambda^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & \lambda q/e, & \lambda q/f \end{matrix}; q, \frac{aq}{ef}\right] \tag{III.23}
$$

$$= \frac{(aq, b, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a;q)_\infty}{(aq/c, aq/d, aq/e, aq/f, \mu q, b\mu/a;q)_\infty}$$

$$\times\, {}_8\phi_7 \left[\begin{matrix} \mu, & q\mu^{\frac12}, & -q\mu^{\frac12}, & aq/bc, & aq/bd, & aq/be, & aq/bf, & b\mu/a \\ & \mu^{\frac12}, & -\mu^{\frac12}, & bc\mu/a, & bd\mu/a, & be\mu/a, & bf\mu/a, & aq/b \end{matrix} ; q, b \right], \tag{III.24}$$

where $\lambda = qa^2/bcd$ and $\mu = q^2a^3/b^2cdef$.

Transformations of a nearly-poised ${}_5\phi_4$ series:

$${}_5\phi_4 \left[\begin{matrix} a, & b, & c, & d, & q^{-n} \\ & aq/b, & aq/c, & aq/d, & a^2q^{-n}/\lambda^2 \end{matrix} ; q, q \right]$$

$$= \frac{(\lambda q/a, \lambda^2 q/a;q)_n}{(\lambda q, \lambda^2 q/a^2;q)_n}\, {}_{12}\phi_{11} \left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & b\lambda/a, & c\lambda/a, & d\lambda/a, \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & aq/b, & aq/c, & aq/d, \end{matrix} \right.$$

$$\left. \begin{matrix} a^{\frac12}, & -a^{\frac12}, & (aq)^{\frac12}, & -(aq)^{\frac12}, & \lambda^2q^{n+1}/a, & q^{-n} \\ \lambda q/a^{\frac12}, & -\lambda q/a^{\frac12}, & \lambda(q/a)^{\frac12}, & -\lambda(q/a)^{\frac12}, & aq^{-n}/\lambda, & \lambda q^{n+1} \end{matrix} ; q, q \right], \tag{III.25}$$

$${}_5\phi_4 \left[\begin{matrix} q^{-n}, & b, & c, & d, & e \\ & q^{1-n}/b, & q^{1-n}/c, & q^{1-n}/d, & eq^{-2n}/\mu^2 \end{matrix} ; q, q \right]$$

$$= \frac{(\mu^2q^{n+1}, \mu q/e;q)_n}{(\mu^2q^{n+1}/e, \mu q;q)_n}$$

$$\times\, {}_{12}\phi_{11} \left[\begin{matrix} \mu, & q\mu^{\frac12}, & -q\mu^{\frac12}, & \mu bq^n, & \mu cq^n, & \mu dq^n, \\ & \mu^{\frac12}, & -\mu^{\frac12}, & q^{1-n}/b, & q^{1-n}/c, & q^{1-n}/d, \end{matrix} \right.$$

$$\left. \begin{matrix} q^{-n/2}, & -q^{-n/2}, & q^{(1-n)/2}, & -q^{(1-n)/2}, & e, & \mu^2q^{n+1}/e \\ \mu q^{(n+2)/2}, & -\mu q^{(n+2)/2}, & \mu q^{(n+1)/2}, & -\mu q^{(n+1)/2}, & \mu q/e, & eq^{-n}/\mu \end{matrix} ; q, q \right], \tag{III.26}$$

where $\lambda = qa^2/bcd$ and $\mu = q^{1-2n}/bcd$.

Transformation of a nearly-poised ${}_7\phi_6$ series:

$${}_7\phi_6 \left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & a^2q^{2-n}/\lambda^2 \end{matrix} ; q, q \right]$$

$$= \frac{(\lambda/aq, \lambda^2/aq;q)_n}{(\lambda q, \lambda^2/a^2q;q)_n} \frac{(1-\lambda^2q^{2n-1}/a)}{(1-\lambda^2/aq)}\, {}_{12}\phi_{11} \left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & b\lambda/a, & c\lambda/a, \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & aq/b, & aq/c, \end{matrix} \right.$$

$$\left. \begin{matrix} d\lambda/a, & (aq)^{\frac12}, & -(aq)^{\frac12}, & qa^{\frac12}, & -qa^{\frac12}, & \lambda^2q^{n-1}/a, & q^{-n} \\ aq/d, & \lambda(q/a)^{\frac12}, & -\lambda(q/a)^{\frac12}, & \lambda/a^{\frac12}, & -\lambda/a^{\frac12}, & aq^{2-n}/\lambda, & \lambda q^{n+1} \end{matrix} ; q, q \right], \tag{III.27}$$

where $\lambda = qa^2/bcd$.

Bailey's $_{10}\phi_9$ transformation formula:

$$
\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n} \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1} \end{matrix}; q, q\right] \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q)_n}\, {}_{10}\phi_9\left[\begin{matrix} \lambda, & q\lambda^{\frac12}, & -q\lambda^{\frac12}, & \lambda b/a, & \lambda c/a, \\ & \lambda^{\frac12}, & -\lambda^{\frac12}, & aq/b, & aq/c, \end{matrix}\right. \\
&\qquad \left.\begin{matrix} \lambda d/a, & e, & f, & \lambda aq^{n+1}/ef, & q^{-n}, \\ aq/d, & \lambda q/e, & \lambda q/f, & efq^{-n}/a, & \lambda q^{n+1} \end{matrix}; q, q\right],
\end{aligned} \tag{III.28}
$$

where $\lambda = qa^2/bcd$.

Transformations of $_{r+2}\phi_{r+1}$ series:

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1}\left[\begin{matrix} a, b, b_1q^{m_1}, \ldots, b_rq^{m_r} \\ bq^{1+m}, b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{m+1-(m_1+\cdots+m_r)}\right] \\
&= \frac{(q, bq/a; q)_\infty}{(bq, q/a; q)_\infty} \frac{(bq;q)_m (b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(q;q)_m (b_1;q)_{m_1}\cdots(b_r;q)_{m_r}} b^{m_1+\cdots+m_r-m} \\
&\quad \times {}_{r+2}\phi_{r+1}\left[\begin{matrix} q^{-m}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix}; q, q\right]
\end{aligned} \tag{III.29}
$$

and

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1}\left[\begin{matrix} a, b, b_1q^{m_1}, \ldots, b_rq^{m_r} \\ bcq, b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right] \\
&= \frac{(bq/a, cq; q)_\infty}{(bcq, q/a; q)_\infty} \frac{(b_1/b;q)_{m_1}\cdots(b_r/b;q)_{m_r}}{(b_1;q)_{m_1}\cdots(b_r;q)_{m_r}} b^{m_1+\cdots+m_r} \\
&\quad \times {}_{r+2}\phi_{r+1}\left[\begin{matrix} c^{-1}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix}; q, cq\right],
\end{aligned} \tag{III.30}
$$

where $m, m_1, \ldots, m_r$ are arbitrary nonnegative integers.

Three-term transformation formulas:

$$
\begin{aligned}
{}_2\phi_1(a, b; c; q, z) &= \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty}\, {}_2\phi_1(c/a, cq/abz; cq/az; q, bq/c) \\
&\quad - \frac{(b, q/c, c/a, az/q, q^2/az; q)_\infty}{(c/q, bq/c, q/a, az/c, cq/az; q)_\infty}\, {}_2\phi_1(aq/c, bq/c; q^2/c; q, z).
\end{aligned} \tag{III.31}
$$

$$
\begin{aligned}
{}_2\phi_1(a, b; c; q, z) &= \frac{(b, c/a, az, q/az; q)_\infty}{(c, b/a, z, q/z; q)_\infty}\, {}_2\phi_1(a, aq/c; aq/b; q, cq/abz) \\
&\quad + \frac{(a, c/b, bz, q/bz; q)_\infty}{(c, a/b, z, q/z; q)_\infty}\, {}_2\phi_1(b, bq/c; bq/a; q, cq/abz).
\end{aligned} \tag{III.32}
$$

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right]\\
&=\frac{(e/b,e/c,cq/a,q/d;q)_\infty}{(e,cq/d,q/a,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}c,d/a,cq/e\\ cq/a,bcq/e\end{matrix};q,\frac{bq}{d}\right]\\
&\quad-\frac{(q/d,eq/d,b,c,d/a,de/bcq,bcq^2/de;q)_\infty}{(d/q,e,bq/d,cq/d,q/a,e/bc,bcq/e;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq/d,bq/d,cq/d\\ q^2/d,eq/d\end{matrix};q,\frac{de}{abc}\right].
\end{aligned}
\tag{III.33}
$$

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right]&=\frac{(e/b,e/c;q)_\infty}{(e,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}d/a,\ b,\ c\\ d,\ bcq/e\end{matrix};q,q\right]\\
&+\frac{(d/a,b,c,de/bc;q)_\infty}{(d,e,bc/e,de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}e/b,e/c,de/abc\\ de/bc,eq/bc\end{matrix};q,q\right].
\end{aligned}
\tag{III.34}
$$

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}a, & b, & c\\ & aq/b, & aq/c\end{matrix};q,\frac{aqx}{bc}\right]\\
&=\frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_5\phi_4\left[\begin{matrix}a^{\frac12}, & -a^{\frac12}, & (aq)^{\frac12}, & -(aq)^{\frac12}, & aq/bc\\ & aq/b, & aq/c, & ax, & q/x\end{matrix};q,q\right]\\
&\quad+\frac{(a,aq/bc,aqx/b,aqx/c;q)_\infty}{(aq/b,aq/c,aqx/bc,x^{-1};q)_\infty}\\
&\quad\times{}_5\phi_4\left[\begin{matrix}xa^{\frac12},-xa^{\frac12},x(aq)^{\frac12},-x(aq)^{\frac12},aqx/bc\\ aqx/b,aqx/c,xq,ax^2\end{matrix};q,q\right].
\end{aligned}
\tag{III.35}
$$

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & f\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right]\\
&=\frac{(aq,aq/de,aq/df,aq/ef;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}\,{}_4\phi_3\left[\begin{matrix}aq/bc,d,e,f\\ aq/b,aq/c,def/a\end{matrix};q,q\right]\\
&\quad+\frac{(aq,aq/bc,d,e,f,a^2q^2/bdef,a^2q^2/cdef;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,a^2q^2/bcdef,def/aq;q)_\infty}\\
&\quad\times{}_4\phi_3\left[\begin{matrix}aq/de,aq/df,aq/ef,a^2q^2/bcdef\\ a^2q^2/bdef,a^2q^2/cdef,aq^2/def\end{matrix};q,q\right].
\end{aligned}
\tag{III.36}
$$

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & f\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right]\\
&=\frac{(aq,aq/de,aq/df,aq/ef,eq/c,fq/c,b/a,bef/a;q)_\infty}{(aq/d,aq/e,aq/f,aq/def,q/c,efq/c,be/a,bf/a;q)_\infty}\\
&\quad\times{}_8\phi_7\left[\begin{matrix}ef/c,q\,(ef/c)^{\frac12},-q\,(ef/c)^{\frac12},aq/bc,aq/cd,ef/a,e,f\\ (ef/c)^{\frac12},-(ef/c)^{\frac12},bef/a,def/a,aq/c,fq/c,eq/c\end{matrix};q,\frac{bd}{a}\right]
\end{aligned}
$$

$$+\frac{b}{a}\frac{(aq,bq/a,bq/c,bq/d,bq/e,bq/f,d,e,f,aq/bc;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,bd/a,be/a;q)_\infty}$$
$$\times\frac{(bdef/a^2,a^2q/bdef;q)_\infty}{(bf/a,def/a,aq/def,q/c,b^2q/a;q)_\infty}$$
$$\times\,{}_8\phi_7\left[\begin{matrix}b^2/a,bqa^{-\frac12},-bqa^{-\frac12},b,bc/a,bd/a,be/a,bf/a\\ ba^{-\frac12},-ba^{-\frac12},bq/a,bq/c,bq/d,bq/e,bq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right]. \tag{III.37}$$

Transformation of an ${}_8\psi_8$ series:

$$\frac{(aq/b,aq/c,aq/d,aq/e,q/ab,q/ac,q/ad,q/ae;q)_\infty}{(fa,ga,f/a,g/a,qa^2,q/a^2;q)_\infty}$$
$$\times\,{}_8\psi_8\left[\begin{matrix}qa, & -qa & ba, & ca, & da, & ea, & fa, & ga\\ a, & -a, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f, & aq/g\end{matrix};q,\frac{q^2}{bcdefg}\right]$$
$$=\frac{(q,q/bf,q/cf,q/df,q/ef,qf/b,qf/c,qf/d,qf/e;q)_\infty}{(fa,q/fa,aq/f,f/a,g/f,fg,qf^2;q)_\infty}$$
$$\times\,{}_8\phi_7\left[\begin{matrix}f^2, & qf, & -qf, & fb, & fc, & fd, & fe, & fg\\ & f, & -f, & fq/b, & fq/c, & fq/d, & fq/e, & fq/g\end{matrix};q,\frac{q^2}{bcdefg}\right]$$
$$+\ \text{idem}\ (f;g). \tag{III.38}$$

Bailey's four-term ${}_{10}\phi_9$ transformation:

$${}_{10}\phi_9\left[\begin{matrix}a,qa^{\frac12},-qa^{\frac12},b,c,d,e,f,g,h\\ a^{\frac12},-a^{\frac12},aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h\end{matrix};q,q\right]$$
$$+\frac{(aq,b/a,c,d,e,f,g,h,bq/c;q)_\infty}{(b^2q/a,a/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a;q)_\infty}$$
$$\times\frac{(bq/d,bq/e,bq/f,bq/g,bq/h;q)_\infty}{(bd/a,be/a,bf/a,bg/a,bh/a;q)_\infty}$$
$$\times\,{}_{10}\phi_9\left[\begin{matrix}b^2/a, & qba^{-\frac12}, & -qba^{-\frac12}, & b, & bc/a, & bd/a, & be/a, & bf/a, & bg/a, & bh/a\\ & ba^{-\frac12}, & -ba^{-\frac12}, & bq/a, & bq/c, & bq/d, & bq/e, & bq/f, & bq/g, & bq/h\end{matrix};q,q\right]$$
$$=\frac{(aq,b/a,\lambda q/f,\lambda q/g,\lambda q/h,bf/\lambda,bg/\lambda,bh/\lambda;q)_\infty}{(\lambda q,b/\lambda,aq/f,aq/g,aq/h,bf/a,bg/a,bh/a;q)_\infty}$$
$$\times\,{}_{10}\phi_9\left[\begin{matrix}\lambda,q\lambda^{\frac12},-q\lambda^{\frac12},b,\lambda c/a,\lambda d/a,\lambda e/a,f,g,h\\ \lambda^{\frac12},-\lambda^{\frac12},\lambda q/b,aq/c,aq/d,aq/e,\lambda q/f,\lambda q/g,\lambda q/h\end{matrix};q,q\right]$$
$$+\frac{(aq,b/a,f,g,h,bq/f,bq/g,bq/h,\lambda c/a,\lambda d/a,\lambda e/a,abq/\lambda c;q)_\infty}{(b^2q/\lambda,\lambda/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a,be/a,bf/a;q)_\infty}$$
$$\times\frac{(abq/\lambda d,abq/\lambda e;q)_\infty}{(bg/a,bh/a;q)_\infty}\,{}_{10}\phi_9\left[\begin{matrix}b^2/\lambda,qb\lambda^{-\frac12},-qb\lambda^{-\frac12},b,bc/a,bd/a,\\ b\lambda^{-\frac12},-b\lambda^{-\frac12},bq/\lambda,abq/c\lambda,abq/d\lambda,\end{matrix}\right.$$
$$\left.\begin{matrix}be/a,bf/\lambda,bg/\lambda,bh/\lambda\\ abq/e\lambda,bq/f,bq/g,bq/h\end{matrix};q,q\right], \tag{III.39}$$

where $a^3q^2 = bcdefgh$ and $\lambda = qa^2/cde$.

Transformation of a ${}_{10}\psi_{10}$ series:

$$\frac{(aq/b, aq/c, aq/d, aq/e, aq/f, q/ab, q/ac, q/ad, q/ae, q/af; q)_\infty}{(ag, ah, ak, g/a, h/a, k/a, qa^2, q/a^2; q)_\infty}$$

$$\times\ {}_{10}\psi_{10}\left[\begin{matrix} qa, -qa, ba, ca, da, ea, fa, ga, ha, ka \\ a, -a, aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, aq/k \end{matrix}; q, \frac{q^3}{bcdefghk}\right]$$

$$= \frac{(q, q/bg, q/cg, q/dg, q/eg, q/fg, qg/b, qg/c, qg/d, qg/e, qg/f; q)_\infty}{(gh, gk, h/g, k/g, ag, q/ag, g/a, aq/g, qg^2; q)_\infty}$$

$$\times\ {}_{10}\phi_9\left[\begin{matrix} g^2, qg, -qg, gb, gc, gd, ge, gf, gh, gk \\ g, -g, qg/b, qg/c, qg/d, qg/e, qg/f, qg/h, qg/k \end{matrix}; q, \frac{q^3}{bcdefghk}\right]$$

$$+\ \text{idem}\ (g; h, k). \tag{III.40}$$

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